5.1 - Fundamental Identities

An equation with one variable is an identity if it is true for all values of the variable for which the two sides of the equation are defined.

$\boxed{\text{Ex}}$ The equation $\frac{x}{x} = 1$ is true for all $x \neq 0$. Since the left side is undefined when $x = 0$, the equation is an identity. ▫

$\boxed{\text{Ex}}$ The equation $\csc\theta = \frac{1}{\sin\theta}$ is true for all $\theta \neq n\pi$, where n is any integer. Since either side of the equation is undefined when $\theta = n\pi$, the equation is an identity. ▫

Below are listed the fundamental identities, all of which we have seen already save for the even-odd identities.

FUNDAMENTAL IDENTITIES

Reciprocal Identities:

$$\cot\theta = \frac{1}{\tan\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}$$

Quotient Identities:

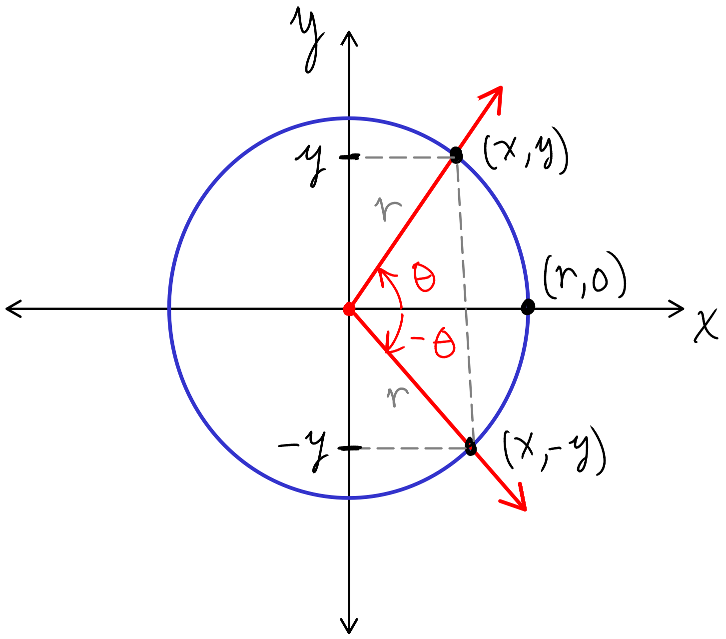
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad \cot^2\theta + 1 = \csc^2\theta$$

Even-Odd Identities:

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta, \quad \sec(-\theta) = \sec\theta, \quad \cot(-\theta) = -\cot\theta$$



- $\sin\theta = \frac{y}{r}$ & $\sin(-\theta) = \frac{-y}{r}$,

  So $\sin(-\theta) = -\frac{y}{r} = -\sin\theta$

- $\cos\theta = \frac{x}{r} = \cos(-\theta)$

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$\boxed{18}$ Find $\sin\theta$, given that $\tan\theta = -\frac{\sqrt{7}}{2}$ & $\sec\theta > 0$.

Method I: Using definitions of trigonometric functions

- $\sec\theta > 0$ in quadrants I & IV $\left(\sec\theta = \frac{r}{x}\right.$, so need $\left.x > 0\right)$
- $\tan\theta < 0$ in quadrants II & IV
- So the terminal side of θ is in quadrant IV, where $x > 0$ & $y < 0$.
- $\tan\theta = \frac{y}{x} = -\frac{\sqrt{7}}{2} \rightarrow$ We can let $\underbrace{y = -\sqrt{7}}_{y<0}$ & $\underbrace{x = 2}_{x>0}$
- So $r = \sqrt{x^2+y^2} = \sqrt{2^2+(-\sqrt{7})^2} = \sqrt{4+7} = \sqrt{11}$
- $\sin\theta = \frac{y}{r} = \boxed{\frac{-\sqrt{7}}{\sqrt{11}}} = \boxed{\frac{-\sqrt{77}}{11}}$

Method II: Using fundamental identities

- The only fundamental identity that features tan and sin is $\tan\theta = \sin\theta / \cos\theta$.
- Solve for $\sin\theta$: $\sin\theta = \cos\theta \tan\theta$, $(*)$

• We're given $\sec\theta > 0$, so get $\sec\theta$ in the equation (*):

$$\sin\theta = \frac{\tan\theta}{\sec\theta}, \quad (\star)$$

• Now get $\sec\theta$ expressed in terms of $\tan\theta$:

$$\sec^2\theta = 1+\tan^2\theta \Rightarrow \sqrt{\sec^2\theta} = \sqrt{1+\tan^2\theta}$$

$$\Rightarrow |\sec\theta| = \sqrt{1+\tan^2\theta} \quad \left(\sqrt{x^2} = |x|\right)$$

$$\Rightarrow \sec\theta = \sqrt{1+\tan^2\theta} \quad (\sec\theta > 0)$$

• Now $(\star)$ becomes:

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{-\sqrt{7}/2}{\sqrt{1+(-\sqrt{7}/2)^2}} = \frac{-\sqrt{7}/2}{\sqrt{1+(-\sqrt{7}/2)^2}} = \boxed{\frac{-\sqrt{77}}{11}}$$

Either method (I or II) is fine on an exam.

$\boxed{64}$ First write the expression below in terms of sine and cosine, and then simplify so that no quotients (i.e. fractions) are present:

$$\frac{1+\cot\theta}{\cot\theta}.$$

$$\frac{1+\cot\theta}{\cot\theta}. = \frac{1+\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\cos\theta}{\sin\theta}} \cdot \frac{\sin\theta}{\sin\theta} = \frac{\sin\theta+\cos\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} + 1 = \boxed{\tan\theta + 1}$$

5.2 - Verifying Trigonometric Identities

To verify a trigonometric identity, DO NOT add a term to both sides of the proposed identity, and DO NOT multiply both sides of the proposed identity by any factor. There should be no moving of quantities across the equal sign!

The reason is that such operations may lead to the conclusion that an equation is an identity when it is in fact not.

The most egregious example: multiplying both sides of an equation by 0 to get the identity 0=0, and so concluding that the original equation is likewise an identity.

$$2x+1 = 3x \Rightarrow 0\cdot(2x+1) = 0\cdot(3x) \Rightarrow 0 = 0$$

But $2x+1 = 3x$ is not an identity since it is only a true statement when $x = 1$. For other values of x the two sides of the equation are defined as real numbers, but the numbers are not equal.

$\boxed{46}$ Verify the identity: $\dfrac{\tan\alpha}{\sec\alpha} = \sin\alpha$

- Use only the fundamental identities in the table in section 5.1 to do these problems!
- We usually start with the more complicated side (here the left side) and simplify it.
- To simplify the left side, we'll get everything in terms of sin and cos...

$$\frac{\frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\alpha}} = \sin\alpha \Rightarrow \frac{\frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\alpha}}\cdot\frac{\cos\alpha}{\cos\alpha} = \sin\alpha \Rightarrow$$

$$\frac{\frac{\sin\alpha}{1}}{\frac{1}{1}} = \sin\alpha \Rightarrow \boxed{\sin\alpha = \sin\alpha}$$

- We're done once both sides look identical. That is, once both sides look identical, the identity is verified.

$\boxed{50}$ Verify $\sin^2\beta\,\underbrace{(1+\cot^2\beta)} = 1$

Can use $\cot^2\beta + 1 = \csc^2\beta \ldots$

$$\sin^2\beta\,\underline{\csc^2\beta} = 1 \Rightarrow \sin^2\beta\left(\frac{1}{\sin\beta}\right)^2 = 1 \Rightarrow \sin^2\beta\cdot\frac{1}{\sin^2\beta} = 1 \Rightarrow$$

$$\boxed{1=1}$$

Alternate Method:

$$\sin^2\beta\cdot\left(1+\frac{\cos^2\beta}{\sin^2\beta}\right) = 1 \Rightarrow \sin^2\beta + \sin^2\beta\cdot\frac{\cos^2\beta}{\sin^2\beta} = 1 \Rightarrow$$

$$\sin^2\beta + \cos^2\beta = 1 \Rightarrow \boxed{1=1}$$

$\boxed{62}$ Verify $\dfrac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$

- LHS is more complicated, so we'll work with it...

