Math 125 Exam #2 Key (Sping 2021)

1 Must have $f(x) = C(x-3)^2(x+4)(x-1)^3$ with C such that $f(-1) = 20$. This requires $C = -\frac{5}{96}$, and so we finally get

$$f(x) = -\tfrac{5}{96}(x-3)^2(x+4)(x-1)^3.$$

2 To have real coefficients the Conjugate Zeros Theorem implies that $1+2i$ must also be a zero, and so we need

$$\begin{aligned} f(x) &= (x-6)[x-(1-2i)][x-(1+2i)] \\ &= (x-6)(x^2-2x+5) \\ &= x^3 - 8x^2 + 17x - 30. \end{aligned}$$

3 The rational zeros that G could possibly have include such values as 1 and -5, which are in fact zeros for G. We use synthetic division to start factoring $G(x)$:

$$\begin{array}{r|rrrr|r} 1 & 2 & 11 & -5 & -43 & 35 \\ & & 2 & 13 & 8 & -35 \\ \hline & 2 & 13 & 8 & -35 & 0 \end{array} \quad \longrightarrow \quad f(x) = (x-1)(2x^3+13x^2+8x-35)$$

$$\begin{array}{r|rrr|r} -5 & 2 & 13 & 8 & -35 \\ & & -10 & -15 & 35 \\ \hline & 2 & 3 & -7 & 0 \end{array} \quad \longrightarrow \quad f(x) = (x-1)(x+5)(2x^2+3x-7)$$

Solving $2x^2+3x-7=0$ using the quadratic formula, we obtain the complete list of real zeros: -5, 1, $\frac{-3\pm\sqrt{65}}{4}$.

The complete factorization is

$$G(x) = (x-1)(x+5)\left(x+\tfrac{3+\sqrt{65}}{4}\right)\left(x+\tfrac{3-\sqrt{65}}{4}\right).$$

4 Let $f(x) = x^3 - 8x^2 + 25x - 26$, so the problem is to find all x such that $f(x) = 0$. Among the possible rational zeros is 2, which turns out to work:

$$\begin{array}{r|rrr|r} 2 & 1 & -8 & 25 & -26 \\ & & 2 & -12 & 26 \\ \hline & 1 & -6 & 13 & 0 \end{array} \quad \longrightarrow \quad \text{So 2 is a zero and } f(x) = (x-2)(x^2-6x+13).$$

Applying ye olde quadratic formula to $x^2-6x+13=0$ yields the zeros $3\pm 2i$. In conclusion, f has zeros 2, $3-2i$, $3+2i$, which are therefore the solutions to the given equation.

5 $-3i$ must be another zero, and so $(x-3i)(x+3i) = x^2+9$ is a factor of $H(x)$. Then

$$\begin{array}{rrrrrr}
 & & 3x^2 & +5x & -2 & \\
\cline{2-6}
x^2+9 \big) & 3x^4 & +5x^3 & +25x^2 & +45x & -18 \\
 & -3x^4 & & -27x^2 & & \\
\cline{2-4}
 & & 5x^3 & -2x^2 & +45x & \\
 & & -5x^3 & & -45x & \\
\cline{3-5}
 & & & -2x^2 & & -18 \\
 & & & 2x^2 & & +18 \\
\cline{4-6}
 & & & & & 0
\end{array}$$

shows $3x^2+5x-2$ is another factor. Now,

$$H(x) = 0 \quad\Rightarrow\quad (x^2+9)(3x^2+5x-2) = 0 \quad\Rightarrow\quad (x-3i)(x+3i)(3x-1)(x+2) = 0,$$

and therefore other zeros of H are $\frac{1}{3}$ and -2.

6 First we have

$$Z(x) = \frac{(x-2)(x^2+2x+4)}{(x-3)(x-2)} = \frac{x^2+2x+4}{x-3},$$

so $x=3$ is a vertical asymptote for Z. Long division then gives $Z(x) = (x+5) + \dfrac{19}{x-3}$, which shows that $y = x+5$ is an oblique asymptote for Z.

7a Write $x^4 - 16 > 0$, then factor to get $(x-2)(x+2)(x^2+4) > 0$. Since $x^2+4>0$ for any x, we divide it out to get $(x-2)(x+2) > 0$, and then use the Intermediate Value Theorem to determine the solution set to be $(-\infty,-2)\cup(2,\infty)$.

7b Employ the Intermediate Value Theorem method as in class, or consider the following approach. Factoring, the inequality $x^3-2x^2-3x<0$ becomes $x(x-3)(x+1)<0$. Now we run through cases.

Case I: $x<0$, $x-3<0$, $x+1<0$, giving $x<-1$. Case II: $x<0$, $x-3>0$, $x+1>0$, giving a contradiction. Case III: $x>0$, $x-3<0$, $x+1>0$, giving $0<x<3$. Case IV: $x>0$, $x-3>0$, $x+1<0$, giving a contradiction. Solution set: $(-\infty,-1)\cup(0,3)$.

7c Get 0 on the right-hand side:

$$\frac{3x-5}{x+2} - 2 \ge 0 \quad\Rightarrow\quad \frac{3x-5}{x+2} - \frac{2(x+2)}{x+2} \ge 0 \quad\Rightarrow\quad \frac{x-9}{x+2} \ge 0.$$

Case I: $x-9\ge 0$ & $x+2>0$, giving $x\ge 9$. Case II: $x-9\le 0$ & $x+2<0$, giving $x<-2$. Solution set: $(-\infty,-2)\cup[9,\infty)$.

8 $(f\circ g)(4) = -\frac{31}{16}$, $(g\circ f)(2) = -7$, $(f\circ f)(1) = 11$, $(g\circ g)(-2) = -2$.

9a $(f \circ g)(x) = \sqrt{6 + \frac{3}{4x}}$, and given that $\text{Dom}\, f = (-\infty, 2]$ and $\text{Dom}\, g = (-\infty, 0) \cup (0, \infty)$, we find

$$\begin{aligned}\text{Dom}(f \circ g) &= \{x : x \in \text{Dom}\, g \;\; \& \;\; g(x) \in \text{Dom}\, f\} \\ &= \{x : x \neq 0 \;\; \& \;\; -\tfrac{1}{4x} \leq 2\} \\ &= (-\infty, -\tfrac{1}{8}] \cup (0, \infty).\end{aligned}$$

9b $(g \circ f)(x) = -\dfrac{1}{4\sqrt{6 - 3x}}$ with

$$\begin{aligned}\text{Dom}(g \circ f) &= \{x : x \in \text{Dom}\, f \;\; \& \;\; f(x) \in \text{Dom}\, g\} \\ &= \{x : x \leq 2 \;\; \& \;\; 6 - 3x \neq 0\} \\ &= (-\infty, 2).\end{aligned}$$

9c $(g \circ g)(x) = -\frac{1}{4(-1/4x)} = x$ with

$$\begin{aligned}\text{Dom}(g \circ g) &= \{x : x \in \text{Dom}\, g \;\; \& \;\; g(x) \in \text{Dom}\, g\} \\ &= \{x : x \neq 0 \;\; \& \;\; g(x) \neq 0\} \\ &= (-\infty, 0) \cup (0, \infty).\end{aligned}$$