Math 125 Exam #2 Key (Spring 2020)

1 In general

$$(x+y)^n = \sum_{k=0}^{12} \binom{12}{k} (2x)^k,$$

so the coefficient of x^3 is

$$\binom{12}{3} 2^3 = 1760.$$

2 $f(x) = x^3(x+1)^2(x-5)$.

3 Must have $f(x) = c(x+4)(x+1)(x-2)$ with c such that $f(0) = 16$. We have

$$c(0+4)(0+1)(0-2) = f(0) = 16 \quad \Rightarrow \quad -8c = 16 \quad \Rightarrow \quad c = -2,$$

so $f(x) = -2(x+4)(x+1)(x-2)$.

4 Possible rational zeros are $\pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$. Synthetic division shows -1 to be a zero:

$$\begin{array}{r|rrrr|r} -1 & 3 & 4 & 7 & 8 & 2 \\ & & -3 & -1 & -6 & -2 \\ \hline & 3 & 1 & 6 & 2 & 0 \end{array} \qquad \longrightarrow \quad 3x^3 + x^2 + 6x + 2.$$

So

$$f(x) = (x+1)(3x^3 + x^2 + 6x + 2) = (x+1)[x^2(3x+1) + 2(3x+1)] = (x+1)(3x+1)(x^2+2)$$

is the factorization of $f(x)$ over the *reals*, and the real zeros of f are -1 and $-\frac{1}{3}$.

5 The possible rational zeros of $f(x) = 2x^4 + 7x^3 + x^2 - 7x - 3$ are $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$. Synthetic division shows 1 to be a zero:

$$\begin{array}{r|rrrr|r} 1 & 2 & 7 & 1 & -7 & -3 \\ & & 2 & 9 & 10 & 3 \\ \hline & 2 & 9 & 10 & 3 & 0 \end{array}$$

Thus $f(x) = (x-1)(2x^3 + 9x^2 + 10x + 3)$, and since $g(x) = 2x^3 + 9x^2 + 10x + 3$ has zero -1, so that synthetic division gives $g(x) = (x+1)(2x^2 + 7x + 3)$, we get

$$f(x) = (x-1)(x+1)(2x^2 + 7x + 3) = (x+1)(x-1)(x+3)(2x+1).$$

Equation is thus $(x+1)(x-1)(x+3)(2x+1) = 0$, so solution set is $\{-1, 1, -3, -\frac{1}{2}\}$.

6 Another zero must be $-3i$ since the polynomial function has real coefficients. This means

$$(x-3i)(x+3i) = x^2 + 9$$

is a factor of $H(x)$. With long division we have

$$\frac{H(x)}{x^2+9} = 3x^2 + 5x - 2,$$

and so

$$H(x) = (x^2+9)(3x^2+5x-2) = (x^2+9)(3x-1)(x+2).$$

The zeros of H are $3i$, $-3i$, $\frac{1}{3}$, -2.

7a Domain of U is $(-\infty, \frac{1}{4}) \cup (\frac{1}{4}, \infty)$.

7b Solving $U(x) = 0$ gets us the x-intercept $-\frac{7}{2}$ (recall that $\frac{1}{4}$ is not in the domain). The y-intercept is $U(0) = 7$.

7c For $x \neq \frac{1}{4}$ we find $U(x) = 2x + 7$, which means no vertical asymptote.

7d For $x \neq \frac{1}{4}$ we find $U(x) = 2x + 7$, which means $y = 2x + 7$ is an oblique asymptote.

8a Write as $(x+8)(x-2) < 0$, which has solution set $(-8, 2)$.

8b Write as follows:

$$\frac{5}{x-1} - \frac{3}{x+2} \geq 0 \quad \Rightarrow \quad R(x) = \frac{2x+13}{(x-1)(x+2)} \geq 0.$$

Zeros of the numerator and denominator are $-\frac{13}{2}$, -2, and 1, which partitions the real line into intervals $(-\infty, -\frac{13}{2})$, $(-\frac{13}{2}, -2)$, $(-2, 1)$, and $(1, \infty)$. Since $R(-10) < 0$, $R(-3) > 0$, $R(0) < 0$, and $R(2) > 0$, the Intermediate Value Theorem implies that $R(x) > 0$ on $(-\frac{13}{2}, -2)$ and $(1, \infty)$. Thus our inequality $R(x) \geq 0$ has solution set $[-\frac{13}{2}, -2) \cup (1, \infty)$.

9 $(f \circ g)(4) = -5$, $(g \circ f)(2) = -2$, $(f \circ f)(1) = 19$, $(g \circ g)(-2) = -2$.

10a We have

$$(f \circ g)(x) = f(g(x)) = f(3/x) = \sqrt{6/x - 12}$$

with domain $(0, \frac{1}{2}]$.

10b Here

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{2x-12}) = \frac{3}{\sqrt{2x-12}}$$

with domain $(6, \infty)$.

10c Here

$$(g \circ g)(x) = g(g(x)) = g(3/x) = \frac{3}{3/x} = x$$

with domain $(-\infty, 0) \cup (0, \infty)$.

11a Solve $y = 9 - 3x$ for x to get $x = \frac{1}{3}(9 - y)$, so $f^{-1}(y) = \frac{1}{3}(9 - y)$. This can also be written as $f^{-1}(x) = \frac{1}{3}(9 - x)$.

11b Let $y = g(x)$. Then, for $x < 0$,

$$y = 2 + \frac{3}{x^2} \quad \Leftrightarrow \quad x^2 = \frac{3}{y-2} \quad \Leftrightarrow \quad |x| = \sqrt{\frac{3}{y-2}} \quad \Leftrightarrow \quad x = -\sqrt{\frac{3}{y-2}},$$

where $\sqrt{x^2} = |x| = -x$ since $x < 0$ is given. Since $y = g(x)$ if and only if $x = g^{-1}(y)$, we now have

$$g^{-1}(y) = -\sqrt{\frac{3}{y-2}},$$

or equivalently

$$g^{-1}(x) = -\sqrt{\frac{3}{x-2}}.$$