Math 125 Exam #3 Key (Spring 2013)

1a We have

$$\text{Dom}(f) = \{x : x^2 - x - 2 \neq 0\} = \{x : (x-2)(x+1) \neq 0\} = \{x : x \neq -1, 2\}.$$

1b The x-intercepts of f are the points $(x, f(x))$ where $f(x) = 0$:

$$\frac{x^2}{(x-2)(x+1)} = 0 \quad \Rightarrow \quad x^2 = 0 \quad \Rightarrow \quad x = 0,$$

so $(0, 0)$ is the only x-intercept. Since $(0, 0)$ is also a y-intercept of f and a function can never have more than one y-intercept, we have found all intercepts.

1c The vertical asymptotes of f are $x = -1$ and $x = 2$.

1d Since $\deg(x^2) = \deg(x^2 - x - 2) = 2$ and

$$\frac{\text{Lead coefficient of } x^2}{\text{Lead coefficient of } x^2 - x - 2} = \frac{1}{1} = 1,$$

we conclude that $y = 1$ is a horizontal asymptote for f.

1e The graph of f intersects the horizontal asymptote $y = 1$ if there is some $x \in \text{Dom}(f)$ for which $f(x) = 1$. This results in the equation

$$\frac{x^2}{x^2 - x - 2} = 1,$$

giving

$$x^2 = x^2 - x - 2 \quad \Rightarrow \quad x + 2 = 0 \quad \Rightarrow \quad x = -2.$$

Thus the graph of f intersects $y = 1$ at $(-2, 1)$.

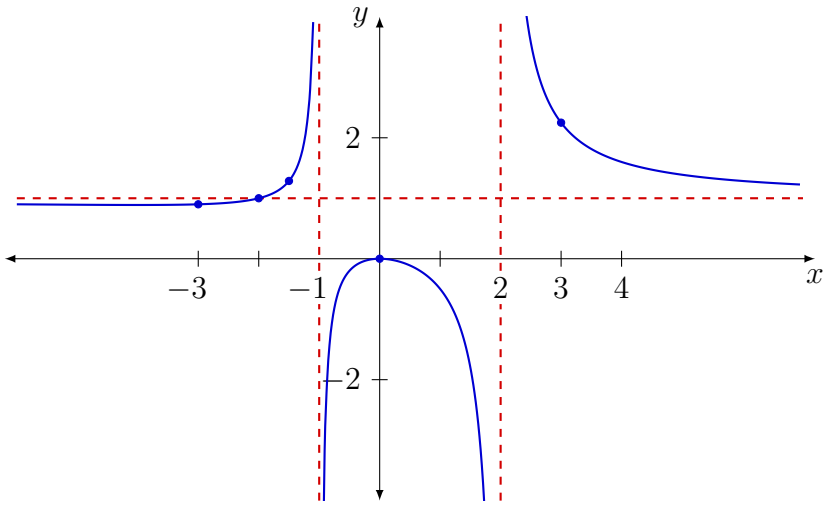
1f The vertical asymptotes partition the plane into three regions:

$$R_1 = \{x : x < -1\}, \quad R_2 = \{x : -1 < x < 2\}, \quad \text{and} \quad R_3 = \{x : x > 2\}.$$

We will want at least one point that lies on the graph of f in each region, and in region R_1 in particular we want points on either side of $(-2, 1)$ where the graph of f intersects the horizontal asymptote $y = 1$. Calculating

$$f(-3) = \tfrac{9}{10}, \quad f\left(-\tfrac{3}{2}\right) = \tfrac{9}{7}, \quad f(3) = \tfrac{9}{4},$$

we obtain the points $\left(-3, \frac{9}{10}\right)$, $\left(-\frac{3}{2}, \frac{9}{7}\right)$, and $\left(3, \frac{9}{4}\right)$. We sketch the graph:



2a Factoring, we get $(x+2)(x+3) > 0$. There are two cases to consider.

Case 1: $x+2 > 0$ & $x+3 > 0$. This gives $x > -2$ & $x > -3$, which is equivalent to $x > -2$.

Case 2: $x+2 < 0$ & $x+3 < 0$. This gives $x < -2$ & $x < -3$, which is equivalent to $x < -3$.

