4

Polynomial & Rational Functions

4.5 – Rational Functions

A function f is a **rational function** if there exist polynomial functions p and q, with q not the zero function, such that

$$f(x) = \frac{p(x)}{q(x)}$$

for all x for which $p(x)/q(x) \in \mathbb{R}$. That is, $f = p/q$. Clearly

$$\text{Dom}(f) = \{x \in \mathbb{R} : q(x) \neq 0\}. \tag{1}$$

Before proceeding with a study of rational functions it will be convenient to establish some new notation. Given any function f (not necessarily a rational function), to write

$$f(x) \to \infty \quad \text{as} \quad x \to c$$

means that as x approaches the number c the value of $f(x)$ grows without bound in the positive direction. For instance we have

$$\frac{1}{x^2} \to \infty \quad \text{as} \quad x \to 0,$$

which is to say the closer x gets to 0 the larger $1/x^2$ gets—*and there is no limit to how large a positive quantity* $1/x^2$ *can become!*

Definition 4.1. *The line* $x = c$ *is a* ***vertical asymptote*** *of a rational function* f *if*

$$|f(x)| \to \infty \quad \textit{as} \quad x \to c.$$

By way of an example, if f is the function given by

$$f(x) = \frac{1}{(x-2)^2},$$

then the vertical line $x = 2$ is the vertical asymptote of f. See Figure 1.

Theorem 4.2. *A rational function* $f(x) = p(x)/q(x)$ *has vertical asymptote* $x = c$ *if and only if* $p(c) \neq 0$ *and* $q(c) = 0$.

Recall the statement of the Factor Theorem: if p is a polynomial function, then $p(c) = 0$ if and only if $x - c$ is a factor of $p(x)$. Thus the statement of Theorem 4.2 can be rephrased as follows.

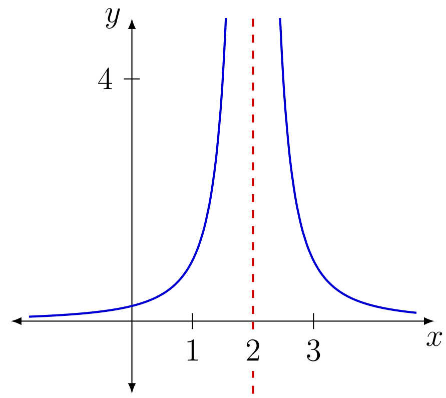

FIGURE 1. A vertical asymptote at $x = 2$

Corollary 4.3. *A rational function $f(x) = p(x)/q(x)$ has vertical asymptote $x = c$ if and only if $x - c$ is only a factor of $q(x)$.*

A rational function $f = p/q$ is said to be in **reduced form** if $p(x)$ and $q(x)$ have no common factors. To go about finding vertical asymptotes of $f = p/q$ we could fully factor $p(x)$ and $q(x)$, cancel all common factors $a_1x + b_1, \ldots, a_kx + b_k$ present, and so render $f(x)$ in reduced form:

$$f(x) = \frac{p(x)}{q(x)} = \frac{\hat{p}(x)(a_1x + b_1)\cdots(a_kx + b_k)}{\hat{q}(x)(a_1x + b_1)\cdots(a_kx + b_k)} = \frac{\hat{p}(x)}{\hat{q}(x)}. \tag{2}$$

We now find by Corollary 4.3 that the vertical asymptotes of f are precisely the zeros of the polynomial function $\hat{q}$, since $\hat{p}$ and $\hat{q}$ have no common factors by construction.

It must be emphasized that the last equality in (2) only applies when x is such that

$$a_kx + b_k \neq 0$$

for $k = 1, \ldots, n$. Put another way, for each $k = 1, \ldots, n$ we only have

$$\frac{a_kx + b_k}{a_kx + b_k} = 1$$

if $x \neq -b_k/a_k$, since otherwise the fraction becomes $0/0$, and $0/0$ does not equal 1 or any other number!

Another important point is that even if a rational function $f(x) = p(x)/q(x)$ can be written in a reduced form $\hat{p}(x)/\hat{q}(x)$, the domain of f is still given by (1). Thus, if $q(c) = 0$ but $\hat{q}(c) \neq 0$, *we still do not admit c into the domain of f!* The domain of any rational function must be found from its *original* form, not its reduced form.