- We use the "multiply by the conjugate of the denominator" strategy. Generally the conjugate of A-B is A+B, and the conjugate of A+B is A-B.

$$\frac{1}{\sec\alpha - \tan\alpha}\cdot\frac{\sec\alpha + \tan\alpha}{\sec\alpha + \tan\alpha} = \sec\alpha + \tan\alpha$$

$$\frac{\sec\alpha + \tan\alpha}{\sec^2\alpha - \tan^2\alpha} = \sec\alpha + \tan\alpha$$

$(A-B)(A+B) = A^2 - B^2$. Now use $\tan^2\alpha + 1 = \sec^2\alpha \ldots$

$$\frac{\sec\alpha+\tan\alpha}{(\tan^2\alpha+1)-\tan^2\alpha}=\sec\alpha+\tan\alpha$$

$$\frac{\sec\alpha+\tan\alpha}{1}=\sec\alpha+\tan\alpha$$

$$\boxed{\sec\alpha+\tan\alpha=\sec\alpha+\tan\alpha}$$

5.3/5.4 - Sum & Difference Identities

SUM AND DIFFERENCE IDENTITIES

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

All of the sum and difference identities are derived in the textbook. They will be provided on future exams, and so do not need to be memorized. The angles α and β may be any real numbers, and so using the difference identities we can show that the cofunction identities of section 2.1 are valid for any angle, and not just acute angles.

Letting $\alpha = 90^\circ$ and $\beta = \theta$ in $\cos(\alpha - \beta)$, the first difference identity displayed above gives:

$$\cos(90^\circ - \theta) = \cos 90^\circ \cos\theta + \sin 90^\circ \sin\theta$$
$$= 0 \cdot \cos\theta + 1 \cdot \sin\theta$$
$$= \sin\theta$$

That is, for any angle θ, we have $\cos(90^\circ - \theta) = \sin\theta$.

In radians, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ for any real number θ

Cofunction Identities (radian versions — θ any real #):

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$, $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

The radian versions of the cofunction identities have 90° in place of $\frac{\pi}{2}$.

Some section 5.3 exercises...

$\boxed{10}$ Find the exact value of $\cos(-15^\circ)$ without a calculator.

Cosine is even, so $\cos(-15^\circ) = \cos(15^\circ)$. Now,

$$\cos(15^\circ) = \cos(45^\circ - 30^\circ)$$
$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$ ■

$\boxed{38}$ Find one value of θ satisfying $\sin\theta = \cos(2\theta + 30^\circ)$

- Use $\sin\theta = \cos(90^\circ - \theta)$ to get $\cos(90^\circ - \theta) = \cos(2\theta + 30^\circ)$
- We could have: $90^\circ - \theta = 2\theta + 30^\circ$

$$-3\theta = -60^\circ$$
$$\boxed{\theta = 20^\circ}$$ ■

$\boxed{50}$ Write $\cos(270^\circ + \theta)$ as a trigonometric function of θ alone.

- Using $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we get:

$$\cos(270^\circ + \theta) = \cos 270^\circ \cos\theta - \sin 270^\circ \sin\theta$$
$$= 0 \cdot \cos\theta - (-1)\sin\theta$$
$$= \boxed{\sin\theta}$$ ■

$\boxed{56}$ Find $\cos(s+t)$ & $\cos(s-t)$, given that $\cos s = \frac{\sqrt{2}}{4}$, $\sin t = -\frac{\sqrt{5}}{6}$, and s and t are both in quadrant IV.

$$\cos(s+t) = \cos s \underbrace{\cos t}_{\text{unknown}} - \underbrace{\sin s}_{\text{unknown}} \sin t$$

To find the unknowns, we use $\sin^2\theta + \cos^2\theta = 1$.

- So: $\sin^2 s + \cos^2 s = 1 \Rightarrow \sin^2 s = 1 - \cos^2 s \Rightarrow$

$$\sin s = -\sqrt{1-\cos^2 s} = -\sqrt{1-(\sqrt{2}/4)^2} = -\sqrt{1-\frac{2}{16}} = -\sqrt{\frac{7}{8}},$$

since $\sin s < 0$ for s in quadrant IV

and hence $\sin s = -\frac{\sqrt{7}}{2\sqrt{2}} = -\frac{\sqrt{14}}{4}$.

- Next, $\cos^2 t = 1 - \sin^2 t \Rightarrow \cos t = +\sqrt{1-\sin^2 t} \Rightarrow$

since $\cos t > 0$ for t in quadrant IV

$$\cos t = \sqrt{1-\left(-\sqrt{5}/6\right)^2} = \sqrt{1-\frac{5}{36}} = \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{6}$$

- Finally, $\cos(s+t) = \cos s \cos t - \sin s \sin t$

$$= \frac{\sqrt{2}}{4}\cdot\frac{\sqrt{31}}{6} - \left(-\frac{\sqrt{14}}{4}\right)\left(-\frac{\sqrt{5}}{6}\right)$$

$$= \frac{\sqrt{62}}{24} - \frac{\sqrt{70}}{24} = \boxed{\frac{\sqrt{62}-\sqrt{70}}{24}}$$

Some section 5.4 exercises...

$\boxed{5.4.26}$ Find the exact value of $\sin 40^\circ \cos 50^\circ + \cos 40^\circ \sin 50^\circ$.

- Use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ with $\alpha = 40^\circ$ & $\beta = 50^\circ$:

$\sin 40^\circ \cos 50^\circ + \cos 40^\circ \sin 50^\circ = \sin(40^\circ + 50^\circ) = \sin 90^\circ = \boxed{1}$

$\boxed{5.4.40}$ Write as an expression involving trigonometric functions of x alone: $\tan\left(\frac{\pi}{4}+x\right)$.

• Use $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ with $\alpha = \frac{\pi}{4}$ & $\beta = x$:

$$\tan\left(\frac{\pi}{4}+x\right) = \frac{\tan\frac{\pi}{4} + \tan x}{1-\tan\frac{\pi}{4}\tan x} = \boxed{\frac{1+\tan x}{1-\tan x}}$$

$\boxed{5.4.54}$ Given that $\cos s = -\frac{15}{17}$, $\sin t = \frac{4}{5}$, s is in quadrant II, and t is in quadrant I, find:

a) $\sin(s+t)$

b) $\tan(s+t)$

c) The quadrant $s+t$ is in.

a $\sin(s+t) = \sin s \cos t + \cos s \sin t$, so we need to find $\sin s$ & $\cos t$.

$\sin s > 0$ for s in QII

$$\sin s = +\sqrt{1-\cos^2 s} = \sqrt{1-\left(-\frac{15}{17}\right)^2} = \sqrt{\frac{64}{289}} = \frac{8}{17}$$

$\cos t > 0$ for t in QI

$$\cos t = +\sqrt{1-\sin^2 t} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

Finally: $\sin(s+t) = \sin s \cos t + \cos s \sin t$

$$= \frac{8}{17}\cdot\frac{3}{5} + \left(-\frac{15}{17}\right)\left(\frac{4}{5}\right) = \boxed{-\frac{36}{85}}$$

b
$$\tan(s+t) = \frac{\tan s + \tan t}{1-\tan s\tan t} = \frac{\frac{\sin s}{\cos s} + \frac{\sin t}{\cos t}}{1-\frac{\sin s}{\cos s}\cdot\frac{\sin t}{\cos t}} = \frac{\frac{8/17}{-15/17} + \frac{4/5}{3/5}}{1-\frac{8/17}{-15/17}\cdot\frac{4/5}{3/5}}$$

$$= \frac{-8/15 + 4/3}{1-\left(-8/15\right)\left(4/3\right)}\cdot\frac{45}{45} = \frac{-24+60}{45-(-32)} = \boxed{\frac{36}{77}}$$

c We found that $\sin(s+t) < 0$, so $s+t$ is in QIII or QIV

Also we have $\tan(s+t) > 0$, so $s+t$ is in QIII or QI

So $s+t$ must be in $\boxed{\text{QIII}}$.

