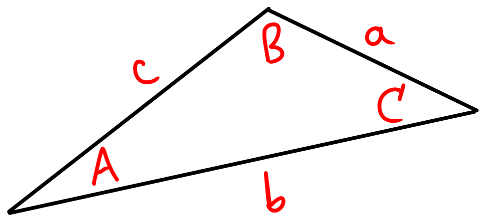
7.1 - Oblique Triangles & the Law of Sines

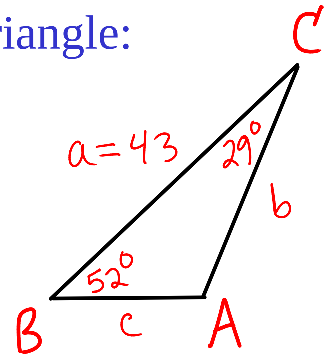
Law of Sines



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Given two angles and a side (the side-angle-angle or SAA case), or given two sides and an angle opposite one of the sides (the angle-side-angle or ASA case), the Law of Sines must be used to solve the triangle.

Ex Solve the triangle:



$A+B+C = 180° \Rightarrow A = 180° - B - C = 180° - 52° - 29° = 99°.$

We know A & a, so we use the Law of Sines with $\frac{\sin A}{a}$ in the equation. For instance:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 99°}{43} = \frac{\sin 52°}{b} \Rightarrow$$

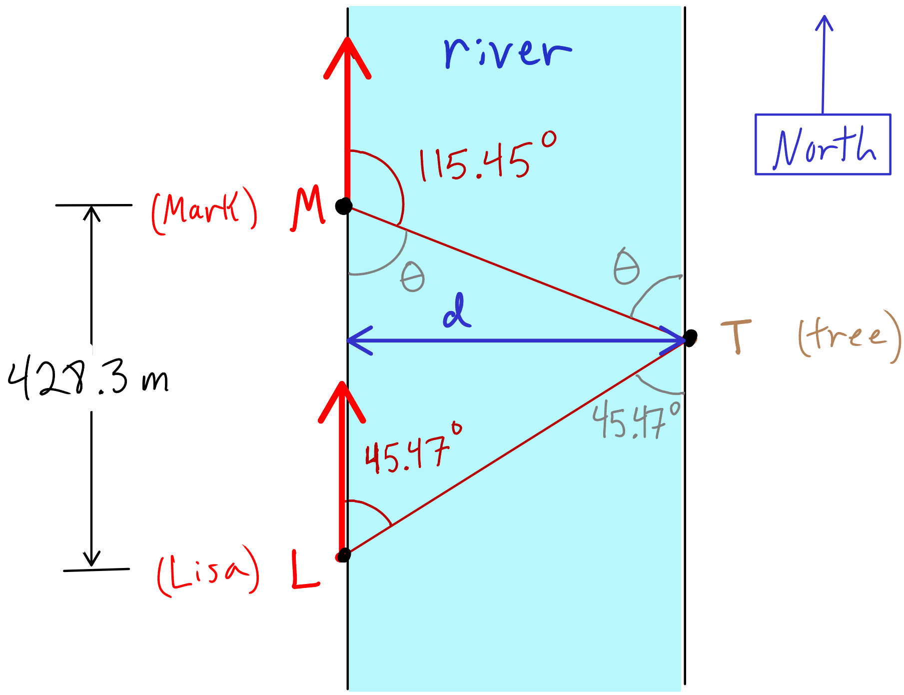
$$b = \frac{43 \sin 52°}{\sin 99°} = 34.307.$$

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow c = \frac{a \sin C}{\sin A} = \frac{43 \sin 29°}{\sin 99°} = 21.11.$$

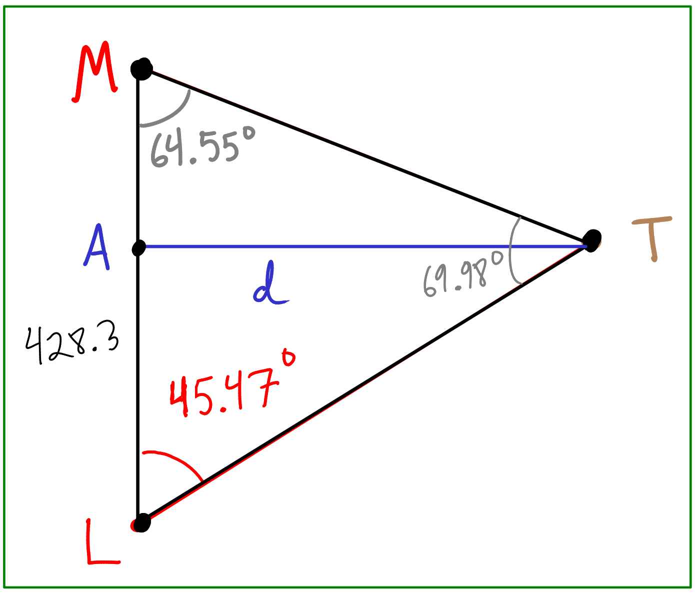
We round to 2 significant digits here: $\boxed{A = 99°,\ b = 34,\ c = 21.}$

$\boxed{38}$ Standing on one bank of a river flowing north, Mark notes a tree on the opposite bank at a bearing of 115.45°. Lisa is on the same bank as Mark, but 428.3 m away. She measures the tree's bearing as 45.47°. The two banks are parallel. What's the distance across the river?