Example 4.4. Find the domain and vertical asymptotes of

$$f(x) = \frac{x^2 + x - 6}{x - 2},$$

and then give the graph of f.

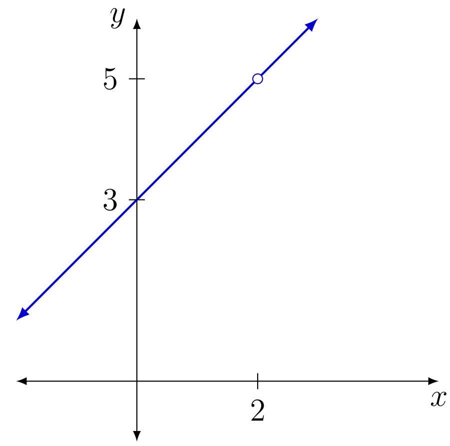

FIGURE 2. A hole at $x = 2$.

Solution. We have

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : x - 2 \neq 0\} = \{x \in \mathbb{R} : x \neq 2\} = (-\infty, 2) \cup (2, \infty).$$

To find any vertical asymptotes, we obtain the reduced form of the fraction:

$$\frac{x^2 + x - 6}{x - 2} = \frac{(x-2)(x+3)}{x-2} = x + 3.$$

Thus we may define f by

$$f(x) = x + 3, \quad x \neq 2, \tag{3}$$

so $f(x) = \hat{p}(x)/\hat{q}(x)$ with $\hat{p}(x) = x + 3$ and $\hat{q}(x) = 1$. Since $x = c$ is a vertical asymptotes of f if and only if $\hat{q}(c) = 0$, we conclude that f has *no* vertical asymptotes.

To graph f we may for the most part simply graph the line $y = x + 3$; however it must be stressed, as it is in (3), that 2 is still not in the domain of f. As a result there is a "hole" in the graph of f at the point $(2, 5)$. See Figure 2. ■

Definition 4.5. *The line $y = c$ is a **horizontal asymptote** of a rational function f if*

$$f(x) \to c \quad \text{as} \quad x \to \pm\infty.$$

Example 4.6. Let

$$f(x) = \frac{x + 2}{x^2 + 2x - 15}.$$

(a) Find the domain of f.
(b) Find the intercepts of f.
(c) Find all vertical asymptotes of f.
(d) Find the horizontal or oblique asymptote of f.
(e) Find all points where f intersects its horizontal or oblique asymptote.
(f) Find additional points on the graph of f as needed.
(g) Sketch the graph of f.

Solution.
(a) The domain of f is

$$\begin{aligned}\mathrm{Dom}(f) &= \{x \in \mathbb{R} : x^2 + 2x - 15 \neq 0\} = \{x \in \mathbb{R} : (x-3)(x+5) \neq 0\} \\ &= \{x \in \mathbb{R} : x \neq -5, 3\} = (-\infty, -5) \cup (-5, 3) \cup (3, \infty).\end{aligned}$$

(b) The x-intercepts of f are the points $(x, f(x))$ where $f(x) = 0$. Now,

$$f(x) = 0 \quad \Rightarrow \quad \frac{x+2}{x^2 + 2x - 15} = 0 \quad \Rightarrow \quad x + 2 = 0 \quad \Rightarrow \quad x = -2,$$

so $(-2, 0)$ is the only x-intercept.
The y-intercept of f is the point $(0, f(0)) = \left(0, -\frac{2}{15}\right)$.

(c) We have

$$f(x) = \frac{x+2}{(x-3)(x+5)},$$

which is already in reduced form. Thus the vertical asymptotes of f are the lines $x = -5$ and $x = 3$.

(d) Since the degree of the polynomial $x^2 + 2x - 15$ in the denominator of $f(x)$ is greater than the degree of $x + 2$ in the numerator, we conclude that f has horizontal asymptote $y = 0$.

(e) The graph of f intersects the horizontal asymptote $y = 0$ if there is some $x \in \mathrm{Dom}(f)$ for which $f(x) = 0$. We found just such a point already, namely the x-intercept $(-2, 0)$, which is only because the horizontal line $y = 0$ happens to be the x-axis.