5.4.66 Verify the identity $\dfrac{\sin(s+t)}{\cos s \cos t} = \tan s + \tan t$

$$\frac{\sin(s+t)}{\cos s \cos t} = \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t}$$

$$= \frac{\sin s \cancel{\cos t}}{\cos s \cancel{\cos t}} + \frac{\cancel{\cos s} \sin t}{\cancel{\cos s} \cos t}$$

$$= \frac{\sin s}{\cos s} + \frac{\sin t}{\cos t}$$

$$= \boxed{\tan s + \tan t}$$

■

5.5 - Double-Angle Identities

DOUBLE-ANGLE IDENTITIES

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

The double-angle identities derive readily from the sum identities of the previous two sections. Using the sum identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

with $\alpha = \beta = \theta$, we get

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta,$$

which is the first double-angle identity listed above. Using

$$\sin^2\theta + \cos^2\theta = 1$$

yields variants of this first double-angle identity:

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta)\end{aligned}$$

$$\boxed{\cos 2\theta = 2\cos^2\theta - 1}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= (1-\sin^2\theta) - \sin^2\theta\end{aligned}$$

$$\boxed{\cos 2\theta = 1 - 2\sin^2\theta}$$

The product-to-sum and sum-to-product identities the textbook presents in this section we will not make use of.

$\boxed{12}$ Find $\sin 2\theta$ & $\cos 2\theta$, given $\cos\theta = \frac{\sqrt{3}}{5}$ & $\sin\theta > 0$.

- $\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{\sqrt{3}}{5}\right)^2 - 1 = 2\left(\frac{3}{25}\right) - 1 = \boxed{-\frac{19}{25}}$

- Next, $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta \Rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$, and so $\sin\theta = \sqrt{1-\cos^2\theta}$ since $\sin\theta > 0$ is given.

- $\sin 2\theta = 2\sin\theta\cos\theta = 2\cos\theta\sqrt{1-\cos^2\theta} = 2\left(\frac{\sqrt{3}}{5}\right)\sqrt{1-\left(\frac{\sqrt{3}}{5}\right)^2}$

$$= \frac{2\sqrt{3}}{5}\sqrt{\frac{22}{25}} = \frac{2\sqrt{3}}{5}\cdot\frac{\sqrt{22}}{5} = \boxed{\frac{2\sqrt{66}}{25}} \quad \blacksquare$$

$\boxed{22}$ Verify the identity $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

$$\frac{2\tan x}{1+\tan^2 x} = \frac{2\tan x}{\sec^2 x} = 2\cdot\frac{\sin x}{\cancel{\cos x}}\cdot\frac{\cos^{\cancel{2}} x}{1} = 2\sin x\cos x = \sin 2x \quad \blacksquare$$

$\boxed{38}$ Find the exact value of $\dfrac{2\tan 15^\circ}{1-\tan^2 15^\circ}$

We get $\dfrac{2\tan 15^\circ}{1-\tan^2 15^\circ} = \tan(2\cdot 15^\circ) = \tan 30^\circ = \boxed{\frac{1}{\sqrt{3}}} = \boxed{\frac{\sqrt{3}}{3}} \quad \blacksquare$

5.6 - Half-Angle Identities

HALF-ANGLE IDENTITIES

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

The half-angle identities can be derived from the double-angle identities of section 5.5. Starting with

$$\cos 2x = 2\cos^2 x - 1,$$

we let $x = \frac{\theta}{2}$ and solve for $\cos\frac{\theta}{2}$:

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}, \qquad (1)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

Next, let $x = \frac{\theta}{2}$ in $\cos 2x = 1 - 2\sin^2 x$ and solve for $\sin\frac{\theta}{2}$:

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}, \qquad (2)$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

With (1) and (2) on the previous page we can derive a half-angle identity for the tangent function:

$$\tan^2\frac{\theta}{2} = \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{\frac{1+\cos\theta}{2}}{\frac{1-\cos\theta}{2}} = \frac{1+\cos\theta}{1-\cos\theta} \Rightarrow$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\,, \qquad (3)$$

An alternative identity without the plus/minus symbol may be derived for the tangent function:

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \cdot \frac{2\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$

$\longleftarrow$ Use $\sin 2x = 2\sin x\cos x$ with $x = \frac{\theta}{2}$.

$\longleftarrow$ Use $\cos 2x = 2\cos^2 x - 1$, or $2\cos^2 x = 1 + \cos 2x$, with $x = \frac{\theta}{2}$.

The cost of eliminating the +/- symbol is that now we must know the value of $\sin\theta$ as well as $\cos\theta$, whereas (3) only requires knowing $\cos\theta$.

$\boxed{24}$ Find $\cos\frac{x}{2}$, given $\cot x = -3$ & $\frac{\pi}{2} < x < \pi$.

We have $\frac{\pi}{2} < x < \pi \Rightarrow \frac{1}{2}\cdot\frac{\pi}{2} < \frac{1}{2}\cdot x < \frac{1}{2}\cdot\pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$,

so $\frac{x}{2}$ is in Quadrant I (so $\cos\frac{x}{2} > 0$). Now...

$$\cos\frac{x}{2} = +\sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1-\frac{\sqrt{99}}{10}}{2}} = \boxed{\sqrt{\frac{10-\sqrt{99}}{20}}}$$

$\frac{x}{2}$ is in QI

$\cot^2 x + 1 = \csc^2 x \Rightarrow \csc^2 x = 1+(-3)^2 = 10 \Rightarrow$

$\sin^2 x = \frac{1}{10}$, and so $\cos^2 x = 1-\sin^2 x = 1-\left(\frac{1}{10}\right)^2 = \frac{99}{100}$.

$\cos x = -\sqrt{\frac{99}{100}} = -\frac{\sqrt{99}}{10}$.

x is in QII

$\boxed{40}$ Simplify $\pm\sqrt{\frac{1+\cos 20\alpha}{2}}$.

We have $\pm\sqrt{\frac{1+\cos 20\alpha}{2}} = \cos\left(\frac{20\alpha}{2}\right) = \boxed{\cos 10\alpha}$

$\boxed{50}$ Verify $\tan\frac{\theta}{2} = \csc\theta - \cot\theta$.

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$$

6.1 - Inverse Circular Functions

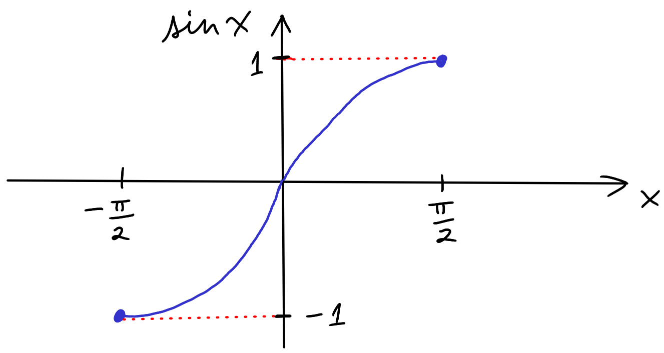
A function f is one-to-one if, for all x_1 and x_2 in the domain of f, we have $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

If f is one-to-one, then it has an inverse denoted by f^{-1}. Note: since f does not represent a number, the -1 in the symbol f^{-1} does not represent an exponent. Read f^{-1} as "f inverse." The inverse of f operates as follows: for all x in the domain and y in the range of f,

$$\boxed{f(x) = y \text{ if and only if } f^{-1}(y) = x}$$

Being periodic, none of the trigonometric (a.k.a. circular) functions are one-to-one. However, we may always restrict the domain of any function that is not one-to-one over some interval on which it is one-to-one. This allows for a kind of "local" inverse of the function.

