Math 125 Exam #1 Key (Fall 2015)

1 We have

$$f(-1) = 1 - \frac{1}{(-1)^2+1} = \frac{1}{2} \quad \text{and} \quad f(2x) = 1 - \frac{1}{(2x)^2+1} = 1 - \frac{1}{4x^2+1}.$$

2a $\mathrm{Dom}(g) = \{x : x - 16 > 0\} = (16, \infty)$.

2b $\mathrm{Dom}(h) = \{t : t + 3 \geq 0 \text{ and } t \neq 2\} = [-3, 2) \cup (2, \infty)$.

3a By definition,

$$(f-g)(x) = f(x) - g(x) = \left(1 + \frac{1}{x}\right) - \frac{1}{x} = 1.$$

Also by definition, since $\mathrm{Dom}(f) = \mathrm{Dom}(g) = (-\infty, 0) \cup (0, \infty)$, we have

$$\mathrm{Dom}(f-g) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g) = (-\infty, 0) \cup (0, \infty).$$

3b By defintion,

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{1 + 1/x}{1/x} = x + 1.$$

Also by definition:

$$\mathrm{Dom}(f/g) = \{x : x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \text{ and } g(x) \neq 0\} = (-\infty, 0) \cup (0, \infty).$$

4a Since $p(-1) = -3(-1)^2 + 5(-1) = -8 \neq 2$, the point $(-1, 2)$ is not on the graph.

4b Here $p(x) = -2$ implies $-3x^2 + 5x = -2$, which becomes $3x^2 - 5x - 2 = 0$, and finally $(3x+1)(x-2) = 0$. Solution set is $\{-\frac{1}{3}, 2\}$. So x may be either $-\frac{1}{3}$ or 2.

4c $\mathrm{Dom}(p) = (-\infty, \infty)$.

4d The y-intercept is $(0, p(0)) = (0, 0)$.

4e In other words we find all x for which $p(x) = 0$, or $-3x^2 + 5x = 0$. Factoring gives $x(5 - 3x) = 0$, and so $x \in \{0, \frac{5}{3}\}$.

5 Definition for f is

$$f(x) = \begin{cases} x, & x \in [-3, 0] \\ -\frac{3}{2}x + 4, & x \in (0, 4]. \end{cases}$$

6 The distance between $P = (x, y)$ and $(0, 0)$ is
$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2-8)^2} = \sqrt{x^4 - 15x^2 + 64},$$
since $y = x^2 - 8$. Thus we have
$$d(x) = \sqrt{x^4 - 15x^2 + 64}.$$

7 We find all x for which $f(x) = g(x)$. Now,
$$\begin{aligned} f(x) = g(x) \quad &\Rightarrow \quad x^2 + 5x - 3 = 2x^2 + 7x - 27 \\ &\Rightarrow \quad x^2 + 2x - 24 = 0 \quad \Rightarrow \quad (x+6)(x-4) = 0, \end{aligned}$$
and so $x = -6, 4$. Since $f(-6) = 3$ and $f(4) = 33$, the points of intersection are thus $(-6, 3)$ and $(4, 33)$.

8a Find x such that $q(x) = 0$:
$$\begin{aligned} q(x) = 0 \quad &\Rightarrow \quad x^2 - 6x - 9 = 0 \quad \Rightarrow \quad (x^2 - 6x + 9) - 18 = 0 \quad \Rightarrow \quad (x-3)^2 = 18 \\ &\Rightarrow \quad |x - 3| = \sqrt{18} \quad \Rightarrow \quad x - 3 = \pm 3\sqrt{2} \quad \Rightarrow \quad x = 3 \pm 3\sqrt{2}. \end{aligned}$$

8b Clearly the domain is $\mathrm{Dom}(q) = (-\infty, \infty)$, and from $q(x) = (x-3)^2 - 18$ it can be seen that the range is $\mathrm{Ran}(q) = [-18, \infty)$.

9a For $f(x) = -x^2 + 4$,
$$f(x) \geq 1 \quad \Rightarrow \quad -x^2 + 4 \geq 1 \quad \Rightarrow \quad x^2 \leq 3 \quad \Rightarrow \quad |x| \leq \sqrt{3} \quad \Rightarrow \quad -\sqrt{3} \leq x \leq \sqrt{3}.$$
Solution set is $\left[-\sqrt{3}, \sqrt{3}\right]$.

9b For $f(x) = -x^2 + 4$ and $g(x) = -x - 2$,
$$f(x) > g(x) \quad \Rightarrow \quad -x^2 + 4 > -x - 2 \quad \Rightarrow \quad x^2 - x - 6 < 0 \quad \Rightarrow \quad (x-3)(x+2) < 0.$$
Two cases follow: either $x - 3 < 0$ and $x + 2 > 0$, or $x - 3 > 0$ and $x + 2 < 0$. The first case gives $x < 3$ and $x > -2$, or equivalently $-2 < x < 3$. The second case gives $x > 3$ and $x < -2$, which is impossible. Thus the solution set consists of all x such that $-2 < x < 3$, which is $(-2, 3)$.

10 If length is x and width is y, then the perimeter is $2x + 2y = 3000$. This implies that $y = 1500 - x$, and so the dimensions of the rectangle are x and $1500 - x$. Area is thus
$$A(x) = x(1500 - x) = -(x^2 - 1500x) = -(x^2 - 1500x + 750^2) + 750^2 =$$
which becomes
$$A(x) = -(x - 750)^2 + 562{,}500.$$
The area $A(x)$ is maximal when $x = 750$, and the maximum area is $A(750) = 562{,}500$ square meters.