(f) The vertical asymptotes partition the plane into three regions:

$$R_1 = \{x : x < -5\}, \quad R_2 = \{x : -5 < x < 3\}, \quad \text{and} \quad R_3 = \{x : x > 3\}.$$

We will want at least one point that lies on the graph of f in each region.
Calculating

$$f(-6) = \frac{-6+2}{(-6)^2 + 2(-6) - 15} = -\frac{4}{9},$$

we find that $\left(-6, -\frac{4}{9}\right)$ is a point on the graph of f in region R_1.
We already have points $(-2, 0)$ and $\left(0, -\frac{2}{15}\right)$ in region R_2. However, the point $(-2, 0)$ lies right on the horizontal asymptote $y = 0$. The point $\left(0, -\frac{2}{15}\right)$ lies to the right of this point, so we should find a point in R_2 that lies to the left of $(-2, 0)$. Calculating

$$f(-4) = \frac{-4+2}{(-4)^2 + 2(-4) - 15} = \frac{2}{7},$$

we obtain $\left(-4, \frac{2}{7}\right)$ as just such a point.
Finally we obtain a point in region R_3. Calculating

$$f(4) = \frac{4+2}{4^2 + 2(4) - 15} = \frac{2}{3},$$

we have the point $\left(4, \frac{2}{3}\right)$.

(g) We sketch the graph of f using the asymptotes and select points as guides. In region R_1 we have the point $\left(-6, -\frac{4}{9}\right)$, which lies below the horizontal asymptote $y = 0$ and to the left of the vertical asymptote $x = -5$. We know the graph of f cannot cross $y = 0$ in this region, so as we move to the left of $\left(-6, -\frac{4}{9}\right)$ we must have

$$f(x) \to 0^- \quad \text{as} \quad x \to -\infty.$$

Also, since the graph of f cannot cross $x = -5$, as we move to the right of $\left(-6, -\frac{4}{9}\right)$ we must have

$$f(x) \to -\infty \quad \text{as} \quad x \to -5^-$$

Moving on to region R_2, as we move to the left of $(-2, 0)$ we must have

$$f(x) \to \infty \quad \text{as} \quad x \to -5^+$$

since $\left(-4, \frac{2}{7}\right)$ is above $y = 0$ and there is no option to cross the horizontal asymptote anywhere to the left of $(-2, 0)$. As we move to the right of $(-2, 0)$ we must have

$$f(x) \to -\infty \quad \text{as} \quad x \to 3^-$$

since $\left(0, -\frac{2}{15}\right)$ is below $y = 0$ and there is no option to cross the horizontal asymptote anywhere to the right of $(-2, 0)$. (Remember: crossing a vertical asymptote is *never* an option!)

Finally, in region R_3 we have the point $\left(4, \frac{2}{3}\right)$, which is above the horizontal asymptote $y = 0$ and to the right of the vertical asymptote $x = 3$. Thus the graph of f is bent upward as we move to the left of $\left(4, \frac{2}{3}\right)$ (to avoid crossing $x = 3$), and bends to avoid crossing $y = 0$ as we move to the right of $\left(4, \frac{2}{3}\right)$. That is, we have

