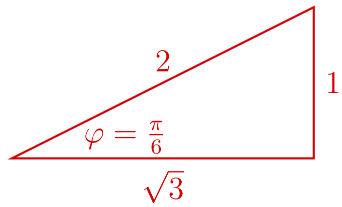
Math 125 Quiz #5 (Fall 2020)

1 Solve $\tan(2\theta) = \frac{1}{\sqrt{3}}$ for $\theta \in [0, 2\pi)$.

So for $\varphi = 2\theta \in [0, 4\pi)$ we solve $\tan\varphi = \frac{1}{\sqrt{3}}$. One solution is $\varphi = \frac{\pi}{6}$, since φ is in the special triangle pictured:



This same triangle can be placed in Quadrant III to give another solution: $\varphi = \frac{7\pi}{6}$. Now, adding 2π to these two solutions, we obtain

$$2\theta = \varphi = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6} \in [0, 4\pi),$$

and therefore the solutions for θ are

$$\theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12} \in [0, 2\pi),$$

2 Solve $\sin^2\theta = 2\cos\theta + 2$ for $\theta \in [0, 2\pi)$.

Recalling $\sin^2 + \cos^2 = 1$, we have $1 - \cos^2\theta = 2\cos\theta + 2$, which with some algebra becomes $(\cos\theta + 1)^2 = 0$, and therefore $\theta = \pi$ is the only solution.

3 Establish the identity

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\sin\theta + 1}{\cos\theta}.$$

Working with the left-hand side, multiply by the conjugate of the denominator:

$$\begin{aligned}\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} &= \frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} \cdot \frac{\sin\theta + \cos\theta + 1}{\sin\theta + \cos\theta + 1} \\ &= \frac{\sin^2\theta - \cos^2\theta + 2\sin\theta + 1}{\sin^2\theta + \cos^2\theta + 2\cos\theta\sin\theta - 1} \qquad (1)\end{aligned}$$

In the numerator of (1) let $\cos^2\theta = 1 - \sin^2\theta$, and in the denominator let $\sin^2\theta + \cos^2\theta = 1$, so (1) becomes

$$\frac{\sin^2\theta - (1 - \sin^2\theta) + 2\sin\theta + 1}{1 + 2\cos\theta\sin\theta - 1} = \frac{2\sin\theta(\sin\theta + 1)}{2\cos\theta\sin\theta} = \frac{\sin\theta + 1}{\cos\theta}.$$