- Recall: "bearing" as given here is defined to be the angle clockwise from due north. See section 2.5.



- Note that $\theta = 180° - 115.45° = 64.55°$
- Also the angle at T is $180° - \theta - 45.47° = 69.98°$

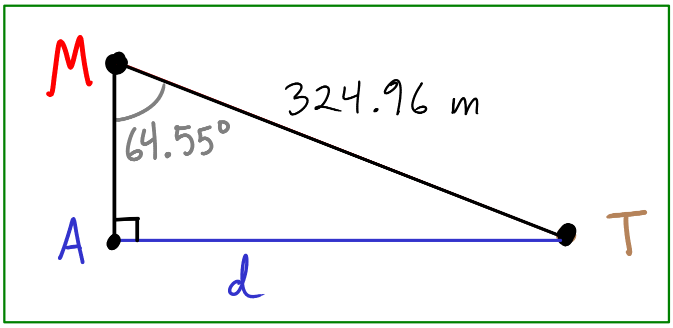


- We find, say, the length of side MT using the Law of Sines. Then we can work with the right triangle ΔMTA to find d, the distance across the river.

• We have: $\dfrac{\sin 45.47^\circ}{MT} = \dfrac{\sin 69.98^\circ}{428.3} \Rightarrow$

$$MT = \frac{428.3 \sin 45.47^\circ}{\sin 69.98^\circ} = 324.96 \text{ m}$$

The right triangle shown at right results. The width of the river is d, where



$\sin 64.55^\circ = \dfrac{d}{324.96} \Rightarrow$

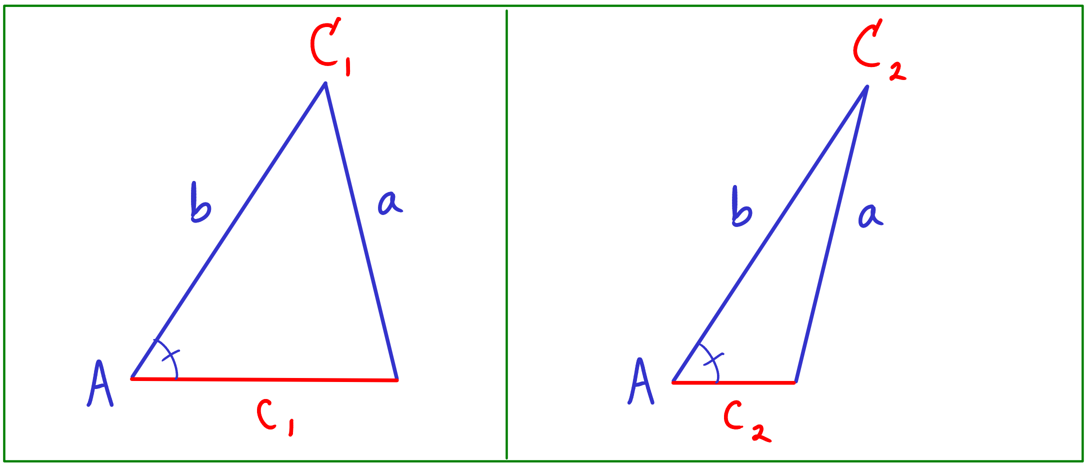
$$d = (324.96 \text{ m}) \sin 64.55^\circ = \boxed{293.4 \text{ m}}$$

7.2 - Ambiguous Case of the Law of Sines

The case when two sides of a triangle and an angle opposite one of the sides is the SSA (side-side-angle) case, also known as the ambiguous case.

This terminology is something of a misnomer. Better said: the SSA case is the only case wherein there is the potential for the given information to yield two possible triangles. But often the SSA case yields no triangle, or precisely one possible triangle — not "ambiguous" at all. Mathematicians are not renowned for their mastery of the language.

Below are depicted two triangles that have sides of length of length a and b, and an angle A opposite the side of length a.



$\boxed{\text{Ex}}$ Solve the triangle ΔABC, given that $C = 82.2^\circ$, $a = 10.9$, and $c = 7.62$.

