Math 140 Exam #2 Key (Spring 2021)

1a $h'(t) = t^{-2/3} - 24t^2 + 9$.

1b Quotient Rule:

$$g'(x) = \frac{(x^2-4)(3x^2) - (x^3+1)(2x)}{(x^2-4)^2} = \frac{x^4 - 12x^2 - 2x}{(x^2-4)^2}.$$

1c Product Rule:

$$y' = \sin x \sec^2 x + \cos x \tan x = \tan x \sec x + \sin x.$$

1d Quotient Rule:

$$y' = \frac{(4+\sin x)(2\sec x \tan x) - (2\sec x)(\cos x)}{(4+\sin x)^2} = \frac{8\sec x \tan x + 2\tan^2 x - 2}{(4+\sin x)^2}.$$

2 First we get $f'(x) = 6x^2 - 6x - 12$. Setting $f'(x) = 60$ gives $6x^2 - 6x - 12 = 60$, or $x^2 - x - 12 = 0$, which factors as $(x-4)(x+3) = 0$. Solutions are $x = -3, 4$. Points on the graph of f with slope 60 are therefore $(-3, -41)$ and $(4, 36)$.

3a $y' = 16(4 - 15x^4)(4x - 3x^5)^{15}$.

3b $y' = -\csc^2(\sqrt{x}\,) \cdot (\sqrt{x}\,)' = -\dfrac{1}{2\sqrt{x}} \csc^2 \sqrt{x}$.

3c Using the Chain Rule:

$$\begin{aligned} h'(x) &= 4\sin^3(\cos(-8x)) \cdot \cos(\cos(-8x)) \cdot (-\sin(-8x)) \cdot (-8) \\ &= 32\sin^3(\cos(-8x))\cos(\cos(-8x))\sin(-8x). \end{aligned}$$

4 By the Quotient Rule:

$$3x^2 = \frac{(x-y)(1+y') - (x+y)(1-y')}{(x-y)^2} \quad \Rightarrow \quad y' = \frac{3x^2(x-y)^2 + 2y}{2x}.$$

Alternative: rewrite equation as $x^4 - x^3 y = x + y$ to avoid having to use the Quotient Rule, if desired. Now differentiate with respect to x:

$$4x^3 - 3x^2 y - x^3 y' = 1 + y' \quad \Rightarrow \quad 4x^3 - 3x^2 y - 1 = x^3 y' + y' = (x^3+1)y' \quad \Rightarrow \quad y' = \frac{4x^3 - 3x^2 y - 1}{x^3+1}.$$

5 Implicit differentiation gives

$$y^{5/2} + \tfrac{5}{2}xy^{3/2}y' + \tfrac{3}{2}x^{1/2}y + x^{3/2}y' = 0,$$

and so

$$y' = -\frac{y^{5/2} + \frac{3}{2}x^{1/2}y}{x^{3/2} + \frac{5}{2}xy^{3/2}} = -\frac{2y^{5/2} + 3x^{1/2}y}{2x^{3/2} + 5xy^{3/2}}$$

At the point $(x, y) = (4, 1)$ we have

$$y' = -\frac{2(1)^{5/2} + 3(4)^{1/2}(1)}{2(4)^{3/2} + 5(4)(1)^{3/2}} = -\frac{2 + 3(2)}{2(8) + 5(4)} = -\frac{2}{9},$$

which is the slope of the tangent line. Equation of tangent line is therefore

$$y - 1 = -\tfrac{2}{9}(x - 4) \quad \Rightarrow \quad 2x + 9y = 17.$$

6 Letting r be the radius of the plate, we're given that $dr/dt = 0.02$ cm/min. The area A of the plate is $A = \pi r^2$, and so, differentiating this equation with respect to t (where r and A are implicitly functions of t), we get

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 0.04\pi r.$$

Finally, when $r = 60$ cm, we find the area to be increasing at the rate

$$\left.\frac{dA}{dt}\right|_{r=60} = 0.04\pi \cdot 60 = 2.4\pi \text{ cm}^2/\text{min} \approx 7.54 \text{ cm}^2/\text{min}.$$

7 Let 0 be the time when the bicycle is directly under the balloon. If the bicycle is at coordinates $(0, 0)$, then the balloon is at $(0, 65)$. At time $t > 0$ (in seconds) the bicycle will be at coordinates $(17t, 0)$ if we take it to be moving in the direction of the positive x-axis, while the balloon will be at coordinates $(0, t + 65)$. The distance between the bicycle and balloon at time t is therefore

$$D = \sqrt{(17t)^2 + (t + 65)^2} = (290t^2 + 130t + 4225)^{1/2}.$$

The rate of change of the distance between bicycle and balloon is dD/dt. Differentiating our equation above with respect to t yields

$$\frac{dD}{dt} = \frac{290t + 65}{\sqrt{290t^2 + 130t + 4225}},$$

and so

$$\left.\frac{dD}{dt}\right|_{t=4} = \frac{290(4) + 65}{\sqrt{290 \cdot 4^2 + 130 \cdot 4 + 4225}} = \frac{1225}{\sqrt{9385}} \approx 12.6 \text{ ft/s}.$$

Another approach is to set up a right triangle with vertical side length y (the height of the balloon), horizontal side length x (the distance the bicycle has traveled to, say, the right of the vertical line the balloon is traveling up). Then $dy/dt = 1$ ft/s and $dx/dt = 17$ ft/s, and $D = \sqrt{x^2 + y^2}$. Now,

$$\frac{dD}{dt} = \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{\sqrt{x^2+y^2}} = \frac{17x + y}{\sqrt{x^2+y^2}},$$

and since we can set $x = 17(4) = 68$ ft and $y = 65 + 4 = 69$ ft when $t = 4$ seconds, we get

$$\left.\frac{dD}{dt}\right|_{t=4} = \frac{17(68) + 69}{\sqrt{68^2 + 69^2}} = \frac{1225}{\sqrt{9385}} \approx 12.6 \text{ ft/s}.$$