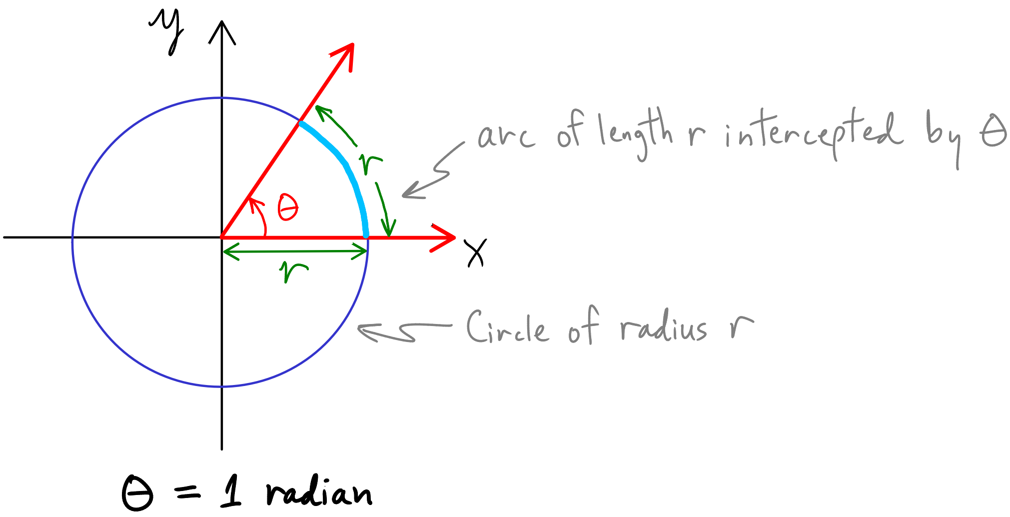
3.1 - Radian Measure

If an angle at most one revolution in measure has its vertex at the center of a circle, then a portion of the circle, called an arc, will lie between the rays that form the angle's initial and terminal sides. We say the angle intercepts the arc.

Definition An angle with its vertex at the center of a circle of radius r that intercepts an arc on the circle having length r has a measure of 1 radian.



We recall that the circumference C of a circle of radius r is

$$\boxed{C = 2\pi r}$$

Now, if an angle with vertex at the center of a circle of radius r has measure 1 radian if it intercepts an arc of length r, then an angle with vertex at the center of the circle that intercepts an arc of length 2π r must have measure 2π radians. But an arc of length 2πr constitutes the entire circumference of the circle, so that the angle must also have measure $360°$. That is,

$$\boxed{360^\circ = 2\pi \text{ radians}} \longrightarrow \boxed{180^\circ = \pi \text{ radians}}$$

The radian measure of an angle with vertex at the center of a circle is the ratio of the length of the intercepted arc to the radius of the circle. The length units cancel, leaving a pure (real) number. That is, "radian" is not really a unit. An angle with measure given in terms of radians is unitless.

Using radian measure for angles enables the six trigonometric functions to have domains consisting of real numbers, like any other kind of function encountered in algebra such as a polynomial or a logarithm.

$$\boxed{180^\circ = \pi \text{ radians}}$$

$\boxed{16}$ Convert 270° to radians

$$\underbrace{(270^\circ)\left(\frac{\pi \text{ radians}}{180^\circ}\right)}_{\text{Degree units "cross cancel"}} = \frac{270}{180}\pi = \boxed{\frac{3\pi}{2}}$$

$\boxed{26}$ Convert -1800° to radians

$$\underbrace{(-1800^\circ)\left(\frac{\pi \text{ radians}}{180^\circ}\right)}_{\text{Degree units "cross cancel"}} = -\frac{1800}{180}\pi = \boxed{-10\pi}$$

$\boxed{38}$ Convert $\frac{11\pi}{15}$ to degrees

$$\left(\frac{11\pi}{15}\right)\left(\frac{180^\circ}{\pi}\right) = \frac{11}{15}(180^\circ) = \left(\frac{11\cdot 180}{15}\right)^\circ = (11\cdot 12)^\circ = \boxed{132^\circ}$$

$\boxed{50}$ Convert $174°50'$ to radians. Round to the nearest thousandth.