- We find angle A: $\dfrac{\sin A}{a} = \dfrac{\sin C}{c} \Rightarrow$

$$\sin A = \frac{a \sin C}{c} = \frac{10.9 \sin 82.2^\circ}{7.62} \approx 1.417.$$

Since it's impossible to have $\sin A > 1$, there is no solution. $\square$

$\boxed{\text{Ex}}$ Solve the triangle $\triangle ABC$ given that $B = 113.72°$, $a = 189.6$, and $b = 243.8$.

- With the information at hand, we must first find angle A:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow$$

$$\sin A = \frac{a \sin B}{b} = \frac{189.6 \sin 113.72°}{243.8} \approx 0.71199$$

Certainly we may have $A = \sin^{-1} 0.71199 \approx 45.397°$

Recall that $\sin A = \sin(180° - A)$, so we should consider whether $A' = 180° - A = 180° - 45.397° = 134.603°$ is another possibility. But no: our triangle would have interior angles $A' = 134.603°$ & $B = 113.72°$, and so $A' + B > 180°$ — impossible! The interior angles of a triangle must add up to exactly 180°. So $A = 45.397°$ is the only possible solution, and we will get precisely one triangle out of our data.

- Next we have $C = 180° - A - B = 180° - 45.397° - 113.72°$, or $C = 20.883°$

- Finally, $\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow$

$$c = \frac{a \sin C}{\sin A} = \frac{189.6 \sin 20.883°}{\sin 45.397°} = 94.924.$$

- We have carried extra digits throughout our work to control roundoff error, but now it's time to give our final results with the proper significant digits observed:

$$\boxed{A = 45.40°, \quad C = 20.88°, \quad c = 94.92}$$

$\boxed{\text{Ex}}$ Solve the triangle $\triangle ABC$ given that $B = 48.2°$, $a = 890$, and $b = 697$.

- With the information at hand, we must first find angle A:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow$$

$$\sin A = \frac{a \sin B}{b} = \frac{890 \sin 48.2°}{697} \approx 0.9519.$$

One solution is $A_1 = \sin^{-1} 0.9519 \approx 72.16°$

Since $\sin\theta = \sin(180° - \theta)$, we consider the possibility that $A_2 = 180° - A_1 = 107.84°$ is another solution. Indeed, because $B + A_2 = 48.2° + 107.84° < 180°$, we find $A_2 = 107.84°$ to be a second possible value for angle A opposite side a! We will get two triangles.

- Case 1: $A = A_1 = 72.16°$. Then for C we get

$C_1 = 180° - A_1 - B = 59.64°$, and for c we get

$$\frac{\sin A_1}{a} = \frac{\sin C_1}{c_1} \Rightarrow$$

$$c_1 = \frac{a \sin C_1}{\sin A_1} = \frac{890 \sin 59.64°}{\sin 72.16°} = 806.7.$$

- Case 2: $A = A_2 = 107.84°$. Then for C we get

$C_2 = 180° - A_2 - B = 23.96°$, and for c we get

$$\frac{\sin A_2}{a} = \frac{\sin C_2}{c_2} \Rightarrow$$

$$c_2 = \frac{a \sin C_2}{\sin A_2} = \frac{890 \sin 23.96°}{\sin 107.84°} = 379.7.$$

- Rounding to the proper number of significant digits (3 digits in this case), we have:

Triangle 1: $A = A_1 = 72.2^\circ$

$C = C_1 = 59.6^\circ$

$c = c_1 = 807$

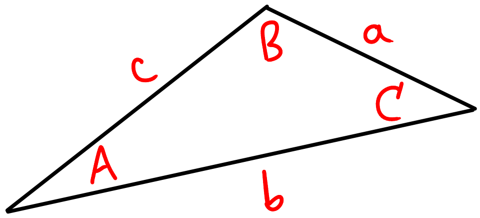
Triangle 2: $A = A_2 = 108^\circ$

$C = C_2 = 24.0^\circ$

$c = c_2 = 380$

7.3 - The Law of Cosines

Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Given a, b, c (the side-side-side or SSS case), or given c, A, b or a, C, b or a, B, c (the side-angle-side or SAS case), we must use the Law of Cosines to solve the triangle. The Law of Sines will lead nowhere.

$\boxed{\text{Ex}}$ Solve ΔABC, given $a = 324$ m, $b = 421$ m, $C = 298$ m.

- We could start with any of the three equations constituting the Law of Cosines. We'll pick the first one in order to find A:

$$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc} = \frac{324^2 - 421^2 - 298^2}{-2(421)(298)} \approx 0.6419.$$

So $A = \cos^{-1} 0.6419 = 50.06°$.

