Math 125 Exam #3 Key (Spring 2022)

1a We have

$$(f \circ g)(x) = f(g(x)) = f(2/\sqrt{x}) = \frac{2/\sqrt{x}}{1 - 4/\sqrt{x}} = \frac{2}{\sqrt{x} - 4}$$

and

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{1-2x}\right) = \frac{2}{\sqrt{\dfrac{x}{1-2x}}}.$$

1b $D_{f\circ g} = \{x : x \in D_g \ \& \ g(x) \in D_f\}$ with $D_g = (0, \infty)$ and $D_f = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$. Thus:

$$D_{f\circ g} = \left\{x : x > 0 \ \ \& \ \ \frac{2}{\sqrt{x}} \neq \frac{1}{2}\right\} = (0, \tfrac{1}{16}) \cup (\tfrac{1}{16}, \infty).$$

1c $D_{g\circ f} = \{x : x \in D_f \ \& \ f(x) \in D_g\}$, so

$$D_{g\circ f} = \left\{x : x \neq \frac{1}{2} \ \ \& \ \ \frac{x}{1-2x} > 0\right\} = (0, \tfrac{1}{2}).$$

2 Need $(f \circ g)(0) = -20$, where $(f \circ g)(0) = f(g(0)) = 3c^2 - 7$. Thus c must be such that $3c^2 - 7 = -20$, or $c^2 = -\frac{13}{3}$. But there is no real c value that can satisfy this, and so there is no solution here.

3a Let $y = T(x)$, so $y = \frac{2x-3}{x+4}$. Solve for x to get $x = \frac{4y+3}{2-y}$, and thus

$$T^{-1}(y) = \frac{4y+3}{2-y}.$$

3b $R_T = D_{T^{-1}} = (-\infty, 2) \cup (2, \infty)$ and $R_{T^{-1}} = D_T = (-\infty, -4) \cup (-4, \infty)$.

4a $5^{x+3} = 5^{-2}$ implies $x + 3 = -2$, so $x = -5$.

4b Write $x^3 = \frac{1}{8}$, so $x = \frac{1}{2}$.

4c Write $\log_3 \dfrac{(x+4)^2}{9} = 2$, so $\dfrac{(x+4)^2}{9} = 3^2$, giving $x = -13, 5$. But -13 is extraneous, so $x = 5$.

4d Write $\log_5(x+3)(x-1) = 1$, so $(x+3)(x-1) = 5$, giving $x = -4, 2$. But -4 is extraneous, so $x = 2$.

4e Write $(3^x)^2 - 3 \cdot 3^x + 1 = 0$. Letting $u = 3^x$ makes this $u^2 - 3u + 1 = 0$, so that

$$3^x = u = \frac{3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2},$$

and thus $x = \log_3 \frac{3 \pm \sqrt{5}}{2}$.

4f With the change-of-base formula:

$$\log_2 x + \frac{\log_2 x}{\log_2 4} + \frac{\log_2 x}{\log_2 8} = 11 \quad \hookrightarrow \quad \log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{3} = 11 \quad \hookrightarrow \quad \log_2 x = 6,$$

so $x = 2^6 = 64$.

5a Set $y = v(x)$, so $y = 8 - \log_6(2x+1)$, giving $x = \dfrac{6^{8-y} - 1}{2}$, and hence $v^{-1}(y) = \dfrac{6^{8-y} - 1}{2}$.

5b $R_{v^{-1}} = D_v = (-\frac{1}{2}, \infty)$ and $R_v = D_{v^{-1}} = (-\infty, \infty)$.

6a $N(0) = 1000$.

6b To the nearest tenth[1]: $N(4) = 1000e^{0.01(4)} \approx 1040.8$.

6c Find t for which $N(t) = 1600$. This means solving $1000e^{0.01t} = 1600$ to get $t = \frac{\ln 1.6}{0.01} \approx 47.0$ hours.

7 Convert $0.444°$ to minutes: $(0.444°)\left(\dfrac{60'}{1°}\right) = 26.64'$.

Convert $0.64'$ to seconds, and round to the nearest second: $(0.64')\left(\dfrac{60''}{1'}\right) = 38.4'' \approx 38''$.
Therefore $44.444° \approx 44°\,26'\,38''$.

8 The point given lies on a circle of radius $\sqrt{10}$, so:

$$\sin\theta = -\frac{3}{\sqrt{10}}, \quad \cos\theta = \frac{1}{\sqrt{10}}, \quad \tan\theta = -3, \quad \csc\theta = -\frac{\sqrt{10}}{3}, \quad \sec\theta = \sqrt{10}, \quad \cot\theta = -\frac{1}{3}.$$

9 With use of the 30-60-90 right triangle in the first quadrant we find the angle to be $\frac{\pi}{3}$. Other angles are given by $\frac{\pi}{3} + 2\pi n = \frac{\pi + 6\pi n}{3}$ for integers n, so letting $n = -2, -1, 0, 1, 2$ we obtain the angles

$$-\frac{11\pi}{3}, \quad -\frac{5\pi}{3}, \quad \frac{\pi}{3}, \quad \frac{7\pi}{3}, \quad \frac{13\pi}{3}.$$

[1] Normally a population would be rounded to the nearest whole number, but this entire problem was handed to me by the college as part of a college-wide "assessment," so we're stuck with this notion of having four fifths of a bacterium.