Math 140 Exam #3 Key (Fall 2020)

1 First,

$$u'(t) = \frac{-\sin^2 t + \cos^2 t - \sin t}{2} = \frac{(2\sin t - 1)(\sin t + 1)}{2},$$

so $u'(t) = 0$ if and only if $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$. These are the critical points in $(0, 2\pi)$, and so we evaluate

$$u(0) = 0, \quad u(\tfrac{\pi}{6}) = \frac{\sqrt{3}-2}{8}, \quad u(\tfrac{5\pi}{6}) = \frac{-\sqrt{3}-2}{8}, \quad u(\tfrac{3\pi}{2}) = \frac{1}{2}, \quad u(2\pi) = 0.$$

The absolute minimum is thus $u(\frac{5\pi}{6}) = \frac{-\sqrt{3}-2}{8}$, and the absolute maximum is $u(\frac{3\pi}{2}) = \frac{1}{2}$.

2 Let $f(x) = x^3 - 7x^2 + 25x + 8$, so c is a solution to the equation if and only if $f(c) = 0$. Since f is everywhere continuous, $f(-1) < 0$, and $f(0) > 0$, the Intermediate Value Theorem implies there exists some $c \in (-1, 0)$ such that $f(c) = 0$, and hence c is a real solution to the equation. That is, the equation has at least one real solution.

Suppose the equation has more than one real solution. Then there are two real solutions c_1 and c_2, and we can assume $c_1 < c_2$. Now, $f(c_1) = f(c_2) = 0$, and since f is everywhere differentiable Rolle's Theorem implies there exists some $r \in (c_1, c_2)$ such that $f'(r) = 0$, or equivalently $3r^2 - 14r + 25 = 0$. But with the quadratic formula we find that

$$r = \frac{14 \pm \sqrt{14^2 - 4(3)(25)}}{2(3)} = \frac{7}{3} \pm \frac{\sqrt{26}}{3}i,$$

and so $r \notin (c_1, c_2)$. As this is a contradiction, we conclude that the equation cannot have more than one real solution, and therefore it has exactly one real solution.

3a Domain is $\mathbb{R}$, the only intercept is $(0, 0)$, and the only asymptote is the horizontal asymptote $y = 10$.

3b $f'(x) = \dfrac{60x}{(x^2+3)^2}$.

3c $f'(x) > 0$ for $x > 0$, and $f'(x) < 0$ for $x < 0$. Thus f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ by the Monotonicity Test, and therefore f has a local minimum at $(0, 0)$ by the First Derivative Test.

3d $f''(x) = \dfrac{180(1-x)(1+x)}{(x^2+3)^3}$

3e We find $f'' < 0$ on $(-\infty, -1)$ and $(1, \infty)$, while $f'' > 0$ on $(-1, 1)$. Therefore f is concave down on $(-\infty, -1)$ and $(1, \infty)$, and concave up on $(-1, 1)$. Inflection points are therefore $(\pm 1, f(\pm 1)) = (\pm 1, \frac{5}{2})$.

4 Find x so that the distance from $(x, 4x-6)$ to $(0,0)$ is minimal. The square of the distance could be minimized instead and is easier to differentiate:

$$D(x) = x^2 + (4x-6)^2 \quad \Rightarrow \quad D'(x) = 34x - 48.$$

The only x for which $D(x) = 0$ is $x = \frac{24}{17}$. This is the x-coordinate of the closest point, which has full coordinates $(\frac{24}{17}, -\frac{6}{17})$.

5 With r the radius of the base of the cone, and h the cone's height, we find that h, r, and 5 are the lengths of the sides of a right triangle such that $r^2 + h^2 = 5^2$, or $r^2 = 25 - h^2$. Now from $V = \frac{\pi}{3} r^2 h$ we obtain a function of h alone:

$$V(h) = \frac{\pi}{3}(25 - h^2)h.$$

Hence $V'(h) = \frac{\pi}{3}(25 - 3h^2)$, so that $V'(h) = 0$ if and only if $h = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$. This is the value of h that maximizes the volume of the cone, and the corresponding r value is $r = \sqrt{25 - h^2} = 5\sqrt{2/3} = \frac{5\sqrt{6}}{3}$.

6 Let $f(x) = \sqrt[3]{x}$, so $f'(x) = \frac{1}{3}x^{-2/3}$. We linearize f at $x = 64$: the point of tangency is $(64, \sqrt[3]{64}) = (64, 4)$, and the slope of the tangent line is $f'(64) = \frac{1}{48}$. From the point-slope formula we find the tangent line to be $y - 4 = \frac{1}{48}(x - 64)$, and therefore the linearization is $L(x) = \frac{1}{48}x + \frac{8}{3}$. Finally,

$$\sqrt[3]{65} = f(65) \approx L(65) = \frac{65}{48} + \frac{8}{3} = \frac{193}{48}.$$

7a The limit could be figured by factoring out $\sqrt{x}$ from the numerator and denominator and reducing. But with L'Hôpital's Rule we have

$$\lim_{x\to\infty} \frac{3}{x} \csc \frac{5}{x} = \lim_{x\to\infty} \frac{3/x}{\sin(5/x)} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{-3/x^2}{(-5/x^2)\cos(5/x)} = \lim_{x\to\infty} \frac{3}{5\cos(5/x)} = \frac{3}{5}.$$

7b From the identity $\sin(2u) = 2 \sin u \cos u$ we get $\csc x = \frac{1}{2}\csc(x/2)\sec(x/2)$, which will help make things a bit easier:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{(\sin x)\sqrt{1-x}}{\sqrt{x}} &\overset{\text{LR}}{=} \lim_{x\to 0^+} \frac{(\cos x)\sqrt{1-x} - (\sin x)\cdot \frac{1}{2}(1-x)^{-1/2}}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to 0^+} 2\sqrt{x}\left[(\cos x)\sqrt{1-x} - \frac{\sin x}{2\sqrt{1-x}}\right] \\
&= 2\sqrt{0}\left[(\cos 0)\sqrt{1-0} - \frac{\sin 0}{2\sqrt{1-0}}\right] = 0.
\end{aligned}$$