Math 140 Exam #3 Key (Fall 2015)

1 We have $f'(x) = 6x^2 - 30x + 24 = 6(x-1)(x-4)$, so $f'(x) = 0$ when $x = 1, 4$. These are the critical points. We evaluate:

$$f(0) = 0, \quad f(1) = 11, \quad f(4) = -16, \quad f(5) = -5.$$

The absolute minimum value of f on $[0,5]$ is $f(4) = -16$, and the absolute maximum value is $f(1) = 11$.

2a Domain is $(-\infty, -1/2) \cup (-1/2, \infty)$. There is no x-intercept, but the y-intercept is $(0, 12)$.

2b By long division we have

$$f(x) = \frac{1}{2}x - \frac{1}{4} + \frac{49}{4(2x+1)},$$

and so $y = \frac{1}{2}x - \frac{1}{4}$ is a slant asymptote for f. There is also vertical asymptote $x = -\frac{1}{2}$

2c Since

$$f'(x) = \frac{2(x+4)(x-3)}{(2x+1)^2},$$

critical points of f are $x = -4, 3$. (Note: $-\frac{1}{2}$ is not a critical point since it is not in the domain of f.)

2d $f'(x) > 0$ implies $(x+4)(x-3) > 0$, which occurs for $x \in \text{Dom}(f)$ if $x > 3$ or $x < -4$. So f is increasing on $(-\infty, -4)$ and $(3, \infty)$, and decreasing on $(-4, -\frac{1}{2})$ and $(-\frac{1}{2}, 3)$. Therefore f has a local maximum at $x = -4$, and a local minimum at $x = 3$.

2e We have $f''(x) = 98(2x+1)^{-3}$, so $f''(x) > 0$ for $x \in (-\frac{1}{2}, \infty)$, and $f''(x) < 0$ for $x \in (-\infty, -\frac{1}{2})$. Therefore f is concave up on $(-\frac{1}{2}, \infty)$, and concave down on $(-\infty, -\frac{1}{2})$. There is no inflection point at $-\frac{1}{2}$ since $-\frac{1}{2} \notin \text{Dom}(f)$.

3 The surface area of a box having square base with sides of length x is given by the function $S(x) = 2x^2 + 400/x$. Then $S'(x) = 4x - 400/x^2$, and setting $S'(x) = 0$ gives $4x^3 - 400 = 0$, whence $x = \sqrt[3]{100}$. The dimensions of the box are therefore $\sqrt[3]{100} \times \sqrt[3]{100} \times \sqrt[3]{100}$.

4 The tangent line to the graph of $V(r) = \frac{4}{3}\pi r^3$ at $(5, V(5)) = (5, \frac{500}{3}\pi)$ is given by the function

$$L(r) = 100\pi r - \frac{1000}{3}\pi$$

For r "near" 5 we have $V(r) \approx L(r)$, so in particular

$$V(5.1) - V(5) \approx L(5.1) - L(5) = 100\pi(5.1) - 100\pi(5) = 10\pi$$

5 By the Mean Value Theorem there exists some $c \in (-2, 14)$ such that

$$f'(c) = \frac{f(14) - f(-2)}{14 - (-2)} = \frac{7 - f(-2)}{16}.$$

Since $f'(c) \leq 10$ is required, we must have $7 - f(-2) \leq 160$, or $f(-2) \geq -153$.

6a We have

$$\lim_{x \to \pi/2^-} \left(\frac{\pi}{2} - x\right) \sec x = \lim_{x \to \pi/2^-} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{x \to \pi/2^-} \frac{-1}{-\sin x} = \frac{1}{\sin(\pi/2)} = 1.$$

6b Using L'Hôpital's Rule three times:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x} = \lim_{x \to 0} \frac{2 \sec^4 x + 4 \tan^2 x \sec^2 x}{6}$$
$$= \frac{2 \sec^4(0) + 4 \tan^2(0) \sec^2(0)}{6} = \frac{2 \cdot 1^4 + 4 \cdot 0^2 \cdot 1^2}{6} = \frac{1}{3}.$$

7a $\displaystyle\int \left(\frac{5}{t^2} + 4t^2\right) dt = \int (5t^{-2} + 4t^2) dt = -\frac{5}{t} + \frac{4}{3} t^3 + C.$

7b $\displaystyle\int (\sin 2y - \cos 6y) dy = -\frac{1}{2} \cos 2y - \frac{1}{6} \sin 6y + C.$