$$f(x) \to \infty \quad \text{as} \quad x \to 3^+$$

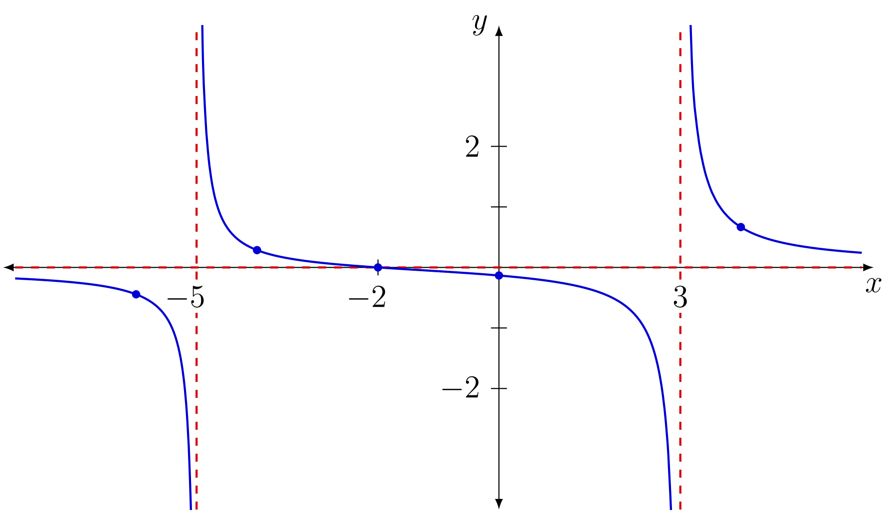

FIGURE 3

and

$$f(x) \to 0^+ \quad \text{as} \quad x \to \infty$$

See Figure 3. ■

Example 4.7. Let

$$f(x) = \frac{x^3 - 4x^2}{x^3 - 5x^2 + 2x + 8}.$$

(a) Find the domain of f.
(b) Find the intercepts of f.
(c) Find all vertical asymptotes of f.
(d) Find the horizontal or oblique asymptote of f.
(e) Find all points where f intersects its horizontal or oblique asymptote.
(f) Find additional points on the graph of f as needed.
(g) Sketch the graph of f.

Solution.
(a) The numerator of $g(x)$ factors as $x^2(x-4)$, and so it would be worthwhile determining whether $x-4$ is also a factor of the denominator. Letting

$$q(x) = x^3 - 5x^2 + 2x + 8,$$

we carry out the divison $q(x) \div (x-4)$:

$$\begin{array}{r|rrr|r} 4 & 1 & -5 & 2 & 8 \\ & & 4 & -4 & -8 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

The remainder is 0, so $q(4) = 0$ by the Remainder Theorem, and by the Factor Theorem we conclude that $x-4$ is a factor of $q(x)$. Indeed,

$$q(x) = (x-4)(x^2 - x - 2) = (x-4)(x-2)(x+1),$$

and we have

$$f(x) = \frac{x^2(x-4)}{(x-4)(x-2)(x+1)}.$$

It is now clear that

$$\text{Dom}(f) = \{x : x \neq -1, 2, 4\} = (-\infty, -1) \cup (-1, 2) \cup (2, 4) \cup (4, \infty).$$

(b) The x-intercepts of f are the points $(x, f(x))$ where $f(x) = 0$. From $f(x) = 0$ we have

$$\frac{x^2(x-4)}{(x-4)(x-2)(x+1)} = 0 \;\Rightarrow\; \frac{x^2}{(x-2)(x+1)} = 0 \;\Rightarrow\; x^2 = 0 \;\Rightarrow\; x = 0,$$

so $(0,0)$ is the only x-intercept. Since $(0,0)$ is also a y-intercept of f and a function can never have more than one y-intercept, we have found all intercepts.

(c) The vertical asymptotes of f are at precisely the values of x that lead to division by 0 in the expression $f(x)$ *when $f(x)$ is in reduced form.* The reduced form for $f(x)$ is

$$\frac{x^2}{(x-2)(x+1)},$$

and so the vertical asymptotes of f are $x=-1$ and $x=2$. Since $4 \notin \mathrm{Dom}(f)$ but $x=4$ is not a vertical asymptote of f, we conclude that there is a hole in the graph of f at the point

$$\left(4, \frac{4^2}{(4-2)(4+1)}\right) = \left(4, \tfrac{8}{5}\right).$$

(d) Since

$$\deg(x^3-4x^2) = \deg(x^3-5x^2+2x+8) = 3,$$

and

$$\frac{\text{Lead coefficient of } x^3-4x^2}{\text{Lead coefficient of } x^3-5x^2+2x+8} = \frac{1}{1} = 1,$$

we conclude that $y=1$ is a horizontal asymptote for f.

(e) The graph of f intersects the horizontal asymptote $y=1$ if there is some $x \in \mathrm{Dom}(f)$ for which $f(x)=1$. This results in the equation

$$\frac{x^2(x-4)}{(x-4)(x-2)(x+1)} = 1,$$

whence

$$\frac{x^2}{(x-2)(x+1)} = 1 \quad\Rightarrow\quad x^2 = (x-2)(x+1) \quad\Rightarrow\quad x+2=0 \quad\Rightarrow\quad x=-2.$$

Thus the graph of f intersects $y=1$ at $(-2, f(-2)) = (-2, 1)$.

