Math 140 Exam #2 Key (Spring 2019)

1a $s'(t) = \frac{3}{2}(2t - \sin t)^{1/2}(2 - \cos t)$

1b $f'(\theta) = -\sin(\sec(\tan\theta)) \cdot \sec(\tan\theta)\tan(\tan\theta) \cdot \sec^2\theta$

2 The given line $y = -3x + \frac{1}{2}$ has slope -3, and so we seek a tangent line with slope $\frac{1}{3}$. This means finding x such that $y' = \frac{1}{3}$; that is,

$$\frac{2}{\sqrt{1+4x}} = \frac{1}{3}.$$

Solving, we get $x = \frac{35}{4}$, and so the point is $\left(\frac{35}{4}, 6\right)$.

3 Since $-\sin(xy) \cdot (xy' + y) = (\cos y)y'$, we have

$$y' = -\frac{y\sin(xy)}{\cos y + x\sin(xy)}.$$

4 From $4y^3y' - 8yy' = 4x^3 - 10x$ we get

$$y' = \frac{2x^3 - 5x}{2y^3 - 4y},$$

and so $y' = 0$ when $(x, y) = (0, -2)$. The tangent line is the horizontal line $y = -2$.

5a $h(t)$ is maximal at t for which $h'(t) = 0$. Since $h'(t) = 0$ implies $80 - 32t = 0$, we find $h(t)$ is maximal when $t = \frac{5}{2}$, and the maximum height is $h(5/2) = 100$ feet.

5b From $h(t) = 96$ we find $t = 2, 3$. When $t = 3$ the ball is on its way down, and the velocity is $h'(3) = -16$ ft/s.

5c From $h(t) = 0$ we find $t = 0, 5$, and so the ball hits the ground when $t = 5$. Velocity is $h'(5) = -80$ ft/s.

6 From $A = \ell w$ we have

$$\frac{dA}{dt} = \ell\frac{dw}{dt} + w\frac{d\ell}{dt} = 3\ell + 8w,$$

and so when $\ell = 20$ cm and $w = 10$ cm we find that $dA/dt = 3(20) + 8(10) = 140$ cm^2/s

7 If h is the height of the conical volume of water, then the radius of the volume of water is $r = h/3$. The volume is thus

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3.$$

Then the volume of water is changing at a rate of

$$\frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}.$$

On the other hand, if c is the constant rate at which water is being pumped in, then $dV/dt = c - 10,000$, and so

$$c = 10,000 + \frac{\pi}{9} h^2 \frac{dh}{dt}$$

We're given that $dh/dt = 20$ cm/min when $h = 200$ cm, so

$$c = 10,000 + \frac{\pi}{9}(200)^2(20) \approx 289,253 \text{ cm}^3/\text{min}.$$

8 Let $f(x) = \sqrt[3]{x}$, so $f'(1000) = \frac{1}{3}(1000)^{-2/3} = \frac{1}{300}$. The tangent line to $f(x) = \sqrt[3]{x}$ at $(1000, 10)$ is thus given by

$$L(x) = \frac{1}{300}x + \frac{20}{3}.$$

Now,

$$\sqrt[3]{1003} = f(1003) \approx L(1003) = \frac{1}{300}(1003) + \frac{20}{3} = 10.01.$$

9 Let $f'(x) = 0$ to get $x^3 - x^2 - 2 = 0$, which has solutions $x = -1, 0, 2$. All these critical points of f lie in $[-2, 3]$. We evaluate f at the critical points as well as the endpoints:

$$f(-2) = 33, \quad f(-1) = -4, \quad f(0) = 1, \quad f(2) = -31, \quad f(3) = 28.$$

By the Closed Interval Method we conclude that $f(2) = -31$ is the absolute minimum value of f on $[-2, 3]$, and $f(-2) = 33$ is the absolute maximum value.

10 Let $f(x) = 2x - 1 - \sin x$, so the job is to show that f has exactly one real zero. Since $f(0) = -1 < 0$ and $f(\pi) = 2\pi - 1 > 0$, the Intermediate Value Theorem implies there exists some $c \in (0, \pi)$ such that $f(c) = 0$, and so f has at least one real zero. Suppose f has two real zeros x_1 and x_2, so $f(x_1) = f(x_2) = 0$. By Rolle's Theorem (or the Mean Value Theorem) there exists some c between x_1 and x_2 such that $f'(c) = 0$. Since $f'(x) = 2 - \cos x$, it follows that $2 - \cos c = 0$, or $\cos c = 2$. However, it is known that $-1 \leq \cos x \leq 1$ for all real x, so c cannot be real, and hence c cannot be between x_1 and x_2. This is a contradiction, and therefore we conclude that f cannot have two real zeros. Therefore f has exactly one real root.

11 Since $f'(x) \geq 2$ for all $x \in [1, 4]$, the function f is continuous on $[1, 4]$ and $(1, 4)$ as required by the Mean Value Theorem, and we can use the theorem to conclude there exists $1 < c < 4$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - 10}{3}.$$

But $f'(c) \geq 2$ also, so that

$$\frac{f(4) - 10}{3} \geq 2,$$

and hence $f(4) \geq 16$. That is, 16 is the smallest that $f(4)$ can possibly be.