- We could now use the 2nd equation in the Law of Cosines to find B, or we could use the Law of Sines instead. Using the Law of Sines will be slightly less computationally intensive.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow$$

$$\sin B = \frac{b\sin A}{a} = \frac{421\sin 50.06°}{324} = 0.9963 \Rightarrow$$

$B = \sin^{-1} 0.9963 = 85.04°.$

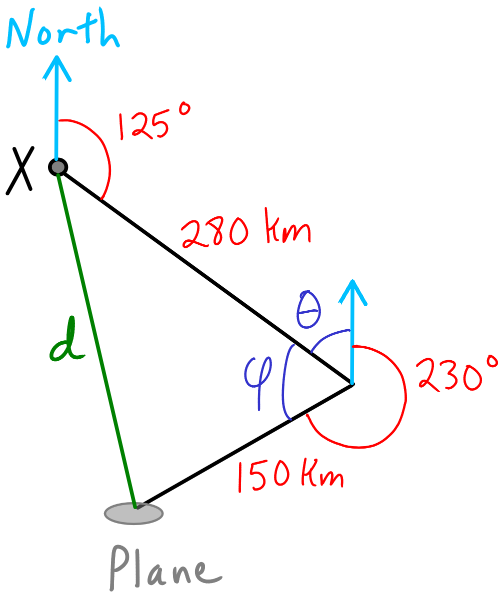
Finally, $C = 180° - A - B = 180° - 50.06° - 85.04° = 44.90°.$

Keeping 3 significant digits, we have:

$$\boxed{A = 50.0°\,,\quad B = 85.0°\,,\quad C = 44.9°}$$

Note: we do not cover Heron's Area Formula.

$\boxed{44}$ An airplane flies 280 km from point X at a bearing of 125°, and then turns and flies at a bearing of 230° for 150 km. How far is the plane from point X?



We have the triangle above. The angle θ is supplementary to 125 , since they are interior angles on the same side of a transversal (see page 129 of the textbook). Thus

$$\theta = 180° - 125° = 55°$$

Now we find φ. We have $\varphi + \theta + 230° = 360°$, and so

$$\varphi = 360° - 230° - \theta = 75°.$$

Knowing φ, we have the SAS case, and use the Law of Cosines to find the distance d between the plane & X.

$$d^2 = 150^2 + 280^2 - 2(150)(280)\cos 75^\circ$$

$$d^2 = 79{,}159.2$$

$$d = 281.4 \approx \boxed{281 \text{ km}}$$

8.1 - Complex Numbers

- We define i to be a number for which $i^2 = -1$, called the imaginary unit. Another symbol for i is $\sqrt{-1}$.

- An imaginary number is a number of the form bi, where b is real.

  Examples: i $(i = 1i)$, 0 $(0 = 0i)$, $-i$ $(-i = (-1)i)$, $\frac{1}{2}i$, $\sqrt{2}i$ (usually written $i\sqrt{2}$), πi, etc.

- A complex number has the standard form $a+bi$, where a & b are both real.

  Examples: $-2+4i$, $3+(-2)i$ (usually written $3-2i$), $5i$ $(5i = 0+5i)$, 8 $(8 = 8+0i)$.

Definition	For any $a>0$, $\sqrt{-a} = i\sqrt{a}$

So $\sqrt{-1} = i\sqrt{1} = i \cdot 1 = i$.

And $\sqrt{-36} = i\sqrt{36} = i \cdot 6 = 6i$.

(Note: $(6i)^2 = 6^2 i^2 = 36 \cdot (-1) = -36$ ✓)

Definition Let a, b, c, d be real numbers. Then:

1) $(a+bi) + (c+di) = (a+c) + (b+d)i$

2) $(a+bi) - (c+di) = (a-c) + (b-d)i$

3) $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

#3 is just FOIL: $(a+bi)(c+di) = ac + adi + bci + bdi^2$

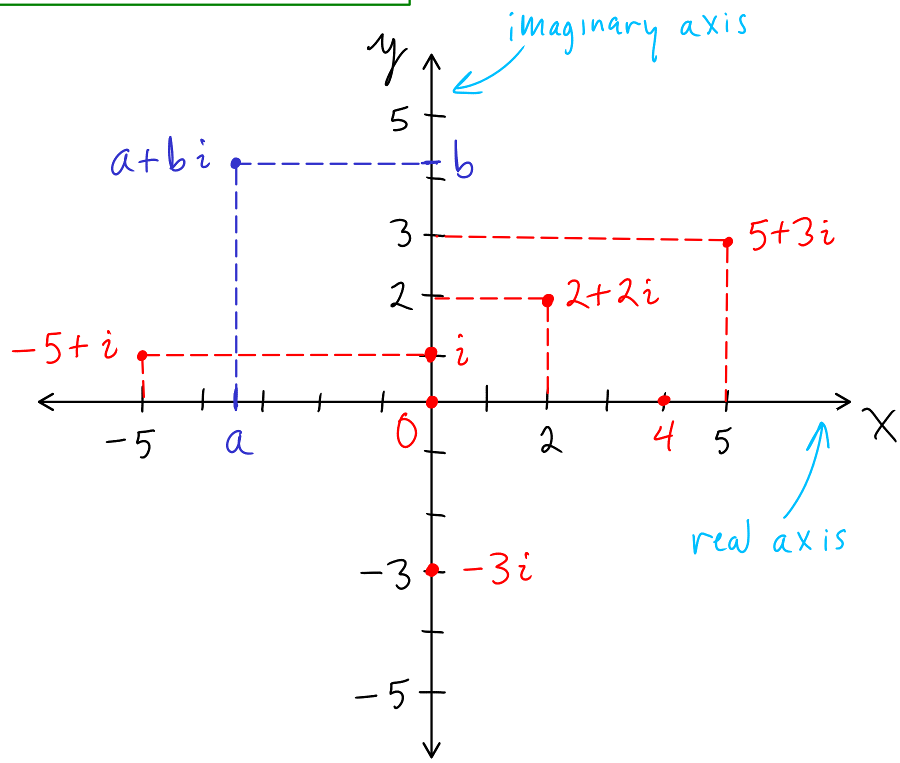
$$= ac + (ad+bc)i + bd(-1)$$

$$= (ac-bd) + (ad+bc)i$$

8.2 - Polar Form of Complex Numbers

Recall: the standard form for a complex number is a+bi, where a and b are real numbers. Thus a complex number is specified with two numbers: a real part a, and an imaginary part b. In this way a complex number a+bi corresponds to a point (a,b) in the usual rectangular coordinate system. In fact, the complex number system is seen to be a two-dimensional number system: analogous to the real number line there is the complex number plane. To locate complex numbers in this plane it is necessary to have two coordinate axes: a horizontal axis called the real axis, and vertical axis called the imaginary axis, each axis being a copy of the real number line. This is precisely the same setup as the rectangular system, only a point (a,b) is interpreted to represent the complex number a+bi.

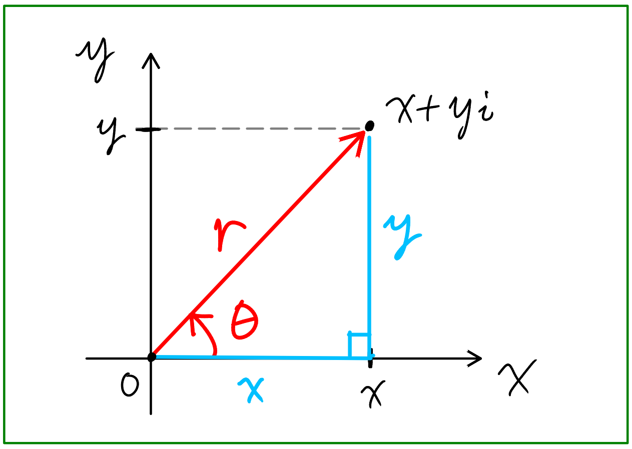
Complex Number Plane



Note that 0 = 0i = 0+0i, so 0 is the only number that is both real and imaginary. Generally we have 0+bi = bi and a+0i = a.

The standard form a+bi of a complex number is also known as the rectangular form.

Suppose a complex number with rectangular form x+yi lies in the complex plane a distance r from the origin, on the terminal side of an angle θ having initial side the positive real axis:



Here we have $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2+y^2}$, so that:

$$x+yi = (r\cos\theta)+(r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

We call $r(\cos\theta + i\sin\theta)$ the polar form of the complex number x+yi, with r the modulus (or absolute value) of x+yi, and θ the argument of x+yi. In this section we always choose θ to be a value in the interval [0°,360°) or [0,2π).

The symbol $\operatorname{cis}\theta$ is a shorthand for $\cos\theta + i\sin\theta$, and so

$$\boxed{x+yi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta}$$

Ex Write $5\operatorname{cis}300°$ in rectangular form.

$5\operatorname{cis}300° = 5(\cos 300° + i\sin 300°) = 5\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$, and so the proper rectangular form is $\boxed{\frac{5}{2} - \frac{5\sqrt{3}}{2}i}$

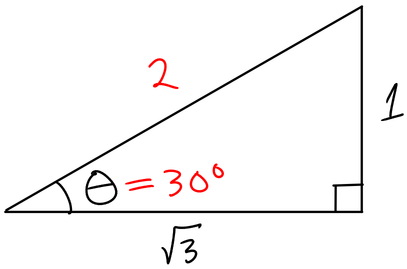
From the figure on the previous page we can see that

$$\tan\theta = \frac{y}{x},$$

which we use to convert a complex number from rectangular to polar form...

$\boxed{\text{Ex}}$ Write $4\sqrt{3}+4i$ in polar form $r(\cos\theta + i\sin\theta)$, with $\theta \in [0°, 360°)$.