- If necessary, first convert from D°M′ format to a decimal degree:

$$174°50' = 174° + \left(\frac{50}{60}\right)^{\circ} = 174° + 0.8\overline{3}^{\circ} = 174.8\overline{3}^{\circ}$$

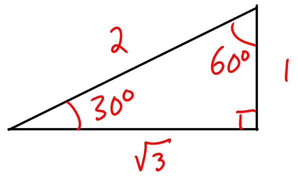
- Now convert to radian measure as in #16 and #26:

$$(174.8\overline{3}^{\circ})\left(\frac{\pi}{180°}\right) = \frac{174.8\overline{3}\,\pi}{180} = 3.0514\ldots \approx \boxed{3.051}$$

$\boxed{68}$ Find the exact value of $\cos\left(\frac{\pi}{6}\right)$

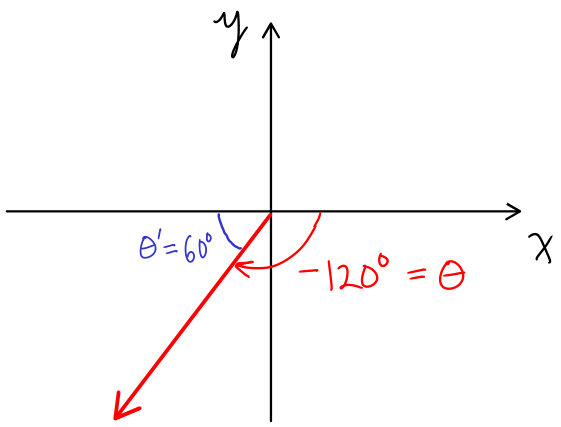
$$\cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\cdot\frac{180°}{\pi}\right) = \cos\left(\frac{1}{6}(180°)\right) = \cos 30° = \boxed{\frac{\sqrt{3}}{2}},$$

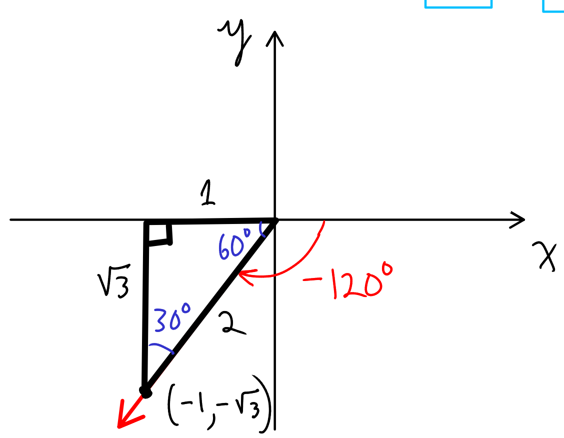
using the 30-60 special triangle:



$\boxed{82}$ Find the exact value of $\cot\left(-\frac{2\pi}{3}\right)$

$$\cot\left(-\frac{2\pi}{3}\right) = \cot\left(-\frac{2\pi}{3}\cdot\frac{180°}{\pi}\right) = \cot(-120°) = \frac{-1}{-\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}} = \boxed{\frac{\sqrt{3}}{3}}$$

