MATH 125 EXAM #2 KEY (SPRING 2019)

1 In general

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

so

$$\begin{aligned}(2x+3)^5 &= \sum_{k=0}^{5} \binom{5}{k} (2x)^{5-k} 3^k \\ &= (2x)^5 + 5(2x)^4(3) + 10(2x)^3(3)^2 + 10(2x)^2(3)^3 + 5(2x)(3)^4 + 3^5 \\ &= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243.\end{aligned}$$

2 $f(x) = (x+1)(x+3)^2(x-5)$.

3a -4 (multiplicity 2), -3 (multiplicity 5).

3b The graph of f touches the x-axis at -4 and crosses it at -3.

3c There are at most 6 turning points, since the polynomial is of degree 7.

3d $f(x) \to -\infty$ as $x \to +\infty$, and $f(x) \to +\infty$ as $x \to -\infty$.

4 Possible rational zeros are $\pm 1, \pm 2, \pm 4, \pm 8$. Synthetic division shows 2 to be a zero:

$$\begin{array}{r|rrrr|r} 2 & 1 & -1 & -6 & 4 & 8 \\ & & 2 & 2 & -8 & -8 \\ \hline & 1 & 1 & -4 & -4 & 0 \end{array} \quad \longrightarrow \quad x^3 + x^2 - 4x - 4.$$

So $f(x) = (x-2)(x^3 + x^2 - 4x - 4)$. For $g(x) = x^3 + x^2 - 4x - 4$ synthetic division shows 2 to be a zero:

$$\begin{array}{r|rrr|r} 2 & 1 & 1 & -4 & -4 \\ & & 2 & 6 & 4 \\ \hline & 1 & 3 & 2 & 0 \end{array} \quad \longrightarrow \quad x^2 + 3x + 2.$$

So

$$\begin{aligned} f(x) &= (x-2) \cdot (x-2)(x^2 + 3x + 2) \\ &= (x-2)^2(x+1)(x+2) \end{aligned}$$

and the zeros of f are -1, 2, and -2.

5 The possible rational zeros of $f(x) = 2x^3 - 11x^2 + 10x + 8$ are $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}$. Synthetic division shows 2 to be a zero:

$$\begin{array}{r|rrr|r} 2 & 2 & -11 & 10 & 8 \\ & & 4 & -14 & -8 \\ \hline & 2 & -7 & -4 & 0 \end{array}$$

Thus $f(x) = (x-2)(2x^2 - 7x - 4) = (x-2)(x-4)(2x+1)$, and equation becomes

$$(x-2)(x-4)(2x+1) = 0.$$

The solution set can now be seen to be $\{-\frac{1}{2}, 2, 4\}$.

6 Another zero must be $1 - 3i$ since the polynomial function has real coefficients. This means that

$$[x - (1+3i)][x - (1-3i)] = x^2 - 2x + 10$$

is a factor of $f(x)$. With long division we have

$$\frac{f(x)}{x^2 - 2x + 10} = x^2 - 5x - 6,$$

and so $f(x) = (x^2 - 2x + 10)(x^2 - 5x - 6)$. The other zeros of f must be the zeros of $x^2 - 5x - 6$. Since $x^2 - 5x - 6 = (x-6)(x+1)$ has zeros -1 and 6, the zeros of f are $1 + 3i$, $1 - 3i$, -1, 6.

7a Domain of T is

$$\{x : 2x^2 + 7x + 5 \neq 0\} = \{x : x \neq -\tfrac{5}{2}, -1\}.$$

7b The x-intercepts are $(-5, 0)$ and $(-1, 0)$, and the y-intercept is $(0, 1)$.

7c $x = -\frac{5}{2}$

7d $y = \frac{1}{2}$ is the horizontal asymptote. No oblique asymptote.

8a We have

$$x^3 - 2x^2 - 3x > 0 \quad\Rightarrow\quad x(x-3)(x+1) > 0,$$

which is satisfied for $x \in (-1, 0) \cup (3, \infty)$.

8b We have

$$\frac{x-4}{2x+4} \geq 1 \quad\Rightarrow\quad \frac{x-4}{2x+4} - \frac{2x+4}{2x+4} \geq 0 \quad\Rightarrow\quad -\frac{x+8}{2x+4} \geq 0 \quad\Rightarrow\quad \frac{x+8}{x+2} \leq 0.$$

There are two cases: either $x + 8 \geq 0$ and $x + 2 < 0$, or $x + 8 \leq 0$ and $x + 2 > 0$. The first case gives $-8 \leq x < -2$, while the second case is impossible. Solution set: $[-8, -2)$.

9 $(f \circ g)(4) = \sqrt{13}$, $(g \circ f)(2) = 3\sqrt{3}$, $(f \circ f)(1) = \sqrt{1+\sqrt{2}}$, $(g \circ g)(0) = 0$.

10a We have $(f \circ g)(x) = f(g(x)) = (\sqrt{x-1}\,)^2 + 1 = x$. Domain:

$$\begin{aligned} \operatorname{Dom} f \circ g &= \big\{x : x \in \operatorname{Dom} g \ \ \& \ \ g(x) \in \operatorname{Dom} f\big\} \\ &= \{x : x \geq 1 \ \ \& \ \ \sqrt{x-1} \in \mathbb{R}\} \\ &= [1, \infty) \end{aligned}$$

10b We have $(g \circ f)(x) = g(f(x)) = g(x^2+1) = \sqrt{x^2} = |x|$. Domain:

$$\begin{aligned} \operatorname{Dom} g \circ f &= \big\{x : x \in \operatorname{Dom} f \ \ \& \ \ f(x) \in \operatorname{Dom} g\big\} \\ &= \big\{x : x \in \mathbb{R} \ \ \& \ \ x^2 + 1 \geq 1\big\} \\ &= (-\infty, \infty) \end{aligned}$$

10c We have $(g \circ g)(x) = g(g(x)) = \sqrt{\sqrt{x-1} - 1}$. Domain:

$$\begin{aligned} \operatorname{Dom} g \circ g &= \big\{x : x \in \operatorname{Dom} g \ \ \& \ \ g(x) \in \operatorname{Dom} g\big\} \\ &= \big\{x : x \geq 1 \ \ \& \ \ \sqrt{x-1} \geq 1\big\} \\ &= \big\{x : x \geq 1 \ \ \& \ \ x \geq 2\big\} \\ &= [2, \infty) \end{aligned}$$

11a Solve $y = x^3 + 1$ for x to get $x = \sqrt[3]{y-1}$, so $f^{-1}(y) = \sqrt[3]{y-1}$. This can also be written as $f^{-1}(x) = \sqrt[3]{x-1}$.

11b Let $y = g(x)$. Then

$$y = \frac{x^2-4}{2x^2} \quad \Leftrightarrow \quad 2x^2y = x^2 - 4 \quad \Leftrightarrow \quad x^2 = \frac{4}{1-2y} \quad \Leftrightarrow \quad x = \sqrt{\frac{4}{1-2y}},$$

where $\sqrt{x^2} = |x| = x$ since $x > 0$ is given. Since $y = g(x)$ if and only if $x = g^{-1}(y)$, we now have

$$g^{-1}(y) = \sqrt{\frac{4}{1-2y}},$$

or equivalently

$$g^{-1}(x) = \sqrt{\frac{4}{1-2x}}.$$