Math 125 Exam #3 Key (Spring 2021)

1a Suppose $\Omega(a) = \Omega(b)$, so that $\dfrac{8}{3-\sqrt{a}} = \dfrac{8}{3-\sqrt{b}}$, and hence $3-\sqrt{a} = 3-\sqrt{b}$. From this it follows that $\sqrt{a} = \sqrt{b}$, and hence $a = b$. Therefore Ω is one-to-one.

1b Let $y = \Omega(x)$, so $y = \dfrac{8}{3-\sqrt{x}}$. Solving for $\sqrt{x}$ gives $\sqrt{x} = \dfrac{3y-8}{y}$, and thus

$$\Omega^{-1}(y) = x = \left(\frac{3y-8}{y}\right)^2.$$

1c $\operatorname{Ran}\Omega = \operatorname{Dom}\Omega^{-1} = (-\infty, 0) \cup (0, \infty)$ and $\operatorname{Ran}\Omega^{-1} = \operatorname{Dom}\Omega = [0, 9) \cup (9, \infty)$.

2 From $2^{x^2+2x} = 2^8$ we have $x^2 + 2x = 8$, and so $x = -4, 2$.

3a Let $y = \Psi(x)$, so $y = \ln(2x-4) + 6$, and then $2x - 4 = e^{y-6}$. Solving for x, we get

$$\Psi^{-1}(y) = x = \frac{e^{y-6}+4}{2}.$$

3b $\operatorname{Ran}\Psi = \operatorname{Dom}\Psi^{-1} = (-\infty, \infty)$ and $\operatorname{Ran}\Psi^{-1} = \operatorname{Dom}\Psi = (2, \infty)$.

4a Write $\log_6(x+3)(x+4) = 1$, so $(x+3)(x+4) = 6$, and then $(x+6)(x+1) = 0$. This results in $x = -6, -1$, but -6 is extraneous, so the solution set is $\{-1\}$.

4b $e^{-2x+3} = 12$ implies $-2x + 3 = \ln 12$, and thus $x = \dfrac{3 - \ln 12}{2}$.

4c Write equation as $(2^x)^2 + 4 \cdot 2^x - 12 = 0$. Let $u = 2^x$ so equation becomes $u^2 + 4u - 12 = 0$, and solve this to get $u = -6, 2$. Thus $2^x = -6, 2$, but since $2^x = -6$ is impossible we're left with $2^x = 2$, or $x = 1$. Solution set: $\{1\}$.

4d With the Change-of-Base Formula we get, since $\log_3 9 = 2$,

$$\frac{\log_3(7x-5)}{\log_3 9} = \log_3(x+1) \quad\Rightarrow\quad \log_3(7x-5) = \log_3(x+1)^2 \quad\Rightarrow\quad 7x - 5 = (x+1)^2.$$

The last equation has solutions $x = 2, 3$, and so the solution set of the original equation is $\{2, 3\}$.

5a $N(t) = N_0 e^{kt}$.

5b Let $t = 0$ for the year 2005, so $N_0 = N(0) = 900,000$, and then $N(t) = 900,000e^{kt}$. Using $N(2) = 800,000$ (since $t = 2$ in 2007) gives

$$800,000 = 900,000e^{2k} \quad \Rightarrow \quad k = \tfrac{1}{2}\ln\tfrac{8}{9} \approx -0.05889,$$

so

$$N(t) = 900,000e^{-0.05889t}.$$

Population in 2009 (when $t = 4$) is thus

$$N(8) = 900,000e^{-0.05889(4)} \approx 711,115.$$

Round to the nearest whole number for a population.

6 We're given $u_0 = 100\,°\text{C}$ (so $u(0) = 100$), $T = 20\,°\text{C}$, and also $u(15) = 75$. With Newton's Law of Cooling:

$$75 = u(15) = 20 + (100 - 20)e^{15k},$$

which gives $k = \frac{1}{15}\ln\frac{55}{80} \approx -0.025$. Thus the model is

$$u(t) = 20 + 80e^{-0.025t}.$$

After another 10 minutes (when $t = 25$), the temperature is

$$u(25) = 20 + 80e^{-0.025(25)} \approx 62.8\,°\text{C}.$$

7 Convert $0.461°$ to minutes:

$$(0.461°)\left(\frac{60'}{1°}\right) = 27.66'.$$

Convert $0.66'$ to seconds, and round to the nearest second:

$$(0.66')\left(\frac{60''}{1'}\right) = 39.6'' \approx 40''.$$

Therefore $87.461° \approx 87°\,27'\,40''$.

8 The point given lies on a circle of radius $\sqrt{29}$, so:

$$\sin\theta = \frac{5}{\sqrt{29}},\quad \cos\theta = -\frac{2}{\sqrt{29}},\quad \tan\theta = -\frac{5}{2},\quad \csc\theta = \frac{\sqrt{29}}{5},\quad \sec\theta = -\frac{\sqrt{29}}{2},\quad \cot\theta = -\frac{2}{5}.$$