MATH 125 EXAM #4 KEY (SPRING 2020)

1a $\pi/3$ (the work of drawing the appropriate triangles and circles is excluded from this key)

1b $2\pi/3$ (the work of drawing the appropriate triangles and circles is excluded from this key)

2a Setting $y = f(x)$, we solve for x to obtain $f^{-1}(y) = \tan^{-1}\left(\dfrac{y+3}{2}\right)$.

2b Ran $f = (-\infty, \infty)$, Dom $f^{-1} = (-\infty, \infty)$, Ran $f^{-1} = (-\frac{\pi}{2}, \frac{\pi}{2})$.

3 Let $\theta = \cos^{-1} u$, so $\theta \in [0, \pi]$ is such that $\cos\theta = x/r = u$, and so we have $x = u$ and $r = 1$. Now, $y^2 = r^2 - x^2 = 1 - u^2$, and so $|y| = \sqrt{1-u^2}$. But $0 \le \theta \le \pi$ implies $y \ge 0$, so that $|y| = y$ and we obtain $y = \sqrt{1-u^2}$. Finally,

$$\tan(\cos^{-1} u) = \tan\theta = \frac{y}{x} = \frac{\sqrt{1-u^2}}{u}.$$

4a Factor: $(2\cos\theta - 1)(\cos\theta + 1) = 0$, so either $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$. Solution set: $\{\pi, \frac{\pi}{3}, \frac{5\pi}{3}\}$.

4b Either $\cot\theta = -1$ or $\csc\theta = \frac{1}{2}$. The latter equation has no solution, but the former gives the solution set $\{\frac{3\pi}{4}, \frac{7\pi}{4}\}$.

5 From $\cos(3x) = -\frac{1}{2}$ we obtain $3x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \cdots$. Solution set: $\{\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}\}$.

6a Get a common denominator on the left-hand side, giving:

$$\frac{\cos^2 v + (1+\sin v)^2}{\cos v(1+\sin v)} = \frac{2 + 2\sin v}{\cos v(1+\sin v)} = \frac{2}{\cos v} = 2\sec v.$$

6b The left-hand side LHS is:

$$\text{LHS} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta} = \sec\theta\csc\theta.$$

7 With the given information we find that $\sin\alpha = \sqrt{3}/2$ and $\cos\beta = 2\sqrt{2}/3$. Now,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{\sqrt{6}}{3} + \frac{1}{6},$$

and

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{-\sqrt{3}-\frac{1}{2\sqrt{2}}}{1-\sqrt{3}\cdot\frac{1}{2\sqrt{2}}}=\frac{2\sqrt{6}+1}{\sqrt{3}-2\sqrt{2}}=-\frac{8\sqrt{2}+9\sqrt{3}}{5}.$$

8a Work with the left-hand side:

$$\text{LHS}=\frac{1}{\cos(\alpha-\beta)}=\frac{1}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}\cdot\frac{\sec\alpha\sec\beta}{\sec\alpha\sec\beta}=\text{RHS}.$$

8b $\tan\frac{v}{2}=\frac{\sin v}{1+\cos v}=\frac{\sin v(1-\cos v)}{1-\cos^2 v}=\frac{\sin v(1-\cos v)}{\sin^2 v}=\frac{1-\cos v}{\sin v}=\csc v-\cot v.$

9 Equation becomes $2\sin\theta\cos\theta=\cos\theta$, and so either $\cos\theta=0$ or $\sin\theta=\frac{1}{2}$. Solution set: $\{\frac{\pi}{2},\frac{3\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}\}$.

10 Height is $22\sin 70^\circ\approx 20.7$ ft.

11a Use Law of Sines: $b=0.65$, $C=100^\circ$, $c=1.29$.

11b Use Law of Sines to find two triangles: $B_1=13.4^\circ$, $C_1=156.6^\circ$, $c_1=6.86$ and $B_2=166.6^\circ$, $C_2=3.4^\circ$, $c_2=1.02$.

11c Requires Law of Cosines, starting with

$$A=\cos^{-1}\left(\frac{a^2-b^2-c^2}{-2bc}\right)=\cos^{-1}(0.1161)=83.3^\circ.$$

Use the law again to get $B=44.1^\circ$, and thus $C=52.6^\circ$.

12 We have a triangle with base 100, and two angles of $B=40^\circ$ and $C=25^\circ$ with the base. The angle opposite the base is thus $A=115^\circ$, and with the Law of Sines we find the side opposite the angle $B=40^\circ$ is $b=70$. Finally, the height of the helicopter is

$$h=b\sin C=70\sin 25^\circ=29.6\approx 30\text{ m}.$$