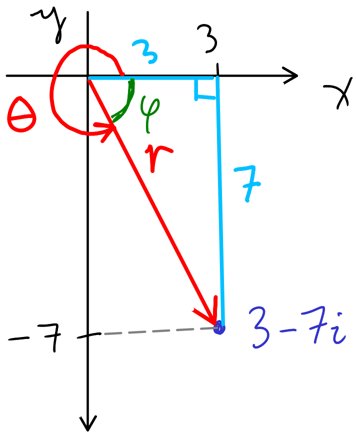
- Here $x+yi = 4\sqrt{3}+4i$, so $x = 4\sqrt{3}$ & $y=4$, and then $\tan\theta = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}}$. The triangle at right results, which we recognize as a 30-60 triangle with θ = 30°.



- Next, $r = \sqrt{x^2+y^2} = \sqrt{(4\sqrt{3})^2+4^2} = 8$.
- Finally, $4\sqrt{3}+4i = \boxed{8(\cos 30° + i\sin 30°)} = \boxed{8 \operatorname{cis} 30°}$ ■

$\boxed{\text{Ex}}$ Write $3-7i$ in polar form $r(\cos\theta + i\sin\theta)$, with $\theta \in [0°, 360°)$. Round to four decimal places, if necessary.

- Here $x+yi = 3-7i$, so $x=3$ & $y=-7$. Now,



$\tan\theta = \frac{y}{x} = -\frac{7}{3}$, with $0° \le \theta < 360°$.

$\tan\varphi = -\frac{7}{3} \Rightarrow \varphi = \tan^{-1}\left(-\frac{7}{3}\right) \approx -66.80141°$

So $\theta = 360° + \varphi \approx 293.19859°$

$r = \sqrt{3^2+7^2} = \sqrt{58} \approx 7.61577$

Polar form:

$$r(\cos\theta + i\sin\theta) = \boxed{7.6158\left(\cos 293.1986^\circ + i\sin 293.1986^\circ\right)}$$

8.3 - The Product & Quotient Theorems

Recalling the identities

$$\cos\alpha\cos\beta \mp \sin\alpha\sin\beta = \cos(\alpha \pm \beta)$$
$$\cos\alpha\sin\beta \pm \sin\alpha\cos\beta = \sin(\alpha \pm \beta),$$

we prove the following theorems.

Product Theorem $$(r_1 \operatorname{cis}\theta_1)(r_2 \operatorname{cis}\theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Proof:

$$\begin{aligned}
&(r_1 \operatorname{cis}\theta_1)(r_2 \operatorname{cis}\theta_2) \\
&\quad = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] \\
&\quad = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&\quad = r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\
&\quad = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\
&\quad = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\
&\quad = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \blacksquare
\end{aligned}$$

Quotient Theorem $$\frac{r_1 \operatorname{cis}\theta_1}{r_2 \operatorname{cis}\theta_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

Proof:

We just need to show that $\dfrac{\operatorname{cis}\theta_1}{\operatorname{cis}\theta_2} = \operatorname{cis}(\theta_1 - \theta_2)$. We have:

$$\frac{\operatorname{cis}\theta_1}{\operatorname{cis}\theta_2} = \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}$$

$$= \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$= \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)$

$= \text{cis}(\theta_1 - \theta_2)$ ■

Ex Find the product, and write the answer in rectangular form:

$$[8(\cos 300° + i \sin 300°)][5(\cos 120° + i \sin 120°)]$$

Using the Product Theorem, we get:

$[8(\cos 300° + i \sin 300°)][5(\cos 120° + i \sin 120°)]$

$= (8 \text{ cis } 300°)(5 \text{ cis } 120°)$

$= (8)(5) \text{ cis } (300° + 120°)$

$= 40 \text{ cis } (420°)$

$= 40(\cos 420° + i \sin 420°)$

$= 40(\cos 60° + i \sin 60°)$

$= 40\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right) = \boxed{20 + 20i\sqrt{3}} = \boxed{20 + 20\sqrt{3}\, i}$ ■

Both are fine forms

Ex Use a calculator to find the following, writing the answer in rectangular form, writing the real and imaginary parts to four decimal places:

$$(4 \text{ cis } 23.09°)(7 \text{ cis } 41.75°)$$

$(4 \text{ cis } 23.09°)(7 \text{ cis } 41.75°) = (4)(7) \text{ cis } (23.09° + 41.75°)$

$= 28 \text{ cis } (64.84°)$

$= 28[\cos(64.84°) + i \sin(64.84°)]$

$= 28(0.42514 + 0.90512i)$

$= \boxed{11.9039 + 25.3435i}$ ■

$\boxed{\text{Ex}}$ Find the quotient, and write the answer in rectangular form:

$$\frac{12(\cos 23^\circ + i\sin 23^\circ)}{6(\cos 293^\circ + i\sin 293^\circ)}$$

$$\frac{12\,\text{cis}\,23^\circ}{6\,\text{cis}\,293^\circ} = \frac{12}{6}\,\text{cis}(23^\circ - 293^\circ) = 2\,\text{cis}(-270^\circ) = 2\,\text{cis}(90^\circ)$$

$$= 2(\cos 90^\circ + i\sin 90^\circ) = 2(0+i) = \boxed{2i}$$

In general, the conjugate of a+ib is a-ib (and the conjugate of a-ib is a+ib). Multiplying a complex number by its conjugate always results in a real number: For real numbers a & b...