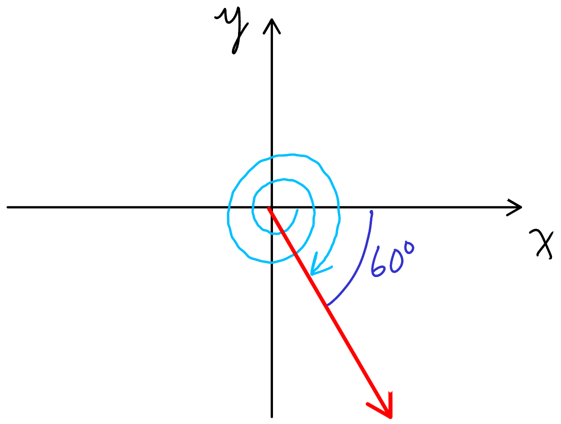


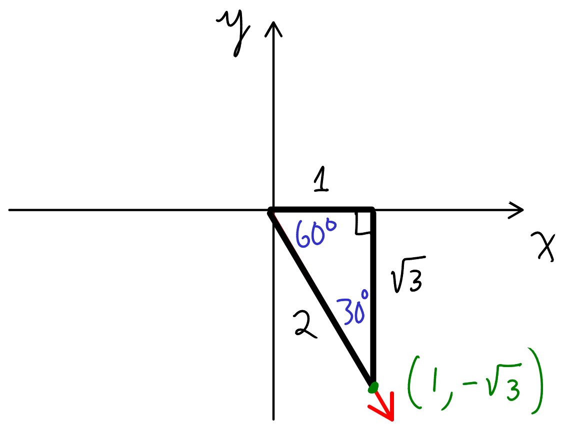


$\boxed{86}$ Find the exact value of $\csc\left(-\frac{13\pi}{3}\right)$.

$$\csc\left(-\frac{13\pi}{3}\right) = \csc\left(-\frac{13\pi}{3}\cdot\frac{180^\circ}{\pi}\right) = \csc(-780^\circ)$$

$\theta = -780^\circ = 2(-360^\circ) + (-60^\circ)$, so reference angle is $\theta' = 60^\circ$





$$\csc\left(-\frac{13\pi}{3}\right) = \csc(-780^\circ) = \frac{2}{-\sqrt{3}} = \boxed{-\frac{2}{\sqrt{3}}} = \boxed{-\frac{2\sqrt{3}}{3}}$$

$\csc\theta = \frac{r}{y}$ with

$r = 2$ & $y = -\sqrt{3}$.

3.2 - Applications of Radian Measure

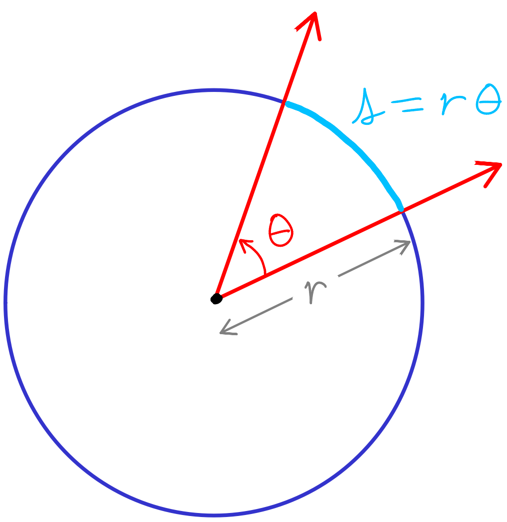
As we have seen in Section 3.1, an angle with measure 2π radians with vertex at the center of a circle of radius r intercepts the entire circumference of the circle, which is an arc of length $2\pi r$. An angle of measure π radians intercepts half the circumference, which is an arc of length πr.

Generally, an angle with measure θ radians intercepts an arc of length θr. If we let s denote arc length, then we obtain the following theorem.

Arc Length Theorem The length s of the arc intercepted on a circle of radius r by a central angle of measure θ radians is given by

$$s = r\theta$$

θ in radians!



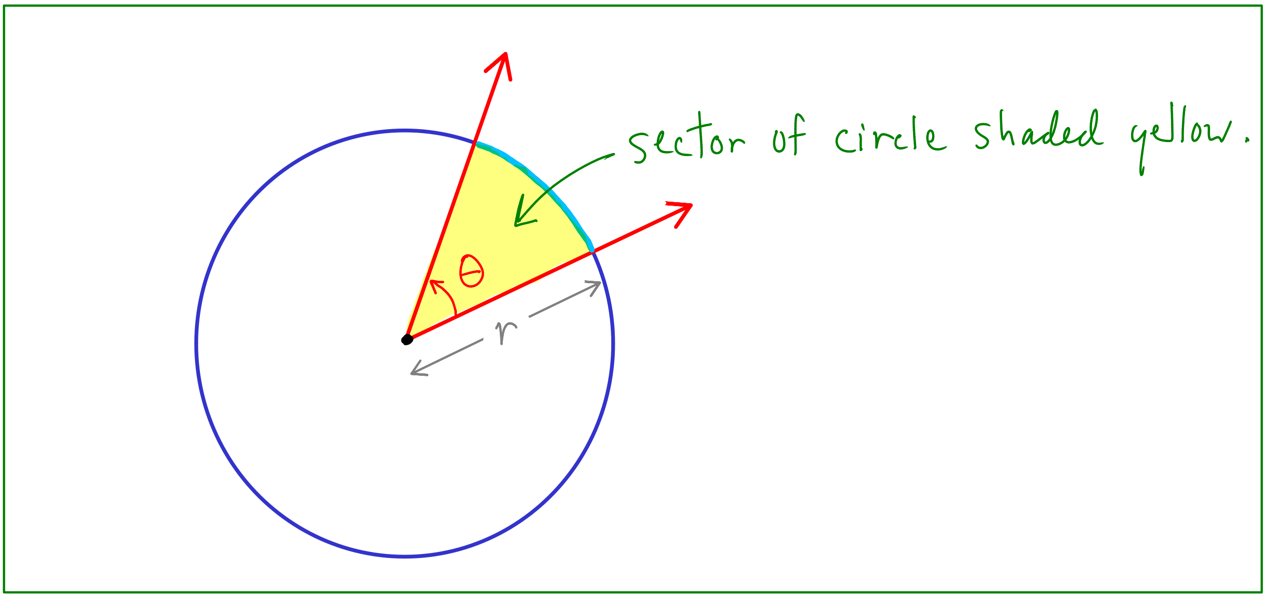
Angle θ in radians has vertex at the center of a circle of radius r, otherwise known as a central angle.