Thus we may have $x < -3$ or $x > -2$. Solution set: $(-\infty, -3) \cup (-2, \infty)$.

2b Manipulate without multiplying by an expression involving x:

$$\frac{2x+1}{x-5} \le 3 \quad \Leftrightarrow \quad \frac{2x+1}{x-5} - 3 \le 0 \quad \Leftrightarrow \quad \frac{2x+1-3(x-5)}{x-5} \le 0 \quad \Leftrightarrow \quad \frac{16-x}{x-5} \le 0.$$

There are two cases to consider.

Case 1: $16 - x \ge 0$ & $x - 5 < 0$. This gives $x \le 16$ & $x < 5$, which is equivalent to $x < 5$. (Note that we cannot have $x = 5$ since division by zero would result.)

Case 2: $16 - x \le 0$ & $x - 5 > 0$. This gives $x \ge 16$ & $x > 5$, which is equivalent to $x \ge 16$. (Letting $x = 16$ results in no division by zero.)

Thus we may have $x < 5$ or $x \ge 16$. Solution set: $(-\infty, 5) \cup [16, \infty)$.

2c Write as $x^2 - 4x + 12 < 0$, and note that the trinomial cannot be factored. Complete the square:

$$x^2 - 4x + 12 < 0 \quad \Rightarrow \quad (x^2 - 4x + 4) + 8 < 0 \quad \Rightarrow \quad (x-2)^2 + 8 < 0 \quad \Rightarrow \quad (x-2)^2 < -8.$$

Since there is no real number x that makes $(x-2)^2$ negatively valued, there is no solution. Solution set: $\varnothing$.

3a Using the formula provided,

$$A(t) = 750\left(1 + \frac{0.08}{4}\right)^{4t} = 750(1.02)^{4t}.$$

3b $750(1.02)^{20} = \$1114.46$ and $750(1.02)^{40} = \$1656.03$.

4a $\log_p(3) = k$

4b $a^{-x} = M$

5 Using a law of logarithms,

$$\ln(x^2 - 9) - \ln(x + 3) = \ln\left(\frac{x^2 - 9}{x + 3}\right) = \ln(x - 3).$$

6a We have

$$5^{4x-7} = 125 \;\Rightarrow\; 5^{4x-7} = 5^3 \;\Rightarrow\; 4x - 7 = 3 \;\Rightarrow\; x = 5/2.$$

6b Take the logarithm of each side:

$$\ln(3^x) = \ln(2^{x-1}) \;\Rightarrow\; x\ln 3 = (x-1)\ln 2 \;\Rightarrow\; x\ln 3 - x\ln 2 = -\ln 2 \;\Rightarrow\; x = \frac{\ln 2}{\ln 2 - \ln 3}.$$

6c Convert to an exponential equation:

$$\log_2(10 + 3x) = 5 \;\Rightarrow\; 2^5 = 10 + 3x \;\Rightarrow\; 3x = 22 \;\Rightarrow\; x = 22/3.$$

6d Consolidate logarithms:

$$\log_2(x+1) + \log_2(x-1) = 3 \;\Rightarrow\; \log_2(x+1)(x-1) = 3 \;\Rightarrow\; 2^3 = x^2 - 1 \;\Rightarrow\; x = \pm 3.$$

But -3 is an extraneous solution (it results in the logarithm of a negative number in the original equation), so $x = 3$ is the only solution.

7 $\sin\varphi = 2/\sqrt{5}$, $\cot\varphi = 1/2$, $\sec\varphi = \sqrt{5}$, $\cos\varphi = 1/\sqrt{5}$, $\csc\varphi = \sqrt{5}/2$.

8 $67^\circ\, 50'\, 24''$

9 Let h be the height of the cloud. We have

$$\tan 62.35^\circ = \frac{h}{75} \;\Rightarrow\; h = 75(\tan 62.35^\circ) \approx 75(1.909) = 143.2.$$

The cloud is about 143 meters high.

10 $469.2°$, $829.2°$, $-250.8°$, $-610.8°$.

11 $\sin\alpha = 3/5$, $\tan\alpha = -3/4$, $\cot\alpha = -4/3$, $\csc\alpha = 5/3$, $\sec\alpha = -5/4$.