MATH 125 EXAM #2 KEY (FALL 2021)

1 Let the rectangle be ℓ long and w wide. Perimeter is $2\ell + 2w = 80$, so $w = 40 - \ell$. Area as a function of ℓ is then $A(\ell) = \ell(40 - \ell) = -\ell^2 + 40\ell$. This is a quadratic function that opens downward, so the vertex is the highest point (representing maximum area). The vertex is at ℓ value $-\frac{b}{2a} = 20$, in which case $w = 40 - 20 = 20$ also. We conclude that the rectangle with maximum area must be $20\,\text{m} \times 20\,\text{m}$, which has area $400\,\text{m}^2$.

2 Solve $f(x) = 0$, or $3x^2 + 6x + 4 = 0$. With the quadratic forumula we obtain $x = -1 \pm \frac{\sqrt{3}}{3}i$.

3 Remove absolute value bars to get $x^2 + 3x = \pm 5$, or $x^2 + 3x \pm 5 = 0$. Using the quadratic formula to solve each equation gives

$$x = -\frac{3}{2} \pm \frac{\sqrt{29}}{2}, \quad -\frac{3}{2} \pm \frac{\sqrt{11}}{2}i.$$

Only the real solutions are required, but the complex-valued solutions are also valid.

4 We must have $x + 6 \geq 7$ or $x + 6 \leq -7$, giving $x \geq 1$ or $x \leq -13$. Solution set is $(-\infty, -13] \cup [1, \infty)$.

5 Must have $f(x) = C(x-3)^2(x+4)(x-1)^3$ with C such that $f(-1) = 20$. This requires $C = -\frac{5}{96}$, and so we finally get

$$f(x) = -\tfrac{5}{96}(x-3)^2(x+4)(x-1)^3.$$

6 To have real coefficients the Conjugate Zeros Theorem implies that $1 + 2i$ must also be a zero, and so we need

$$\begin{aligned} f(x) &= (x-6)[x-(1-2i)][x-(1+2i)] \\ &= (x-6)(x^2-2x+5) \\ &= x^3 - 8x^2 + 17x - 30. \end{aligned}$$

7 The rational zeros that G could possibly have include such values as 1 and -5, which are in fact zeros for G. We use synthetic division to start factoring $G(x)$:

$$\begin{array}{r|rrrr|r} 1 & 2 & 11 & -5 & -43 & 35 \\ & & 2 & 13 & 8 & -35 \\ \hline & 2 & 13 & 8 & -35 & 0 \end{array} \quad \longrightarrow \quad f(x) = (x-1)(2x^3 + 13x^2 + 8x - 35)$$

$$\begin{array}{r|rrr|r} -5 & 2 & 13 & 8 & -35 \\ & & -10 & -15 & 35 \\ \hline & 2 & 3 & -7 & 0 \end{array} \quad \longrightarrow \quad f(x) = (x-1)(x+5)(2x^2 + 3x - 7)$$

Solving $2x^2 + 3x - 7 = 0$ using the quadratic formula, we obtain the complete list of real zeros: -5, 1, $\frac{-3 \pm \sqrt{65}}{4}$.

The complete factorization is

$$G(x) = (x-1)(x+5)\left(x + \tfrac{3+\sqrt{65}}{4}\right)\left(x + \tfrac{3-\sqrt{65}}{4}\right).$$

8 Let $f(x) = x^3 - 8x^2 + 25x - 26$, so the problem is to find all x such that $f(x) = 0$. Among the possible rational zeros is 2, which turns out to work:

$$\begin{array}{r|rrr|r} 2 & 1 & -8 & 25 & -26 \\ & & 2 & -12 & 26 \\ \hline & 1 & -6 & 13 & 0 \end{array} \quad \longrightarrow \quad \text{So 2 is a zero and } f(x) = (x-2)(x^2-6x+13).$$

Applying ye olde quadratic formula to $x^2 - 6x + 13 = 0$ yields the zeros $3 \pm 2i$. In conclusion, f has zeros 2, $3 - 2i$, $3 + 2i$, which are therefore the solutions to the given equation.

9 $-3i$ must be another zero, and so $(x - 3i)(x + 3i) = x^2 + 9$ is a factor of $H(x)$. Then

$$\begin{array}{r|l} & 3x^2 + 5x - 2 \\ \hline x^2+9 & 3x^4 + 5x^3 + 25x^2 + 45x - 18 \\ & -3x^4 - 27x^2 \\ \hline & 5x^3 - 2x^2 + 45x \\ & - 5x^3 - 45x \\ \hline & - 2x^2 - 18 \\ & 2x^2 + 18 \\ \hline & 0 \end{array}$$

shows $3x^2 + 5x - 2$ is another factor. Now,

$$H(x) = 0 \ \Rightarrow \ (x^2+9)(3x^2+5x-2) = 0 \ \Rightarrow \ (x-3i)(x+3i)(3x-1)(x+2) = 0,$$

and therefore other zeros of H are $\frac{1}{3}$ and -2.

10 $K(x) = \dfrac{x(x^2+1)}{(x-3)(x-2)}$ shows the fraction is reduced, so there are vertical asymptotes $x = 3$, $x = 2$. Long division also shows $y = x + 5$ is an oblique asymptote:

$$\begin{array}{r|l} & x + 5 \\ \hline x^2 - 5x + 6 & x^3 + x \\ & -x^3 + 5x^2 - 6x \\ \hline & 5x^2 - 5x \\ & - 5x^2 + 25x - 30 \\ \hline & 20x - 30 \end{array}$$

11a Write $x^4 - 16 > 0$, then factor to get $(x-2)(x+2)(x^2+4) > 0$. Since $x^2 + 4 > 0$ for any x, we divide it out to get $(x-2)(x+2) > 0$, and then use the Intermediate Value Theorem to determine the solution set to be $(-\infty, -2) \cup (2, \infty)$.

11b Employ the Intermediate Value Theorem method as in class, or consider the following approach. Factoring, the inequality $x^3 - 2x^2 - 3x < 0$ becomes $x(x-3)(x+1) < 0$. Now we run through cases.

Case I: $x < 0$, $x - 3 < 0$, $x + 1 < 0$, giving $x < -1$. Case II: $x < 0$, $x - 3 > 0$, $x + 1 > 0$, giving a contradiction. Case III: $x > 0$, $x - 3 < 0$, $x + 1 > 0$, giving $0 < x < 3$. Case IV: $x > 0$, $x - 3 > 0$, $x + 1 < 0$, giving a contradiction. Solution set: $(-\infty, -1) \cup (0, 3)$.

11c Get 0 on the right-hand side:

$$\frac{3x-5}{x+2} - 2 \geq 0 \quad \Rightarrow \quad \frac{3x-5}{x+2} - \frac{2(x+2)}{x+2} \geq 0 \quad \Rightarrow \quad \frac{x-9}{x+2} \geq 0.$$

Case I: $x - 9 \geq 0$ & $x + 2 > 0$, giving $x \geq 9$. Case II: $x - 9 \leq 0$ & $x + 2 < 0$, giving $x < -2$. Solution set: $(-\infty, -2) \cup [9, \infty)$.