The area A of a circle of radius r is given by $A = \pi r^2$.

We could write $A = \frac{1}{2}(2\pi r)r$, where $2\pi r$ is the circumference of the circle. In fact:

$A = \frac{1}{2}$(arc length intercepted by central angle of 2π radians)r.

A sector of a circle is a region bounded by the initial side, terminal side, and intercepted arc of a central angle:



If the central angle has measure θ radians and the circle has radius r, then we obtain what is called a circular sector of radius r and central angle θ.

The area A of a circular sector of a circle of radius r and central angle θ radians, where $0 \le \theta \le 2\pi$, is given by

$$A = \frac{1}{2}(\text{arc length intercepted by central angle of } \theta \text{ radians})\, r.$$

$$= \frac{1}{2}(r\theta)r = \frac{1}{2}r^2\theta$$

using the arc length theorem for the second equality.

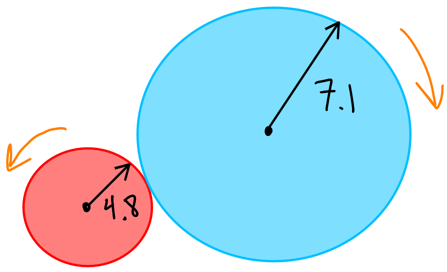
Sector Area Theorem The area A of a circular sector of radius r and central angle θ is

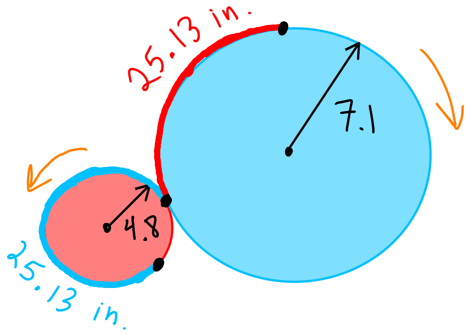
$$A = \frac{1}{2}r^2\theta$$

14 Find the length to three significant digits of the arc intercepted by a central angle $\theta = 11\pi/10$ radian in a circle of radius $r = 0.892$ cm.

We have $s = r\theta = (0.892\text{ cm})(11\pi/10) = 3.0825\ldots \approx \boxed{3.08\text{ cm}}$

$\boxed{30}$ Two gears are adjusted so that the smaller gear drives the larger gear. The smaller gear has radius 4.8 inches and the larger gear has radius 7.1 inches. If the smaller gear rotates through an angle of 300°, through how many degrees does the larger gear rotate?





$s = r\theta$, where $\theta = 300° = 300°\left(\frac{\pi}{180°}\right) = \frac{15}{9}\pi$ radians

θ must be in radians!

