Math 140 Exam #3 Key (Spring 2019)

1a Show your work, but the answer is $-\frac{4}{3}$.

1b Multiply numerator and denominator by $\sqrt{x^2+ax}+\sqrt{x^2+bx}$ and simplify to get

$$\lim_{x\to\infty} \frac{ax-bx}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} = \lim_{x\to\infty} \frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}} = \frac{a-b}{2}.$$

2a Domain is $(-\infty,1)\cup(1,2)\cup(2,\infty)$, and the only intercept is $(0,0)$.

2b Vertical asymptote: $x=2$. Horizontal asymptote: $y=-1$. There is a hole in the graph at $(1,1)$.

2c Since $f'(x)=2/(x-2)^2$, there are no critical points.

2d Since $f'(x)>0$ for all $x\in\text{Dom}(f)$, we conclude that f is increasing on its domain.

2e $f''(x)=-4/(x-2)^3$, so $f''>0$ on $(-\infty,1)\cup(1,2)$ and $f''<0$ on $(2,\infty)$. Thus f is concave up on $(-\infty,1)$ and $(1,2)$, and concave down on $(2,\infty)$. There are no inflection points.

3 The square of the distance between $(x,\sqrt{x})$ and $(3,0)$ is

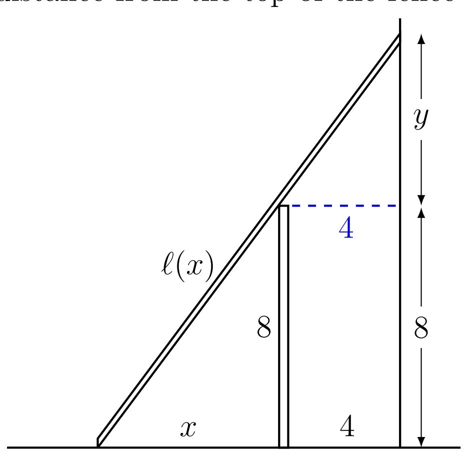
$$D(x)=x^2-5x+9,$$

and since

$$D'(x)=0 \;\Rightarrow\; 2x-5=0 \;\Rightarrow\; x=\frac{5}{2},$$

we conclude that the point $\left(5/2,\sqrt{5/2}\right)$ is closest.

4 For minimum length we must assume the ladder touches the top of the fence. If the base of the ladder is x away from the fence then the ladder forms the hypotenuse of a right triangle with length $\ell(x)$, while the base of the triangle has length $x+4$. The height of the triangle is $y+8$, where y is the vertical distance from the top of the fence to the top of the ladder.



By similar triangles we have

$$\frac{y+8}{x+4} = \frac{y}{4},$$

and hence $y = 32/x$. Now, letting $L(x) = \ell^2(x)$, the Pythagorean Theorem gives

$$L(x) = (x+4)^2 + (y+8)^2 = (x+4)^2 + \left(\frac{32}{x} + 8\right)^2.$$

Now,

$$L'(x) = 2x + 8 - \frac{2048}{x^3} - \frac{512}{x^2},$$

so $L'(x) = 0$ if and only if

$$2x^4 + 8x^3 - 512x - 2048 = 2x^3(x+4) - 512(x+4) = (2x^3 - 512)(x+4) = 0,$$

which yields solutions $x = -4$ (not physically interesting) and $x = 4\sqrt[3]{4}$. The minimum length is $\ell\big(4\sqrt[3]{4}\big)$.

5 For C arbitrary: $\frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C$.

6 $f'(\theta) = -\cos\theta + \sin\theta + C$ implies $f'(0) = -1 + C$, so with $f'(0) = 4$ we must have $C = 5$. Now,

$$f'(\theta) = -\cos\theta + \sin\theta + 5 \quad\Rightarrow\quad f(\theta) = -\sin\theta - \cos\theta + 5\theta + D,$$

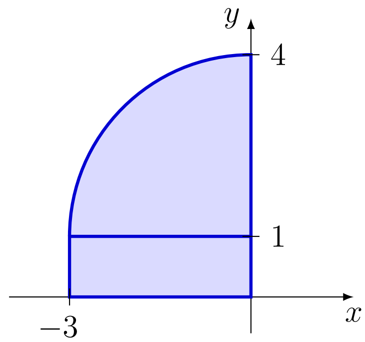
so $3 = f(0) = -1 + D$, and thus $D = 4$. Therefore

$$f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4.$$

7 Using right endpoints,

$$\begin{aligned}
\int_{-2}^{0} f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-2 + \frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\left(-2 + \frac{2i}{n}\right)^2 + \left(-2 + \frac{2i}{n}\right)\right] \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(2 - \frac{6}{n}i + \frac{4}{n^2}i^2\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\left[2n - \frac{6}{n}\cdot\frac{n(n+1)}{2} + \frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty}\frac{2n^2 - 6n + 4}{3n^2} = \frac{2}{3}.
\end{aligned}$$

8 The integral represents the area of a rectangle, plus a quarter of the area of a circle of radius 3 with center $(0,1)$. Pretty picture below.



Thus

$$\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx=(1)(3)+\frac{1}{4}(\pi\cdot 3^2)=\frac{9}{4}\pi+3.$$

9 By the Fundamental Theorem of Calculus,

$$y'=-\sqrt{x}\tan\sqrt{x}\cdot\frac{d}{dx}\sqrt{x}=-\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}}=-\frac{1}{2}\tan\sqrt{x}.$$

10a We obtain $\left[v-2v^4+2v^8\right]_0^1=1$.

10b We have $\left[-\csc t\right]_{\pi/6}^{\pi/2}=2-1=1$.