$\boxed{\text{Ex}}$ The sine function is not one-to-one, since $\sin 0 = \sin \pi = 0$, yet $0 \neq \pi$. But if we restrict the natural domain $(-\infty, \infty)$ of sine to, say, $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $y = \sin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ is one-to-one (see the figure below) and so has an inverse.



The conventions for forming "local" inverses for the six trigonometric functions are given below, and should be memorized.

Definitions of the Inverse Trigonometric Functions

$y = \cos^{-1} x \Leftrightarrow \cos y = x$ for $x \in [-1, 1]$ and $y \in [0, \pi]$.

$y = \sin^{-1} x \Leftrightarrow \sin y = x$ for $x \in [-1, 1]$ and $y \in [-\pi/2, \pi/2]$.

$y = \tan^{-1} x \Leftrightarrow \tan y = x$ for $x \in (-\infty, \infty)$ and $y \in (-\pi/2, \pi/2)$.

$y = \sec^{-1} x \Leftrightarrow \sec y = x$ for $x \in (-\infty, -1] \cup [1, \infty)$ and $y \in [0, \pi/2) \cup (\pi/2, \pi]$.

$y = \csc^{-1} x \Leftrightarrow \csc y = x$ for $x \in (-\infty, -1] \cup [1, \infty)$ and $y \in [-\pi/2, 0) \cup (0, \pi/2]$.

$y = \cot^{-1} x \Leftrightarrow \cot y = x$ for $x \in (-\infty, \infty)$ and $y \in (0, \pi)$.

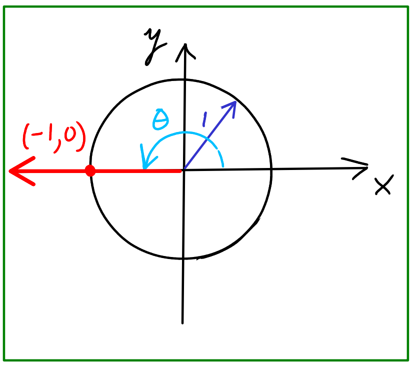
Highlighted in yellow are the domains that we restrict each of the six trigonometric functions in order to obtain a one-to-one function.

Alternate symbols for the inverse trigonometric functions are:

$$\boxed{\begin{array}{l} \cos^{-1} = \text{arccos}, \ \sin^{-1} = \text{arcsin}, \ \tan^{-1} = \text{arctan}, \\ \sec^{-1} = \text{arcsec}, \ \cos^{-1} = \text{arccsc}, \ \cot^{-1} = \text{arccot}. \end{array}}$$

Ex Given that $\cos^{-1}(-1) = \theta$, find θ.

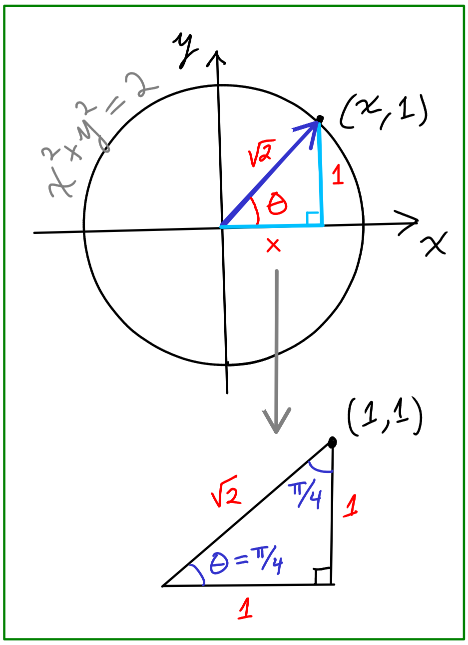
- By the definition of the inverse cosine function, we have $\cos\theta = -1$ for some $\theta \in [0, \pi]$.
- Viewing cos as a circular function (the textbook's approach — note the title of this section), we play out our analysis on the unit circle.
- We have $\cos\theta = \frac{x}{r}$, so $\frac{x}{r} = -1$. Since $r = 1$ on the unit circle, we get $x = -1$. This corresponds to the point $(-1, 0)$, shown in the figure, which lies on the terminal side of angle θ. Clearly $\theta = \pi$



$\therefore \ \boxed{\cos^{-1}(-1) = \pi}$ ■

$\boxed{\text{Ex}}$ Given $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \theta$, find θ.

- By definition $\sin^{-1} y = \theta$ if & only if $\sin\theta = y$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $y \in [-1,1]$. (So θ must be in quadrant I or IV.)
- So: $\sin\theta = \frac{\sqrt{2}}{2}$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- Of course, $\sin\theta < 0$ in quadrant IV. Here $\sin\theta > 0$, so we must have $0 < \theta \le \frac{\pi}{2}$



- This time we won't restrict ourselves to the unit circle: $\frac{y}{r} = \sin\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, so let $y = 1$ & $r = \sqrt{2}$. We have the triangle shown in the figure, in a circle of radius $\sqrt{2}$, with $x > 0$ such that $x^2 + 1^2 = (\sqrt{2})^2$, or $x = 1$. This is a 45-45 triangle, so $\theta = \frac{\pi}{4}$.

$\therefore$ $\boxed{\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}}$ ∎

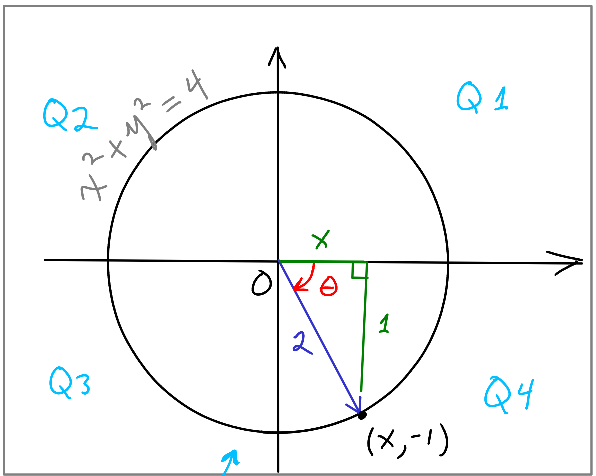
$\boxed{\text{Ex}}$ Evaluate $\cos\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$.

- By definition, for $x \in [0,\pi]$ & $y \in [-1,1]$, we have $\cos x = y$ if & only if $\cos^{-1} y = x$.
- This implies that $\cos(\cos^{-1} y) = y$ for $-1 \le y \le 1$.
- Since $-1 \le -\frac{2}{3} \le 1$, we have $\cos\left(\cos^{-1}\left(-\frac{2}{3}\right)\right) = \boxed{-\frac{2}{3}}$ ∎

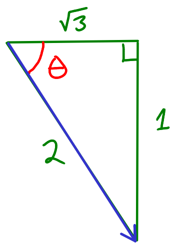
$\boxed{\text{Ex}}$ Evaluate $\cos(\cos^{-1}(5))$.

- By definition, if $\cos^{-1} 5 = x$, then $\cos x = 5$. But this is impossible since the range of cosine is $[-1, 1]$. $\boxed{\text{Undefined.}}$ ■

$\boxed{\text{Ex}}$ Find $\cot\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$.