(f) The vertical asymptotes partition the plane into three regions:

$$R_1 = \{x : x < -1\}, \quad R_2 = \{x : -1 < x < 2\}, \quad \text{and} \quad R_3 = \{x : x > 2\}.$$

We will want at least one point that lies on the graph of f in each region, and in region R_1 in particular we want points on either side of $(-2, 1)$ where the graph of f intersects the horizontal asymptote $y=1$. Calculating

$$f(-3) = \tfrac{9}{10}, \quad f\left(-\tfrac{3}{2}\right) = \tfrac{9}{7}, \quad f(3) = \tfrac{9}{4},$$

we obtain the points $\left(-3, \frac{9}{10}\right)$, $\left(-\frac{3}{2}, \frac{9}{7}\right)$, and $\left(3, \frac{9}{4}\right)$.

(g) We sketch the graph of f using the asymptotes and our few choice points as guides. In region R_1 we have $\left(-3, -\frac{9}{10}\right)$ lying below the horizontal asymptote $y=1$, so that we must have

$$f(x) \to 1^- \quad \text{as} \quad x \to -\infty.$$

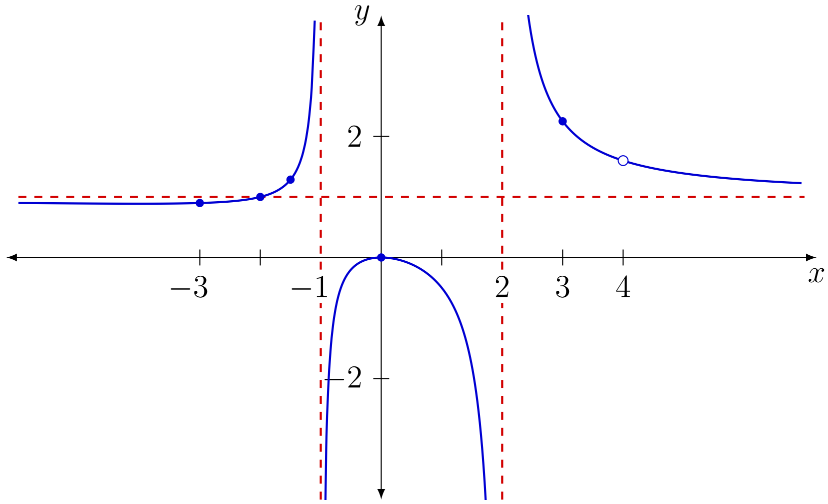

FIGURE 4

In contrast we have $\left(-\frac{3}{2}, \frac{9}{7}\right)$ lying above $y = 1$, so as we move to the right of this point we must have

$$f(x) \to \infty \quad \text{as} \quad x \to -1^-$$

in order to avoid crossing the asymptotes $y = 1$ and $x = -1$.

The situation in region R_2 is fairly simple: $(0, 0)$ lies on the graph of f, and since there are no other intercepts for f and the horizontal asymptote (which f does not intersect in R_2) lies above this point, it must be that the graph of f bends downward as we move to the left or right of $(0, 0)$. In fact we must have

$$f(x) \to -\infty \quad \text{as} \quad x \to -1^+ \quad \text{and} \quad x \to 2^-$$

to avoid intersecting $x = -1$ and $x = 2$.

Finally, in region R_3 we have $\left(4, \frac{9}{4}\right)$, which is above the horizontal asymptote $y = 1$ and to the right of the vertical asymptote $x = 2$. Thus the graph of f is bent upward as we move to the left of $\left(4, \frac{9}{4}\right)$ (to avoid crossing $x = 2$), and bends to avoid crossing $y = 1$ as we move to the right of $\left(4, \frac{9}{4}\right)$. That is, we have

$$f(x) \to \infty \quad \text{as} \quad x \to 2^+$$

and

$$f(x) \to 1^+ \quad \text{as} \quad x \to \infty$$

Finally, don't forget that there is a hole at $\left(4, \frac{8}{5}\right)$! See Figure 4. ■

Example 4.8. Let

$$f(x) = \frac{x^3 + 2x^2 - 3x}{x^2 - 25}.$$

(a) Find the domain of f.
(b) Find the intercepts of f.
(c) Find all vertical asymptotes of f.

(d) Find the horizontal or oblique asymptote of f.
(e) Find all points where f intersects its horizontal or oblique asymptote.
(f) Find additional points on the graph of f as needed.
(g) Sketch the graph of f.