$$(a+bi)(a-bi) = a^2 - a\cancel{b}i + a\cancel{b}i - b^2i^2 \quad \text{(FOIL)}$$

$$= a^2 - b^2i^2$$

$$= a^2 - b^2(-1)$$

$$= a^2 + b^2$$

$\boxed{\text{Ex}}$ Write in rectangular form: $\dfrac{1}{2-2i}$

The easiest approach is to multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{1}{2-2i}\cdot\frac{2+2i}{2+2i} = \frac{2+2i}{4+4\cancel{i}-4\cancel{i}-4i^2} = \frac{2+2i}{4-4i^2} = \frac{2+2i}{4-4(-1)}$$

$$= \frac{2+2i}{8} = \frac{2}{8}+\frac{2}{8}i = \boxed{\frac{1}{4}+\frac{1}{4}i}$$

8.4 - De Moivre's Theorem; Powers & Roots of Complex Numbers

Pronunciation: De Moivre = "Deh MWAH-veh," approximately.

De Moivre's Theorem	For any real number t,

$$(r\,\mathrm{cis}\,\theta)^t = r^t \mathrm{cis}\,t\theta$$

$\boxed{\text{Ex}}$ Write $(2-2i\sqrt{3})^4$ in rectangular form.

- First write $2-2i\sqrt{3}$ in polar form $r(\cos\theta + i\sin\theta)$.
  - Find r. It is always the absolute value of a complex number in rectangular form:

    $$r = |x+iy| = \sqrt{x^2+y^2}.$$

    So here,

    $$r = |2-2i\sqrt{3}| = \sqrt{2^2+(-2\sqrt{3})^2} = \sqrt{16} = 4$$

    Thus $2-2i\sqrt{3} = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$.
  - Find θ. Given $x+iy$, $\tan\theta = \frac{y}{x}$ for $\theta \in [0°, 360°)$.

    Here $x = \frac{1}{2}$ & $y = -\frac{\sqrt{3}}{2}$, which puts (x,y) in Quadrant IV.

    $$\tan\theta = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3} \text{ for } \theta \in [270°, 360°), \text{ giving}$$

    $\theta = 300°$. (Note: we could just as well choose to have θ in radians)
  - Polar form: $4(\cos 300° + i\sin 300°)$

- Use De Moivre's Theorem to find the 4th power of the polar form, and convert back to rectangular form.

$$(2-2i\sqrt{3})^4 = [4(\cos 300° + i\sin 300°)]^4$$
$$= 4^4[\cos(4\cdot 300°) + i\sin(4\cdot 300°)]$$

$= 256(\cos 1200° + i\sin 1200°)$

$= 256(\cos 120° + i\sin 120°)$ 1200° & 120° are coterminal

$= 256\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \boxed{-128 + 128i\sqrt{3}}$

Definition Let n be a positive integer. The complex number a+bi is an nth root of x+yi if $(a+bi)^n = x+yi$.

nth Root Theorem Let n be a positive integer, r>0, and θ be in degrees. Then the complex number $r(\cos\theta + i\sin\theta)$ has precisely n distinct nth roots of the form

$$\sqrt[n]{r}(\cos\alpha + i\sin\alpha),$$

where

$$\alpha = \frac{\theta + 360° \cdot k}{n} \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1.$$

If θ is in radians, then

$$\alpha = \frac{\theta + 2\pi k}{n} \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1.$$

When using the theorem to find square roots we have n=2, and when finding cube roots we have n=3.

Ex Find all cube roots of $2 - 2i\sqrt{3}$ in polar form, and also in rectangular form to four decimal places.

- First write $2-2i\sqrt{3}$ in polar form $r(\cos\theta + i\sin\theta)$.

  In the previous example we found that

  $2-2i\sqrt{3} = 4(\cos 300° + i\sin 300°)$, so $\theta = 300°$

- By the nth Root Theorem there are precisely three cube roots of the form

$$\sqrt[3]{4}(\cos\alpha + i\sin\alpha),$$

where we have (since $\theta = 300°$ & $n=3$):

$$\alpha = \frac{300° + 360° \cdot k}{3} = 100° + 120° k \quad \text{for } k = 0, 1, 2.$$

Thus we have

$$\alpha = 100° + 120° \cdot 0,\ 100° + 120° \cdot 1,\ 100° + 120° \cdot 2 = 100°, 220°, 340°.$$

- The cube roots are therefore

$$\boxed{\begin{aligned} \sqrt[3]{4}\left(\cos 100° + i \sin 100°\right) &= -0.2756 + 1.5633i \\ \sqrt[3]{4}\left(\cos 220° + i \sin 220°\right) &= -1.2160 - 1.0204i \\ \sqrt[3]{4}\left(\cos 340° + i \sin 340°\right) &= 1.4917 - 0.5429i \end{aligned}}$$

$\boxed{\text{Ex}}$ Find all solutions to the equation $z^4 + 16 = 0$, both real and complex. Write answers in rectangular form.