- Let $\theta = \sin^{-1}(-1/2)$
- So $\sin\theta = -\frac{1}{2}$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\sin\theta = \frac{y}{r}$, so $\frac{y}{r} = -\frac{1}{2}$, which we have if we let $y=-1$ & $r=2$
- Note: $y=-1 \Rightarrow$ In Q3 or Q4.
  $\theta \in [-\pi/2, \pi/2] \Rightarrow$ In Q1 or Q4. $\rightarrow$ Must be in Q4.
- Have (see figure): $x^2 + (-1)^2 = 4 \Rightarrow x = \pm\sqrt{3}$. But $x > 0$ since $(x,-1)$ is in Q4, so $x = \sqrt{3}$. So the point $(x,-1)$ is $(\sqrt{3},-1)$, and the triangle in the figure is:



$\longrightarrow$ 30-60-90 Δ, so $\theta = 30^\circ = \frac{\pi}{6}$, which means $\theta = -\pi/6$ in figure above.

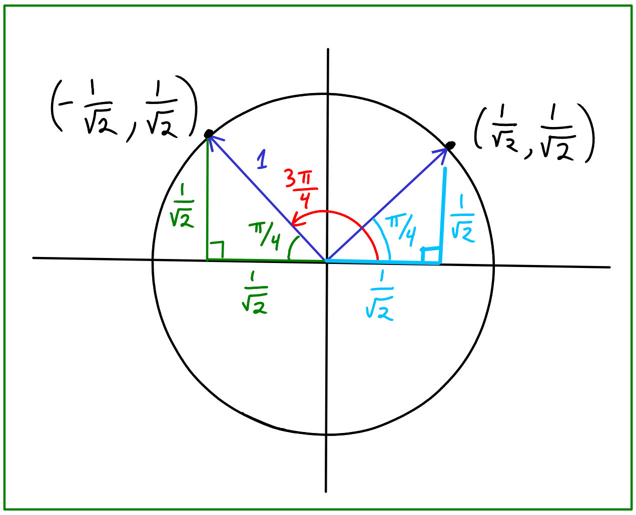
- Hence $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, and therefore we obtain

$$\cot\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \cot\left(-\frac{\pi}{6}\right) = \frac{x}{y} = \frac{\sqrt{3}}{-1} = \boxed{-\sqrt{3}} \quad ■$$

The next example is one that will subvert our expectations, and shows how great care must be taken to not always "cancel out" a trigonometric function and its inverse. That is, it is not always the case that, for instance, $\sin^{-1}(\sin\theta) = \theta$.

Ex Evaluate $\sin^{-1}\left(\sin \frac{11\pi}{4}\right)$.

- Let $\theta = \sin^{-1}\left(\sin\frac{11\pi}{4}\right)$, so that $\sin\theta = \sin\frac{11\pi}{4}$.
- By definition, $\theta = \sin^{-1} y$ if & only if $\sin\theta = y$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- So we cannot have $\theta = \frac{11\pi}{4}$, since $\frac{11\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We must find a suitable θ such that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
- Note that $\frac{11\pi}{4} = \frac{8\pi}{4} + \frac{3\pi}{4} = 2\pi + \frac{3\pi}{4}$, so angle $\frac{3\pi}{4}$ is coterminal with $\frac{11\pi}{4}$.
- So, $\sin\theta = \sin\frac{11\pi}{4} = \sin\frac{3\pi}{4}$, but since $\frac{3\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we cannot have $\theta = \frac{3\pi}{4}$ either!
- To find our solution, we must work on a circle. We'll use the unit circle (i.e. $r=1$). Then: $\sin\frac{11\pi}{4} = \sin\frac{3\pi}{4} = y$, but also $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$. So we are on the unit circle at a point with y-coordinate $y = \frac{1}{\sqrt{2}}$.



- So, θ is an angle with terminal side containing a point on the unit circle having y-coordinate $\frac{1}{\sqrt{2}}$. If x is the x-coordinate, then $x^2+y^2=1 \Rightarrow x^2 + (1/\sqrt{2})^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$
- The terminal side of θ contains either point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. See the figure.
- If the terminal side contains $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then the angle is $\frac{3\pi}{4}$, but again $\frac{3\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- If the terminal side contains $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then the angle is $\frac{\pi}{4}$, and since $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have $\theta = \frac{\pi}{4}$.

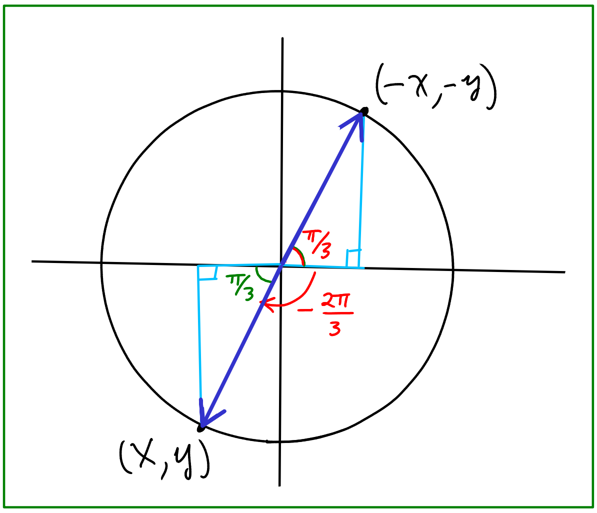
$\therefore \sin^{-1}\left(\sin\frac{11\pi}{4}\right) = \boxed{\frac{\pi}{4}}$ ∎

$\boxed{\text{Ex}}$ Find $\tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right)$.

- Let $\theta = \tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right)$
- By definition, $\theta = \tan^{-1} y$ if & only if $\tan\theta = y$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- So $\tan\theta = \tan\left(-\frac{2\pi}{3}\right)$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

  Since $-\frac{2\pi}{3} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we cannot have $\theta = -\frac{2\pi}{3}$.
- $-\frac{2\pi}{3}$ is an angle in Q3, where tangent is positive. So $\tan\theta = \tan\left(-\frac{2\pi}{3}\right) > 0$.
- But we need $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; that is, θ must be in Q1 or Q4. Since $\tan\theta > 0$, θ must be in Q1.
- $\tan\left(-\frac{2\pi}{3}\right) = \frac{y}{x}$, where $x<0$ & $y<0$ ($-\frac{2\pi}{3}$ is in Q3).
- Now, $\frac{y}{x} = \frac{-y}{-x}$, where $-x>0$ & $-y>0$, and $(-x,-y)$ is a point in Q1. See figure. From the triangles we see

$$\tan\left(-\frac{2\pi}{3}\right) = \frac{y}{x} = \frac{-y}{-x} = \tan\left(\frac{\pi}{3}\right)$$



- Finally,

$$\tan\theta = \tan\left(-\frac{2\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right),$$

  so we can have $\theta = \frac{\pi}{3}$.

  Since $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we conclude that

$$\tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right) = \theta = \boxed{\frac{\pi}{3}}$$

6.2 - Trigonometric Equations I

A trigonometric equation is an equation in which the variable appears in the argument of at least one trigonometric function, and the equation is not satisfied for at least some values of the variable. It may be that a trigonometric equation has no real-valued solution.

$\boxed{\text{Ex}}$ Solve $2\sec\theta + 1 = \sec\theta + 3$ exactly over $[0, 2\pi)$.

