MATH 122 EXAM #3 KEY (SPRING 2024)

1 $\sin\theta = y/r = -\frac{4}{5}$, so we let $y = -4$ and $r = 5$. Now, $\cos\theta < 0$ implies θ is in QII or QIII, where $x < 0$, and so from $x^2 + y^2 = r^2$ comes $x = -\sqrt{r^2 - y^2} = -\sqrt{9} = -3$. Finally: $\cos\theta = -\frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = -\frac{5}{3}$, $\cot\theta = \frac{3}{4}$.

2 There are different ways to do this. Using a couple Pythagorean identities,

$$\frac{\sec^2\theta - 1}{\csc^2\theta - 1} = \frac{(1+\tan^2\theta) - 1}{(1+\cot^2\theta) - 1} = \frac{\tan^2\theta}{\cot^2\theta} = \frac{\sin^2\theta/\cos^2\theta}{\cos^2\theta/\sin^2\theta} = \frac{\sin^4\theta}{\cos^4\theta}$$
$$= \frac{(1-\cos^2\theta)^2}{\cos^4\theta} = \frac{1 - 2\cos^2\theta + \cos^4\theta}{\cos^4\theta} = \sec^4\theta - 2\sec^2\theta + 1.$$

3 Again with a Pythagorean identity:

$$\frac{\sin^2\theta}{\cos\theta} = \frac{1-\cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta} - \cos\theta = \sec\theta - \cos\theta.$$

4 $\sin s = y/r = \frac{2}{3}$, so let $y = 2$ and $r = 3$. Then, with s in QII implying $x < 0$, we have $x = -\sqrt{r^2 - y^2} = -\sqrt{5}$, and so $\cos s = -\frac{\sqrt{5}}{3}$.

Next, $\sin t = y/r = -\frac{1}{3}$, so let $y = -1$ and $r = 3$. Then, with t in QIV implying $x > 0$, we have $x = +\sqrt{r^2 - y^2} = 2\sqrt{2}$, and so $\cos t = \frac{2\sqrt{2}}{3}$.

Finally,

$$\cos(s+t) = \cos s\cos t - \sin s\sin t = \left(-\frac{\sqrt{5}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right) = \frac{2 - 2\sqrt{10}}{9}.$$

5 With a half-angle identity,

$$\tan 15^\circ = \frac{1 - \cos 30^\circ}{\sin 30^\circ} = \frac{1 - \sqrt{3}/2}{1/2} = 2 - \sqrt{3}.$$

6 Working with the left-hand side (LHS), we have

$$\text{LHS} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\cos\alpha\sin\beta} = \frac{\cos\beta}{\sin\beta} + \frac{\sin\alpha}{\cos\alpha} = \cot\beta + \tan\alpha = \tan\alpha + \cot\beta.$$

7 $\sin\theta = y/r = -\frac{\sqrt{5}}{7}$, so let $y = -\sqrt{5}$ and $r = 7$. We have $x > 0$ since $\cos\theta > 0$, so $x = +\sqrt{r^2 - y^2} = \sqrt{44} = 2\sqrt{11}$. Thus $\cos\theta = \frac{2\sqrt{11}}{7}$, so that

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{\sqrt{5}}{7}\right)\left(\frac{2\sqrt{11}}{7}\right) = -\frac{4\sqrt{55}}{49} \quad \text{and} \quad \cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{39}{49}.$$

8 Using identities, we can get the expression written entirely in terms of $\cos x$:

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= (\cos^2 x - \sin^2 x)\cos x - 2\sin^2 x \cos x\\ &= (2\cos^2 x - 1)\cos x - 2(1 - \cos^2 x)\cos x\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

9 Starting with a reciprocal identity,

$$\cot^2(u/2) = \frac{1}{\tan^2(u/2)} = \frac{1}{\left(\dfrac{\sin u}{1+\cos u}\right)^2} = \frac{(1+\cos u)^2}{\sin^2 u}.$$

10 Let $\theta = \tan^{-1}(-2)$, so $\tan\theta = -2$ for $\theta \in (-\pi/2, \pi/2)$. Thus the terminal side of θ may be construed as containing the point $(x, y) = (1, -2)$, with $r = \sqrt{1^2 + 2^2} = \sqrt{5}$. Now,

$$\cos(\tan^{-1}(-2)) = \cos\theta = \frac{x}{r} = \frac{1}{\sqrt{5}}.$$

11 Factor: $(2\cos\theta - \sqrt{3})\cos\theta = 0$, so $\cos\theta = 0$ or $\cos\theta = \sqrt{3}/2$. From $\cos\theta = 0$ comes $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and from $\cos\theta = \sqrt{3}/2$ comes $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$. Solution set: $\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{11\pi}{6}\}$.

12 Write $\sin\theta = 2\sin\theta\cos\theta$, so $(2\cos\theta - 1)\sin\theta = 0$, and either $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. Solution set: $\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\}$.