Math 125 Exam #2 Key (Fall 2019)

1 In general

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

so

$$\begin{aligned}(2x-1)^5 &= \sum_{k=0}^{5} \binom{5}{k} (2x)^{5-k}(-1)^k \\ &= (2x)^5 + 5(2x)^4(-1) + 10(2x)^3(-1)^2 + 10(2x)^2(-1)^3 + 5(2x)(-1)^4 + (-1)^5 \\ &= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1.\end{aligned}$$

2 $f(x) = x^2(x+2)(x-4)$.

3 Possible rational zeros are $\pm1, \pm2, \pm4, \pm8$. Synthetic division shows 2 to be a zero:

$$\begin{array}{r|rrrr|r} 2 & 1 & -1 & -6 & 4 & 8 \\ & & 2 & 2 & -8 & -8 \\ \hline & 1 & 1 & -4 & -4 & 0 \end{array} \quad \longrightarrow \quad x^3 + x^2 - 4x - 4.$$

So $f(x) = (x-2)(x^3+x^2-4x-4)$. For $g(x) = x^3+x^2-4x-4$ synthetic division shows 2 to be a zero:

$$\begin{array}{r|rrr|r} 2 & 1 & 1 & -4 & -4 \\ & & 2 & 6 & 4 \\ \hline & 1 & 3 & 2 & 0 \end{array} \quad \longrightarrow \quad x^2 + 3x + 2.$$

So

$$\begin{aligned} f(x) &= (x-2)\cdot(x-2)(x^2+3x+2) \\ &= (x-2)^2(x+1)(x+2) \end{aligned}$$

and the zeros of f are -1, 2, and -2.

4 The possible rational zeros of $f(x) = 2x^3 - 11x^2 + 10x + 8$ are $\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{2}$. Synthetic division shows 2 to be a zero:

$$\begin{array}{r|rrr|r} 2 & 2 & -11 & 10 & 8 \\ & & 4 & -14 & -8 \\ \hline & 2 & -7 & -4 & 0 \end{array}$$

Thus $f(x) = (x-2)(2x^2-7x-4) = (x-2)(x-4)(2x+1)$, and equation becomes

$$(x-2)(x-4)(2x+1) = 0.$$

The solution set can now be seen to be $\{-\frac{1}{2}, 2, 4\}$.

5 Another zero must be $1-3i$ since the polynomial function has real coefficients. This means

$$[x-(1+3i)][x-(1-3i)] = x^2 - 2x + 10$$

is a factor of $f(x)$. With long division we have

$$\frac{f(x)}{x^2 - 2x + 10} = x^2 - 5x - 6,$$

and so $f(x) = (x^2 - 2x + 10)(x^2 - 5x - 6)$. The other zeros of f must be the zeros of $x^2 - 5x - 6$. Since $x^2 - 5x - 6 = (x-6)(x+1)$ has zeros -1 and 6, the zeros of f are $1+3i$, $1-3i$, -1, 6.

6a Domain of T is

$$\{x : 2x^2 + 7x + 5 \neq 0\} = \{x : x \neq -\tfrac{5}{2}, -1\}.$$

6b The x-intercepts are $(-5, 0)$ and $(-1, 0)$, and the y-intercept is $(0, 1)$.

6c $x = -\frac{5}{2}$

6d $y = \frac{1}{2}$ is the horizontal asymptote. No oblique asymptote.

7a We have

$$x^3 - 2x^2 - 3x > 0 \quad \Rightarrow \quad x(x-3)(x+1) > 0,$$

which is satisfied for $x \in (-1, 0) \cup (3, \infty)$.

7b We have

$$\frac{x+6}{2x-1} \geq -1 \quad \Rightarrow \quad \frac{x-4}{2x+4} + \frac{2x-1}{2x-1} \geq 0 \quad \Rightarrow \quad \frac{3x+5}{2x-1} \geq 0.$$

There are two cases: either $3x + 5 \geq 0$ and $2x - 1 > 0$, or $3x + 5 \leq 0$ and $2x - 1 < 0$. The first case gives $x > 1/2$, and the second case gives $x \leq -5/3$. Solution set: $(-\infty, -\frac{5}{3}] \cup (\frac{1}{2}, \infty)$.

8 $(f \circ g)(4) = \sqrt{13}$, $(g \circ f)(2) = 3\sqrt{3}$, $(f \circ f)(1) = \sqrt{1 + \sqrt{2}}$, $(g \circ g)(0) = 0$.

9a We have $(f \circ g)(x) = f(g(x)) = (\sqrt{x+3}\,)^2 - 2 = x + 1$. Domain:

$$\begin{aligned} \operatorname{Dom} f \circ g &= \big\{x : x \in \operatorname{Dom} g \ \ \& \ \ g(x) \in \operatorname{Dom} f\big\} \\ &= \{x : x \geq -3 \ \ \& \ \ \sqrt{x+3} \in \mathbb{R}\} \\ &= [-3, \infty) \end{aligned}$$

9b We have $(g \circ f)(x) = g(f(x)) = g(x^2 - 2) = \sqrt{x^2 + 1}$. Domain:

$$\begin{aligned} \operatorname{Dom} g \circ f &= \{x : x \in \operatorname{Dom} f \;\;\&\;\; f(x) \in \operatorname{Dom} g\} \\ &= \{x : x \in \mathbb{R} \;\;\&\;\; x^2 - 2 \geq -3\} \\ &= (-\infty, \infty) \end{aligned}$$

9c We have $(g \circ g)(x) = g(g(x)) = \sqrt{\sqrt{x + 3} + 3}$. Domain:

$$\begin{aligned} \operatorname{Dom} g \circ g &= \{x : x \in \operatorname{Dom} g \;\;\&\;\; g(x) \in \operatorname{Dom} g\} \\ &= \{x : x \geq -3 \;\;\&\;\; \sqrt{x + 3} \geq -3\} \\ &= \{x : x \geq -3 \;\;\&\;\; x \geq -3\} \\ &= [-3, \infty) \end{aligned}$$

10a Solve $y = x^3 - 3$ for x to get $x = \sqrt[3]{y + 3}$, so $f^{-1}(y) = \sqrt[3]{y + 3}$. This can also be written as $f^{-1}(x) = \sqrt[3]{x + 3}$.

10b Let $y = g(x)$. Then

$$y = \frac{x^2 - 1}{3x^2} \quad \Leftrightarrow \quad 3x^2 y = x^2 - 1 \quad \Leftrightarrow \quad x^2 = -\frac{1}{3y - 1} \quad \Leftrightarrow \quad x = \sqrt{\frac{1}{1 - 3y}},$$

where $\sqrt{x^2} = |x| = x$ since $x > 0$ is given. Since $y = g(x)$ if and only if $x = g^{-1}(y)$, we now have

$$g^{-1}(y) = \sqrt{\frac{1}{1 - 3y}},$$

or equivalently

$$g^{-1}(x) = \sqrt{\frac{1}{1 - 3x}}.$$