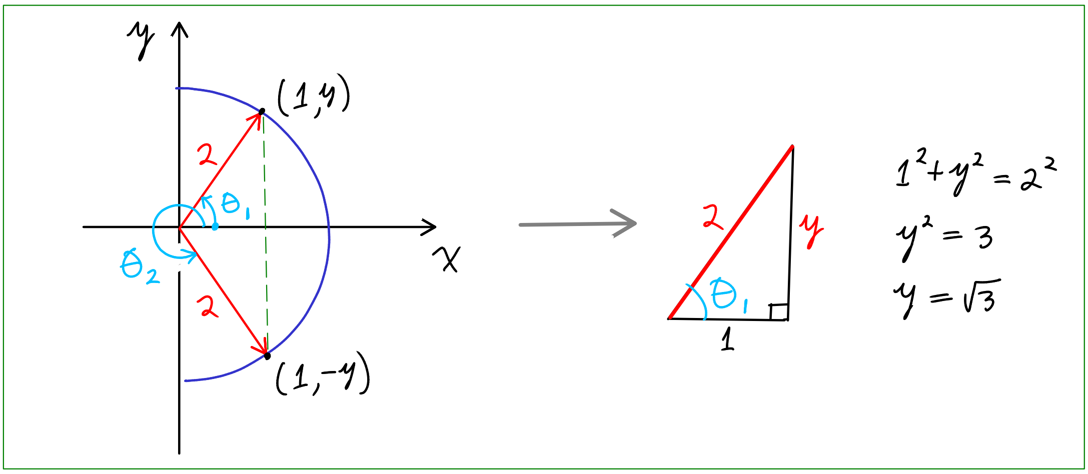
- Every trigonometric term in this equation consists of only sec(θ), so isolate sec(θ) to get:

  $\sec\theta = 2$.

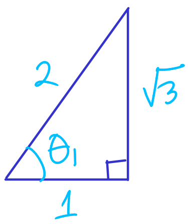
- We now find all $\theta \in [0, 2\pi)$ for which $\sec\theta = 2$. Here $\sec\theta > 0$, so the angle θ must be in quadrant I or IV. We have:

  $\sec\theta = \frac{r}{x}$, so $\frac{r}{x} = 2$, which implies we may let $r = 2$ & $x = 1$.

  This indicates that the terminal side of θ intersects a circle of radius 2 at points with x-coordinate 1:



- We obtain the right triangle with sides of length $1, \sqrt{3}, 2$ (the 30-60 Δ)



  $\rightarrow$ We see that $\theta_1 = 60^\circ = \frac{\pi}{3}$, so $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

- Solution set is $\boxed{\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}}$ ■

$\boxed{\text{Ex}}$ Find all solutions to $2\sec\theta + 1 = \sec\theta + 3$.

In the previous example we found the two solutions in the interval $[0,2\pi)$: $\frac{\pi}{3}, \frac{5\pi}{3}$.

Adding any integer multiple of 2π to either of these two solutions will yield another solution.

So $\frac{\pi}{3} + 2\pi n$ & $\frac{5\pi}{3} + 2\pi n$ are solutions for any integer n.

Solution set:

$$\boxed{\left\{\frac{\pi}{3} + 2\pi n \;\middle|\; n = 0, \pm 1, \pm 2, \dots\right\} \cup \left\{\frac{5\pi}{3} + 2\pi n \;\middle|\; n = 0, \pm 1, \pm 2, \dots\right\}}$$

$\boxed{\text{Ex}}$ Solve $\cos^2\theta - \sin^2\theta + \sin\theta = 0$ exactly over $[0, 2\pi)$.

- Since $\cos^2\theta = 1 - \sin^2\theta$, we can write

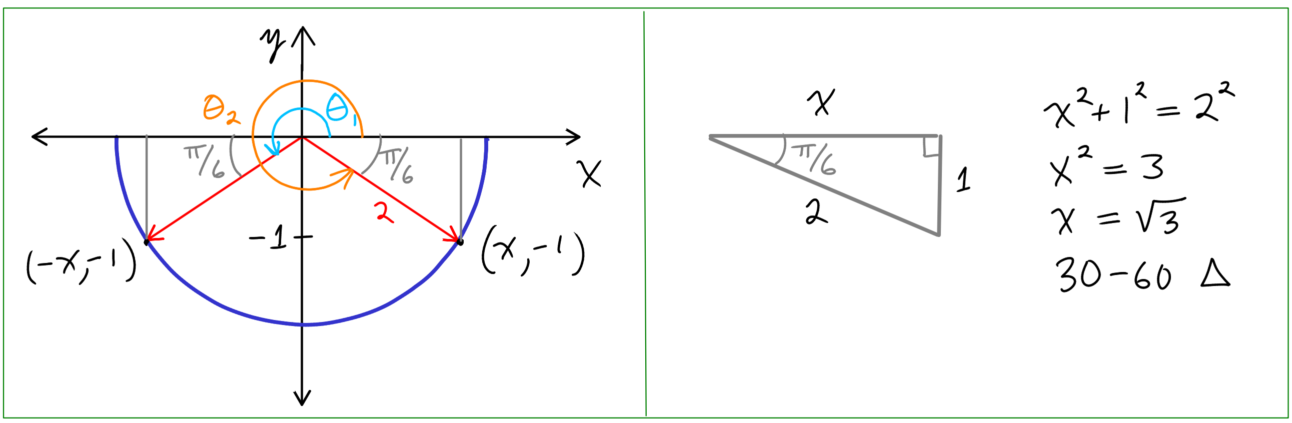
$$(1 - \sin^2\theta) - \sin^2\theta + \sin\theta = 0 \Rightarrow$$

$$-2\sin^2\theta + \sin\theta + 1 = 0$$

$$2\sin^2\theta - \sin\theta - 1 = 0 \quad \left(\text{Factor like } 2x^2 - x - 1 = (2x+1)(x-1)\right)$$

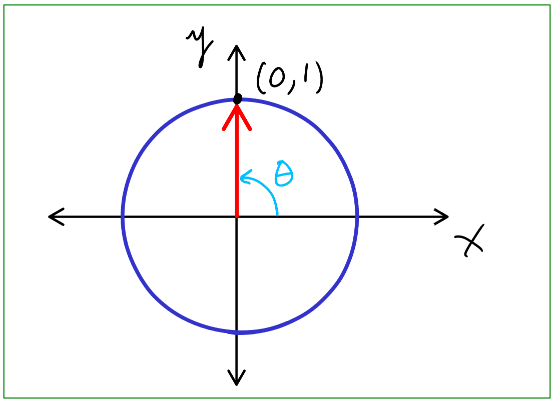
$$(2\sin\theta + 1)(\sin\theta - 1) = 0.$$

- So either $2\sin\theta + 1 = 0$ or $\sin\theta - 1 = 0$
- Isolate $\sin\theta$: $\sin\theta = -\frac{1}{2}$ or $\sin\theta = 1$
- Solving $\sin\theta = -\frac{1}{2}$: By definition $\sin\theta = \frac{y}{r}$, so $\frac{y}{r} = -\frac{1}{2}$ & we can let $y = -1$ & $r = 2$. The terminal side of θ intersects the circle of radius 2 (given by $x^2 + y^2 = 4$) at points with y-coordinate -1.



We have $\theta_1 = \pi + \frac{\pi}{6} = \boxed{\frac{7\pi}{6}}$ & $\theta_2 = 2\pi - \frac{\pi}{6} = \boxed{\frac{11\pi}{6}}$.

- Solving $\sin\theta = 1$: Again $\sin\theta = \frac{y}{r}$, so $\frac{y}{r} = 1$. We can let $y = r = 1$. Thus the terminal side of θ intersects the unit circle at a point with y-coordinate 1. There is only one such point: (0,1). Thus $\theta = \boxed{\pi/2}$.

