Math 140 Exam #1 Key (Spring 2019)

1a $\lim_{x\to 1^-} h(x) = 2$

1b $\lim_{x\to 1^+} h(x) = 4$

1c $\lim_{x\to 1} h(x) = \text{DNE}$

1d $\lim_{x\to 3^-} h(x) = 1$

1e $\lim_{x\to -2} h(x) = 1$

2a $$\lim_{x\to -1} \frac{2x^2+3x+1}{x^2-2x-3} = \lim_{x\to -1} \frac{(2x+1)(x+1)}{(x+1)(x-3)} = \lim_{x\to -1} \frac{2x+1}{x-3} = \frac{2(-1)+1}{-1-3} = \frac{1}{4}.$$

2b $$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} = \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1.$$

3 $\lim_{t\to 2^-} G(t) = \lim_{t\to 2^-} (4-t/2) = 3$ and $\lim_{t\to 2^+} G(t) = \lim_{t\to 2^+} \sqrt{t+c} = \sqrt{2+c}$, so the limit can only exist if $3 = \sqrt{2+c}$, which requires that $c = 7$.

4 Let $\epsilon > 0$. Choose $\delta = \min\{1, 2\epsilon\}$. Suppose x is such that $0 < |x-2| < \delta$, so both $|x-2| < 1$ and $|x-2| < 2\epsilon$ hold. From $|x-2| < 1$ we find that $x > 1$, and so $0 < 1/|x| < 1$ in particular. Now,

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{2|x|} = \frac{1}{|x|} \cdot \frac{|x-2|}{2} < 1 \cdot \frac{2\epsilon}{2} = \epsilon,$$

and the proof is done.

5 Continuity from the left at 1 requires that $\lim_{x\to 1^-} g(x) = g(1)$. Since

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (x^2+x) = 1^2+1 = 2$$

and $g(1) = a$, we set $a = 2$ to secure continuity from the left at 1.

Continuity from the right at 1 requires that $\lim_{x\to 1^+} g(x) = g(1)$. Since

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (3x+5) = 3(1)+5 = 8$$

and $g(1) = a$, we set $a = 8$ to secure continuity from the right at 1.

6 Since $f(0) = 0 < 1000$ and $f(100) = 10{,}000 + 10\sin 100 > 1000$, and f is continuous on $[0, 100]$, the Intermediate Value Theorem implies there exists some $c \in (0, 100)$ such that $f(c) = 1000$.

7 By definition,

$$f'(x) = \lim_{h\to 0} \frac{[2.5(x+h)^2 + 6(x+h)] - (2.5x^2+6x)}{h} = \lim_{h\to 0}(5x + 2.5h + 6) = 5x + 6.$$

8 By the definition of derivative,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty,$$

and so the derivative does not exist.

9a $f'(t) = \dfrac{1}{2\sqrt{t}} + t^{-2}$.

9b $h'(x) = \dfrac{(3-4x)(3) - (1+3t)(-4)}{(3-4x)^2} = \dfrac{13}{(3-4x)^2}$.

9c $y' = \sec^3\theta + \sec^2\theta \tan\theta$.

10 Set $f'(x) = 0$ to get $3x^2 + 6x + 1 = 0$, and solve to get $x = \dfrac{-3 \pm \sqrt{6}}{3}$.

11 Since $y' = \cos x - (1+x)\sin x$, the slope of the line is $y'(0) = \cos 0 - \sin 0 = 1$. The line contains the point $(0, 1)$, and so the equation of the line is $y = x + 1$.

12 Using the result $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$, we have

$$\lim_{t\to 0} \left(\frac{6\sin 6t}{6t} \cdot \frac{2t}{2\sin 2t} \cdot \frac{1}{\cos 6t} \right) = 6 \lim_{t\to 0} \frac{\sin 6t}{6t} \cdot \frac{1}{2} \lim_{t\to 0} \frac{2t}{\sin 2t} \cdot \lim_{t\to 0} \frac{1}{\cos 6t} = 6 \cdot \frac{1}{2} \cdot 1 = 3.$$