Solution.
(a) We have

$$\text{Dom}(f) = \{x : x^2 - 25 \neq 0\} = \{x : x \neq \pm 5\} = (-\infty, -5) \cup (-5, 5) \cup (5, \infty)$$

as the domain for f.

(b) The x-intercepts of f are the points $(x, f(x))$ where $f(x) = 0$, from which we get

$$\frac{x^3 + 2x^2 - 3x}{x^2 - 25} = 0 \quad \Rightarrow \quad x^3 + 2x^2 - 3x = x(x-1)(x+3) = 0 \quad \Rightarrow \quad x = -3, 0, 1.$$

Thus $(-3, 0)$, $(0, 0)$, and $(1, 0)$ are x-intercepts. Since $(0, 0)$ is also a y-intercept of f and a function can never have more than one y-intercept, we have found all intercepts.

(c) We have

$$f(x) = \frac{x(x-1)(x+3)}{(x-5)(x+5)},$$

which is already in reduced form and so the vertical asymptotes are $x = -5$ and $x = 5$.

(d) Since the degree of the polynomial in the numerator of $f(x)$ is one greater than the degree of the polynomial in the denominator, there will be an oblique asymptote. Employing long division, we find that

$$f(x) = (x^3 + 2x^2 - 3x) \div (x^2 - 25) = x + 2 + \frac{22x + 50}{x^2 - 25}, \tag{4}$$

and so the oblique asymptote is the line $y = x + 2$.

(e) The graph of f intersects the oblique asymptote $y = x + 2$ if there is some $x \in \text{Dom}(f)$ for which $f(x) = x + 2$. Using the expression for $f(x)$ given in (4), we obtain the equation

$$x + 2 + \frac{22x + 50}{x^2 - 25} = x + 2,$$

whence

$$\frac{22x + 50}{x^2 - 25} = 0 \quad \Rightarrow \quad 22x + 50 = 0 \quad \Rightarrow \quad x = -\frac{25}{11}.$$

Thus the graph of f intersects $y = x + 2$ at $\left(-\frac{25}{11}, -\frac{25}{11} + 2\right) = \left(-\frac{25}{11}, -\frac{3}{11}\right)$.

(f) The vertical asymptotes partition the plane into three regions:

$$R_1 = \{x : x < -5\}, \quad R_2 = \{x : -5 < x < 5\}, \quad \text{and} \quad R_3 = \{x : x > 5\}.$$

We have plenty of points that lie on the graph of f in region R_2, so it remains to find at least one point in each of R_1 and R_3. In R_1 we have $\left(-7, -\frac{28}{3}\right)$, and in R_3 we have $\left(7, \frac{35}{2}\right)$.

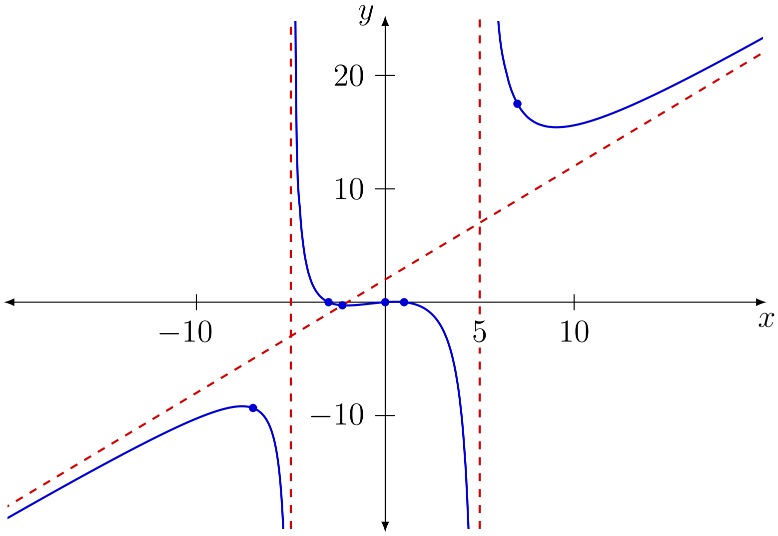

Figure 5

(g) Using the points and asymptotes we have in hand, we finally sketch the graph of f. See Figure 5. ■