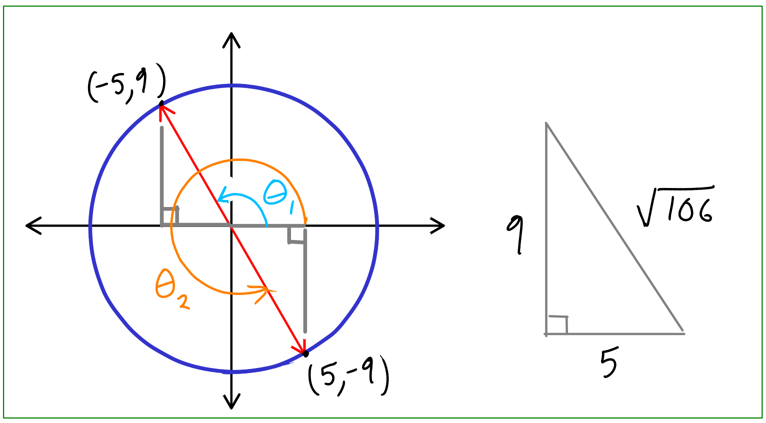


- Solution set for the original equation is:

$$\boxed{\left\{ \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2} \right\}}$$

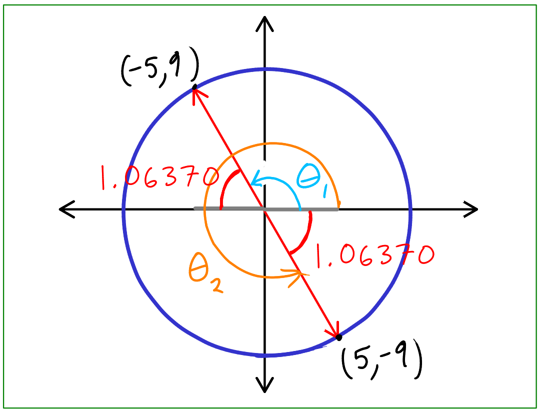
Ex Solve $5\tan\theta + 9 = 0$ over $[0, 2\pi)$, rounding to 4 decimal places if necessary (i.e. if exact solutions cannot be found).

- Isolate $\tan\theta$: $\tan\theta = -\frac{9}{5}$.
- Since $\tan\theta = \frac{y}{x}$, we have $\frac{y}{x} = -\frac{9}{5}$. We may let $x = 5$ & $y = -9$, or $x = -5$ & $y = 9$. The terminal side of θ can contain point (5,-9) or (-5,9). Neither yields a special triangle, so we will have to get decimal approximations to the solutions!



- Use a calculator to find $\tan^{-1}(-9/5)$.
  We have $\tan^{-1}(-9/5) \approx -1.06370$

(Recall that the range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$)



- We have $\theta_1 = \pi - 1.06370 = 2.07789 \approx 2.0779,$

  $\theta_2 = 2\pi - 1.06370 = 5.21948 \approx 5.2195$

Solution set is $\boxed{\{2.0779, 5.2195\}}$ ∎

6.3 - Trigonometric Equations II

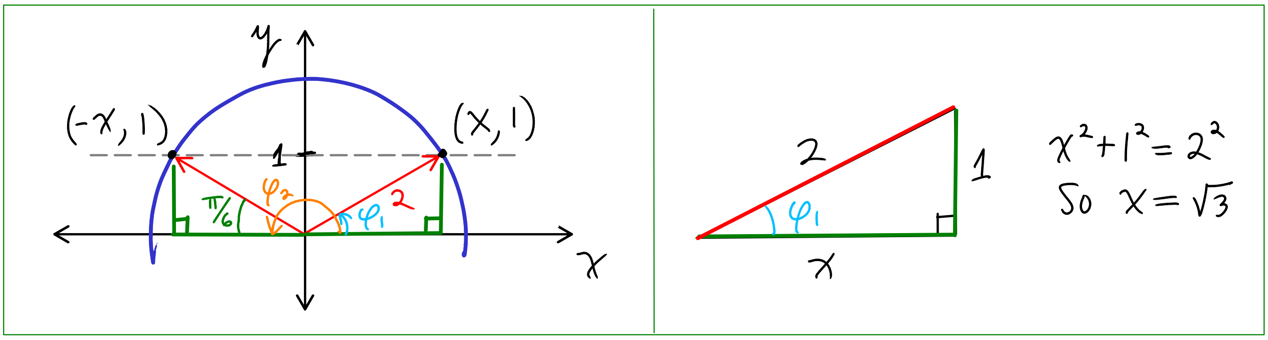
The trigonometric equations in this section will either feature constant multiples of an unknown angle, or else necessitate exponentiating both sides of the equation in order to use a trigonometric identity.

Ex Solve $2\sqrt{3}\sin 2\theta = \sqrt{3}$ over $[0, 2\pi)$

- Isolate $\sin 2\theta$: $\sin 2\theta = \frac{1}{2}$.
- We find $0 \le \theta < 2\pi$ for which $\sin 2\theta = \frac{1}{2}$.

  That is, we find $0 \le 2\theta < 4\pi$ for which $\sin 2\theta = \frac{1}{2}$.

  It might be convenient to let $\varphi = 2\theta$, and solve $\sin\varphi = \frac{1}{2}$ over $[0, 4\pi)$.
- $\sin\varphi = \frac{y}{r} = \frac{1}{2}$, so let $y = 1$ & $r = 2$.



φ_1 is in a 30-60 △ opposite side of length 1: $\varphi_1 = \frac{\pi}{6}$.

Then $\varphi_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Since $0 \le \varphi < 4\pi$, we find that $\varphi_3 = \varphi_1 + 2\pi = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ is another solution, as is $\varphi_4 = \varphi_2 + 2\pi = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$

- So we have $2\theta = \varphi = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$.

  Then $\theta = \frac{1}{2}\left(\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\right) = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
- Solution set: $\boxed{\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\right\}}$

$\boxed{\text{Ex}}$ Solve $\sin\theta\cos\theta = \frac{1}{2}$ over $[0, 2\pi)$.

- Here there are two trigonometric functions floating around, and there is no way to get one converted into an expression in terms of the other without introducing a radical having a +/- symbol attached. The way forward is to square both sides of the equation and use the identity $\sin^2\theta + \cos^2\theta = 1$:

$$\sin^2\theta\cos^2\theta = \frac{1}{4} \Rightarrow \sin^2\theta\left(1 - \sin^2\theta\right) = \frac{1}{4}$$

$$\Rightarrow \sin^4\theta - \sin^2\theta + \frac{1}{4} = 0$$

Factor like $x^4 - x^2 + \frac{1}{4}$

$$\Rightarrow \left(\sin^2\theta - \frac{1}{2}\right)^2 = 0$$

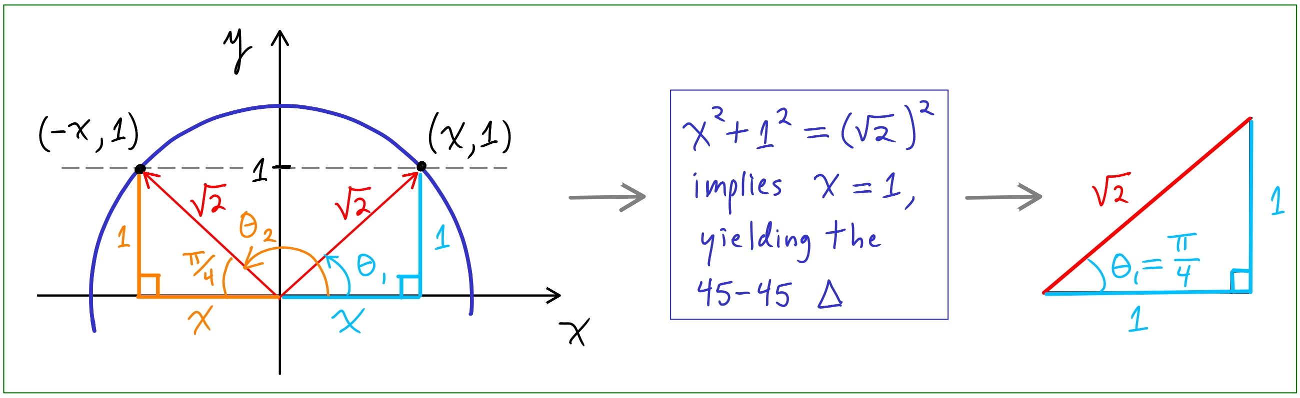
$$\Rightarrow \sin^2\theta - \frac{1}{2} = 0$$

$$\Rightarrow \sin^2\theta = \frac{1}{2}$$

$$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}}$$

- Solving $\sin\theta = \frac{1}{\sqrt{2}}$: Here $\frac{y}{r} = \frac{1}{\sqrt{2}}$, so let $y = 1$ & $r = \sqrt{2}$.

