MATH 125 EXAM #2 KEY (SPRING 2011)

1. Let $y = f(x)$, so $y = x^2 + 16$. Solve for x: $x^2 = y - 16 \quad \Rightarrow \quad x = \sqrt{y - 16}$, since $x \geq 0$. Now, from $f^{-1}(y) = x$ we obtain $f^{-1}(y) = \sqrt{y - 16}$.

2a. Let $y = g(x)$, so $\dfrac{5x-3}{x+6}$. Solve for x: $xy + 6y = 5x - 3 \;\Rightarrow\; 5x - xy = 6y + 3 \;\Rightarrow\; x(5 - y) = 6y + 3 \;\Rightarrow\; x = \dfrac{6y+3}{5-y}$. Since $g^{-1}(y) = x$ we get $g^{-1}(y) = \dfrac{6y+3}{5-y}$.

2b. $\text{Ran}(g^{-1}) = \text{Dom}(g) = (-\infty, -6) \cup (-6, \infty)$, and $\text{Ran}(g) = \text{Dom}(g^{-1}) = (-\infty, 5) \cup (5, \infty)$.

3. Slope: 0; y-intercept: $(0, 3)$; $\text{Dom}(h) = \mathbb{R}$; $\text{Ran}(h) = \{3\}$.

4a. $f(x) = 2x^2 - x + 2 = 2\left(x^2 - \frac{1}{2}x\right) + 2 = 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) + 2 - 2\left(\frac{1}{16}\right) = 2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8}$

4b. Vertex is at $\left(\frac{1}{4}, \frac{15}{8}\right)$; $\text{Dom}(f) = \mathbb{R}$; $\text{Ran}(f) = \left[\frac{15}{8}, \infty\right)$.

5a. $x^2 + x - 30 < 0 \;\Rightarrow\; (x+6)(x-5) < 0$. Case 1: $x + 6 > 0$ & $x - 5 < 0 \;\Rightarrow\; x > -6$ & $x < 5 \;\Rightarrow\; -6 < x < 5$. Case 2: $x + 6 < 0$ & $x - 5 > 0 \;\Rightarrow\; x < -6$ & $x > 5 \;\Rightarrow\; \varnothing$. Solution set: $(-6, 5)$.

5b. $2x^3 + 8x^2 > 0 \;\Rightarrow\; 2x^2(x+4) > 0 \;\Rightarrow\; x + 4 > 0$ & $x^2 \neq 0 \;\Rightarrow\; x > -4$ & $x \neq 0$. Solution set: $(-4, 0) \cup (0, \infty)$.

5c. $\dfrac{x+4}{x-2} \leq 1 \;\Rightarrow\; \dfrac{x+4}{x-2} - 1 \leq 0 \;\Rightarrow\; \dfrac{(x+4)-(x-2)}{x-2} \leq 0 \;\Rightarrow\; \dfrac{6}{x-2} \leq 0 \;\Rightarrow\; x - 2 < 0 \;\Rightarrow\; x < 2$. Solution set: $(-\infty, 2)$.

6. $f(x) = (x-2)(x+1)(x+3) = (x^2 - x - 2)(x+3) = x^3 - x^2 - 2x + 3x^2 - 3x - 6 = x^3 + 2x^2 - 5x - 6$. (Extra Credit: we need to find a constant C with $f(x) = C(x^3 + 2x^2 - 5x - 6)$ such that $f(4) = 10$. Now, $f(4) = C[4^3 + 2(4^2) - 5(4) - 6] = 70C$, so we must have $C = 1/7$ and conclude with $f(4) = \frac{1}{7}(x^3 - 10x^2 + 33x - 34)$.)

7. Vertical asymptotes are $x = 2$ and $x = -2$. Long division yields $Q(x) = x + \dfrac{4x}{x^2 - 4}$, so there's also the oblique asymptote $y = x$.

8. We have $M_1 = \max\{1, |9| + |-5| + |3|\} = \max\{1, 17\} = 17$ and $M_2 = 1 + \max\{|9|, |-5|, |3|\} = 1 + \max\{9, 5, 3\} = 1 + 9 = 10$. A bound on the zeros of f is, by the Bounds on Zeros Theorem, $m = \min\{M_1, M_2\} = \min\{17, 10\} = 10$. That is, if c is a real number such that $f(c) = 0$, then $c \in [-10, 10]$. Isn't that wonderful?

9a.

$$\begin{array}{r|rrrr|r} 1 & 2 & 11 & -5 & -43 & 35 \\ & & 2 & 13 & 8 & -35 \\ \hline & 2 & 13 & 8 & -35 & 0 \end{array} \quad \longrightarrow \quad f(x) = (x-1)(2x^3+13x^2+8x-35)$$

$$\begin{array}{r|rrr|r} -5 & 2 & 13 & 8 & -35 \\ & & -10 & -15 & 35 \\ \hline & 2 & 3 & -7 & 0 \end{array} \quad \longrightarrow \quad f(x) = (x-1)(x+5)(2x^2+3x-7)$$

Complete list of zeros: -5, 1, $\dfrac{-3 \pm \sqrt{65}}{4}$. (The last two zeros can be found using, say, the quadratic formula or completing the square to solve $2x^2 + 3x - 7 = 0$.)

9b. Complete factorization: $f(x) = 2(x-1)(x+5)\left(x + \dfrac{3+\sqrt{65}}{4}\right)\left(x + \dfrac{3-\sqrt{65}}{4}\right)$. (Important: the factor of 2 must be inserted because if you solved $2x^2+3x-7=0$ by completing the square you'll recall that the first step was to divide the 2 out. So it must be restored.)

10. By the Conjugate Zeros Theorem $4+i$ must also be a zero in order to have real coefficients. So we have $f(x) = (x-2)[x-(4-i)][x-(4+i)]$, which simplifies to become $f(x) = x^3 - 10x^2 + 33x - 34$.

11. $f(x) = x^3 - 1 = (x-1)(x^2+x+1)$. Solving $x^2+x+1=0$ yields $x = \dfrac{-1 \pm i\sqrt{3}}{2}$. Full factorization of $f(x)$ is: $f(x) = (x-1)\left(x - \dfrac{-1+i\sqrt{3}}{2}\right)\left(x - \dfrac{-1-i\sqrt{3}}{2}\right) = (x-1)\left(x + \dfrac{1-i\sqrt{3}}{2}\right)\left(x + \dfrac{1+i\sqrt{3}}{2}\right)$.

E.C. 1 is a zero, and dividing $f(x)$ by $x-1$ gives

$$\begin{aligned} f(x) &= (x-1)(x^7+8x^4+x^3+8) \\ &= (x-1)[x^4(x^3+8)+(x^3+8)] \\ &= (x-1)(x^3+8)(x^4+1) \\ &= (x-1)(x+2)(x^2-2x+4)(x^4+1) \end{aligned}$$

It's seen from this that -2 is also a zero. Solving $x^2-2x+4=0$ yields $x = \dfrac{-1 \pm i\sqrt{3}}{2}$. As for $x^4+1=0$, this can be rewritten as $(x^2)^2 - i^2 = 0$ to give us $(x^2-i)(x^2+i)=0$; so either $x^2 = i$ or $x^2 = -i$, whence $x = \pm\sqrt{i}$ or $x = \pm\sqrt{-i} = \pm i\sqrt{i}$. Set of zeros of f are:

$$\left\{1, -2, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}, \sqrt{i}, -\sqrt{i}, i\sqrt{i}, -i\sqrt{i}\right\}$$