- We write the equation as $z^4 = -16$.
- Thus z must be a 4th root of -16. By the nth Root Theorem there are 4 such roots.
- To use the nth Root Theorem we need -16 in polar form.
  - $-16 = x + yi$ for $x = -16$ & $y = 0$.
  - $r = \sqrt{x^2 + y^2} = \sqrt{(-16)^2 + 0^2} = 16$.
  - $\tan\theta = \frac{y}{x} = \frac{0}{-16} = 0$ for $\theta \in [0°, 360°)$, and so $\theta = 180°$.
  - Finally, $-16 = 16(\cos 180° + i \sin 180°)$.
- The 4th roots of -16 are given by

$$\sqrt[4]{16}\left(\cos\alpha + i \sin\alpha\right) = 2\left(\cos\alpha + i\sin\alpha\right),$$

where

$$\alpha = \frac{\theta + 360° \cdot k}{n} = \frac{180° + 360° \cdot k}{4} = 45° + 90° k \quad \text{for } k = 0, 1, 2, 3.$$

So we find that

$$\alpha = 45^\circ + 90^\circ\cdot 0,\ 45^\circ+90^\circ\cdot 1,\ 45^\circ+90^\circ\cdot 2,\ 45^\circ+90^\circ\cdot 3$$
$$= 45^\circ,\ 135^\circ,\ 225^\circ,\ 315^\circ.$$

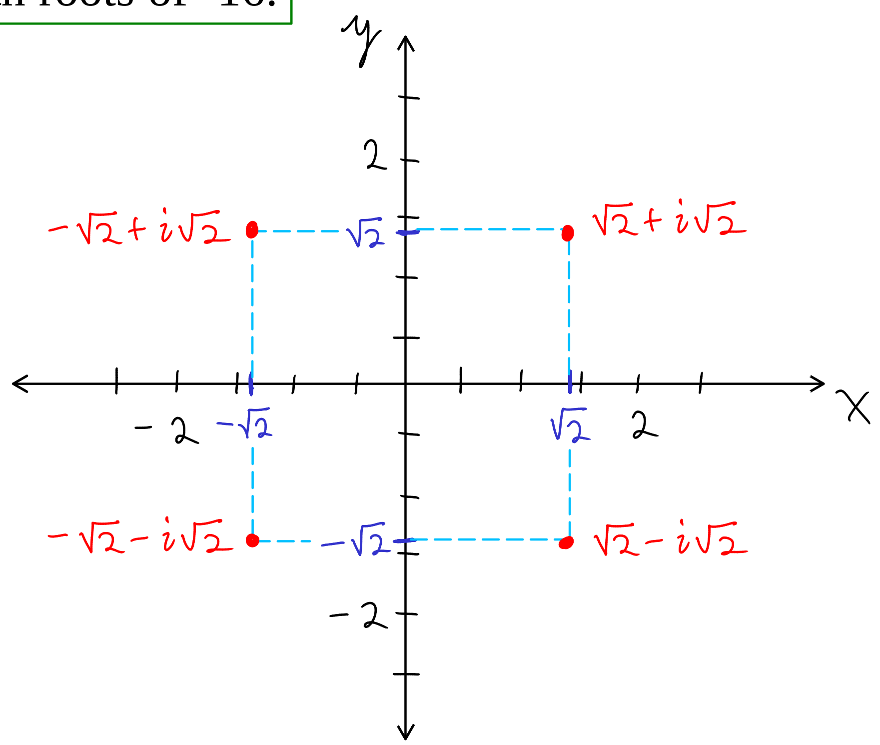
- The 4th roots are therefore

$$2(\cos 45^\circ + i\sin 45^\circ) = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2} + i\sqrt{2}$$
$$2(\cos 135^\circ + i\sin 135^\circ) = 2\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -\sqrt{2} + i\sqrt{2}$$
$$2(\cos 225^\circ + i\sin 225^\circ) = 2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -\sqrt{2} - i\sqrt{2}$$
$$2(\cos 315^\circ + i\sin 315^\circ) = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2} - i\sqrt{2}$$

- The solution set to the equation is:

$$\boxed{\left\{\sqrt{2}+i\sqrt{2},\ -\sqrt{2}+i\sqrt{2},\ -\sqrt{2}-i\sqrt{2},\ \sqrt{2}-i\sqrt{2}\right\}}$$

The 4th roots of -16:



Note: In the assignment for section 8.4, exercises #19 - 30, disregard part (b). Just do part (a).