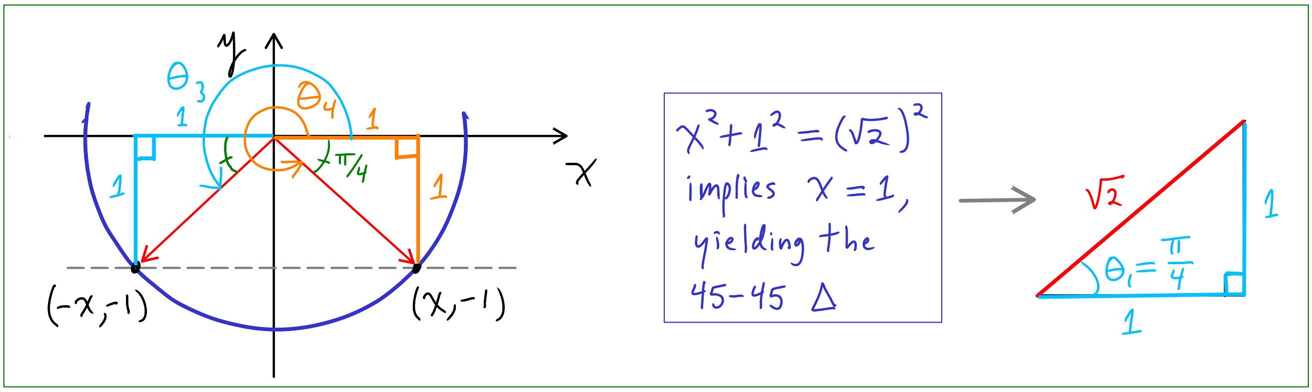
The terminal side of θ intersects the circle of radius $\sqrt{2}$ at points with y-coordinate 1. See the figure.



We have $\theta_1 = \frac{\pi}{4}$ & $\theta_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ as solutions to $\sin\theta = \frac{1}{\sqrt{2}}$.

- Solving $\sin\theta = -\frac{1}{\sqrt{2}}$: Here $\frac{y}{r} = \frac{-1}{\sqrt{2}}$, so let $y = -1$ & $r = \sqrt{2}$.

The terminal side of θ intersects the circle of radius $\sqrt{2}$ at points with y-coordinate -1. Again 45-45 triangles result...



We have $\theta_3 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ & $\theta_4 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ as solutions to $\sin\theta = -\frac{1}{\sqrt{2}}$.

- Are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ all solution to $\sin\theta\cos\theta = \frac{1}{2}$?

  The squaring operation we had done earlier may have resulted in extraneous solutions appearing, so we must check our four proposed solutions using the original equation $\sin\theta\cos\theta = \frac{1}{2}$ ...

  Check $\theta = \frac{\pi}{4}$: $\sin\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{1}{2} \Rightarrow \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$ ✓

  Check $\theta = \frac{3\pi}{4}$: $\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} = \frac{1}{2} \Rightarrow \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{2} \Rightarrow -\frac{1}{2} = \frac{1}{2}$ ✗

  Check $\theta = \frac{5\pi}{4}$: $\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} = \frac{1}{2} \Rightarrow \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$ ✓

  Check $\theta = \frac{7\pi}{4}$: $\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} = \frac{1}{2} \Rightarrow \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \Rightarrow -\frac{1}{2} = \frac{1}{2}$ ✗

  We find only $\frac{\pi}{4}$ & $\frac{5\pi}{4}$ to be valid solutions.

- Solution set: $\boxed{\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}}$ ■

6.4 - Equations Involving Inverse Trigonometric Functions

Ex Solve $y = 3\sin\frac{x}{2}$ for $x \in [-\pi, \pi]$.

- We note that $-\pi \le x \le \pi$ implies that $-\frac{\pi}{2} \le \frac{x}{2} \le \frac{\pi}{2}$, and by the definition of the inverse sine function we have

$$v = \sin u \iff u = \sin^{-1} v$$

whenever $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Therefore:

$$y = 3\sin\frac{x}{2} \Rightarrow \frac{y}{3} = \sin\frac{x}{2} \Rightarrow \frac{x}{2} = \sin^{-1}\frac{y}{3} \Rightarrow \boxed{x = 2\sin^{-1}\frac{y}{3}}$$

Ex Solve $y = -\sqrt{3} + 2\csc\frac{x}{2}$ for $x \in [-\pi, 0) \cup (0, \pi]$.

- Here $x \in [-\pi, 0) \cup (0, \pi]$ implies $\frac{x}{2} \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$. Since

$$v = \csc u \iff u = \csc^{-1} v$$

for $u \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, we have

$$y = -\sqrt{3} + 2\csc\frac{x}{2} \Rightarrow \frac{y+\sqrt{3}}{2} = \csc\frac{x}{2}$$

$$\Rightarrow \frac{x}{2} = \csc^{-1}\left(\frac{y+\sqrt{3}}{2}\right)$$

$$\Rightarrow \boxed{x = 2\csc^{-1}\left(\frac{y+\sqrt{3}}{2}\right)}$$

$\boxed{\text{Ex}}$ Solve $6 \arccos x = 5\pi$ for x exactly.

- First we isolate the inverse trigonometric function:

$$\arccos x = \frac{5\pi}{6}$$

- We use the definition of arccos to isolate x:

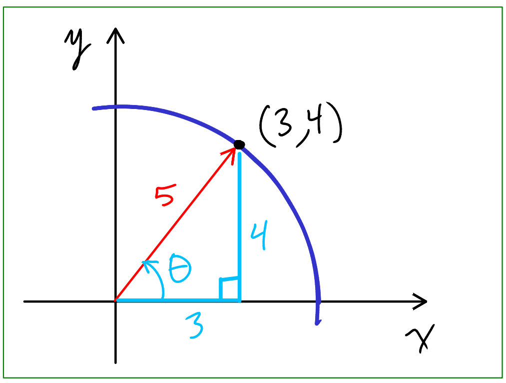
$$x = \cos\frac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2}} \quad \blacksquare$$

$\boxed{\text{Ex}}$ Solve $\cot^{-1} x = \tan^{-1}\frac{4}{3}$ for x exactly.

- The range of $\cot^{-1}$ is $(0,\pi)$, & by the definition of $\tan^{-1}$ the value of $\tan^{-1}\frac{4}{3}$ is $(-\frac{\pi}{2},\frac{\pi}{2})$. Indeed, if we let $\theta = \tan^{-1}\frac{4}{3}$, then $\tan\theta = \frac{4}{3} > 0$, and hence we must have $0 < \theta < \frac{\pi}{2}$ (to have $-\frac{\pi}{2} < \theta < 0$ would result in having $\tan\theta < 0$). So, $\tan^{-1}\frac{4}{3} = \theta \in (0,\frac{\pi}{2})$, which is to say $\tan^{-1}\frac{4}{3}$ lies in the range of $\cot^{-1}$ & the given equation **has a solution!** Moreover, by the definition of $\cot^{-1}$,

$$\cot^{-1} x = \tan^{-1}\frac{4}{3} \quad \Longrightarrow \quad x = \cot\left(\tan^{-1}\frac{4}{3}\right).$$

- Again, $\theta = \tan^{-1}\frac{4}{3}$ implies $\tan\theta = \frac{4}{3}$ for $\theta \in (0,\frac{\pi}{2})$.



Recall $\tan\theta = y/x$, so let $y=4$ & $x=3$. See the figure. We obtain

$$x = \cot\left(\tan^{-1}\frac{4}{3}\right)$$
$$= \cot\theta$$
$$= \boxed{\frac{3}{4}} \quad \blacksquare$$