Math 140 Exam #1 Key (Fall 2015)

1 Let $\epsilon > 0$. Choose $\delta = \epsilon/3$. Suppose $x \in \mathbb{R}$ is such that $0 < |x-7| < \delta$. Then $|x-7| < \epsilon/3$, and since

$$|x-7| < \frac{\epsilon}{3} \quad \Rightarrow \quad 3|x-7| < \epsilon \quad \Rightarrow \quad |3x-21| < \epsilon \quad \Rightarrow \quad |(3x-8)-13| < \epsilon,$$

we conclude that $3x - 8 \to 13$ as $x \to 7$.

2a Since $\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-}(0) = 0$ and

$$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \sqrt{25-x^2} = \sqrt{25-(-5)^2} = 0,$$

we have $\lim_{x\to -5} f(x) = 0$.

2b $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \sqrt{25-x^2} = \sqrt{25-5^2} = 0$.

2c $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} 3x = 3(5) = 15$.

2d From above, $\lim_{x\to 5^-} f(x) = 0 \neq 15 = \lim_{x\to 5^+} f(x)$, and so $\lim_{x\to 5} f(x)$ does not exist.

2e $\lim_{x\to 3} f(x) = \lim_{x\to 3} \sqrt{25-x^2} = \sqrt{25-3^2} = 4$.

3a Factoring,

$$\lim_{t\to 2} \frac{3t^2-7t+2}{2-t} = \lim_{t\to 2} \frac{(3t-1)(t-2)}{2-t} = \lim_{t\to 2}(1-3t) = 1-3(2) = -5.$$

3b Factoring again,

$$\lim_{x\to 49} \frac{\sqrt{x}-7}{x-49} = \lim_{x\to 49} \frac{\sqrt{x}-7}{(\sqrt{x}-7)(\sqrt{x}+7)} = \lim_{x\to 49} \frac{1}{\sqrt{x}+7} = \frac{1}{\sqrt{49}+7} = \frac{1}{14}.$$

4 Since $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-}(3x+b) = 6+b$ and $\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+}(x-2) = 0$, we need $6+b=0$, or $b=-6$. Then the two-sided limit exists and has value 0.

5 Since

$$f(x) = \frac{(x-7)(x-2)}{(x-3)(x-2)} = \frac{x-7}{x-3},$$

f has a vertical asymptote at $x=3$. Indeed, we find that

$$\lim_{x\to 3^+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 3^-} f(x) = +\infty.$$

All we can say about $\lim_{x\to 3} f(x)$ is that it does not exist.

6 Divide by the highest power of x in the denominator:

$$\lim_{x\to\infty} \frac{4x^2-7}{8x^2+5x+2}\cdot\frac{x^{-2}}{x^{-2}} = \lim_{x\to\infty}\frac{4-7x^{-2}}{8+5x^{-1}+2x^{-2}} = \frac{4-0}{8+0+0} = \frac{1}{2}.$$

7 Recall that $\sqrt{x^2}=|x|$, so $\sqrt{x^2}=x$ if $x\geq 0$ and $\sqrt{x^2}=-x$ if $x<0$. Now, since $x\to\infty$ implies $x>0$,

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to\infty}\frac{\sqrt{x^2(1+x^{-2})}}{2x+1} = \lim_{x\to\infty}\frac{x\sqrt{1+x^{-2}}}{2x+1} = \lim_{x\to\infty}\frac{\sqrt{1+x^{-2}}}{2+x^{-1}} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2};$$

and since $x\to-\infty$ implies $x<0$,

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to-\infty}\frac{\sqrt{x^2(1+x^{-2})}}{2x+1} = \lim_{x\to\infty}\frac{-x\sqrt{1+x^{-2}}}{2x+1} = \lim_{x\to\infty}\frac{-\sqrt{1+x^{-2}}}{2+x^{-1}} = -\frac{1}{2}.$$

Horizontal asymptotes are $y=\frac{1}{2}$ and $y=-\frac{1}{2}$.

8 Since

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(x^2-5) = 4^2-5 = 11 \neq 13 = f(4),$$

f is not continuous at 4.

9 Continuity from the left at 1 requires that $\lim_{x\to 1^-} g(x) = g(1)$. Since

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(x^2+x) = 1^2+1 = 2$$

and $g(1)=a$, we set $a=2$ to secure continuity from the left at 1.

Continuity from the right at 1 requires that $\lim_{x\to 1^+} g(x) = g(1)$. Since

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}(3x+5) = 3(1)+5 = 8$$

and $g(1)=a$, we set $a=8$ to secure continuity from the right at 1.

We see that there can be no value for a that results in continuity from the left and right at 1 simultaneously, which means there is no a value which will make g continuous at 1.

10a By definition,

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{[3(1+h)^2-4(1+h)]-[3(1)^2-4(1)]}{h} \\ &= \lim_{h\to 0}\frac{3h^2+2h}{h} = \lim_{h\to 0}(3h+2) = 2. \end{aligned}$$

10b Slope of the tangent line is $f'(1)=2$, so by the point-slope formula we have

$$y-(-1) = 2(x-1) \quad\Rightarrow\quad y+1 = 2x-2 \quad\Rightarrow\quad y = 2x-3$$

is the equation of the line.