The arc length of the motion of the smaller gear is:

$$s = (4.8 \text{ in.})\left(\frac{15\pi}{9}\right) = 25.13 \text{ in.}$$

The arc length of the motion of the larger gear will also be 25.13 in. We want to find the larger gear's angle of rotation in degrees. For the larger gear we have s=25.13 and r=7.1, and we find θ . . .

$$s = r\theta \Rightarrow \theta = \frac{s}{r} = \frac{25.13 \text{ in.}}{7.1 \text{ in.}} = 3.539 \text{ radians.}$$

Angle of rotation of larger gear in degrees is...

$$(3.539)\left(\frac{180°}{\pi}\right) = 202.8° \approx \boxed{203°}$$

3.3 - The Unit Circle and Circular Functions

The unit circle is the circle with center at the origin and radius 1.

By the arc length theorem of Section 3.2, the length s of the arc intercepted by a central angle of θ radians in the unit circle is

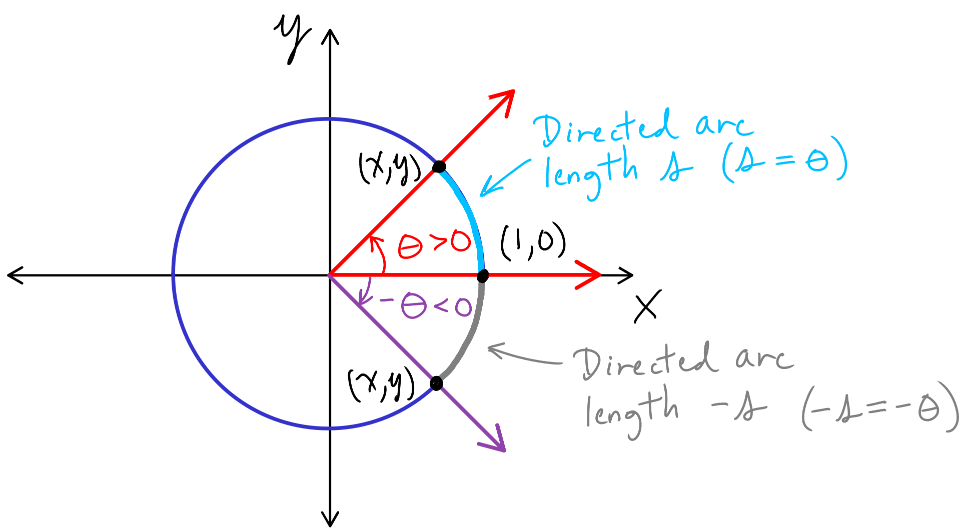
$$s = \theta$$

since r=1 for the unit circle.

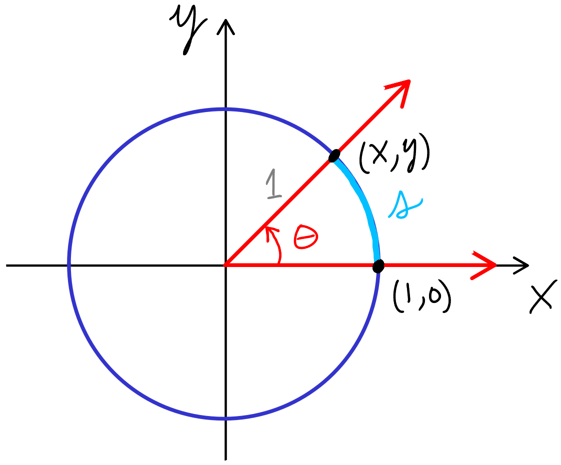
So on the unit circle an angle of s radians corresponds to an intercepted arc of length s units. In Section 3.1 we defined the trigonometric functions as functions of a real-valued variable θ representing an angle of measure θ radians. On the unit circle we could just as well define the trigonometric functions as being functions of a real-valued variable s representing a "directed arc length."

Suppose an angle θ in standard position at the center of the unit circle intercepts an arc of length s. If θ is a positive angle, then the directed arc length intercepted by θ is s. If θ is a negative angle, then the directed arc length intercepted by θ is -s.

Thus, if (x,y) is a point on the terminal side of θ, then a directed arc length of s means one travels s units counterclockwise from (1,0) to (x,y) along the unit circle, and a directed arc length of -s means one travels s units clockwise from (1,0) to (x,y).



Generally we let the symbol s denote directed arc length, so s is a variable that can assume an real number value: positive, negative, or zero. And since $s = \theta$ on the unit circle when θ is measured in radians, we can define the trigonometric functions as functions of s, in which case they are called circular functions.

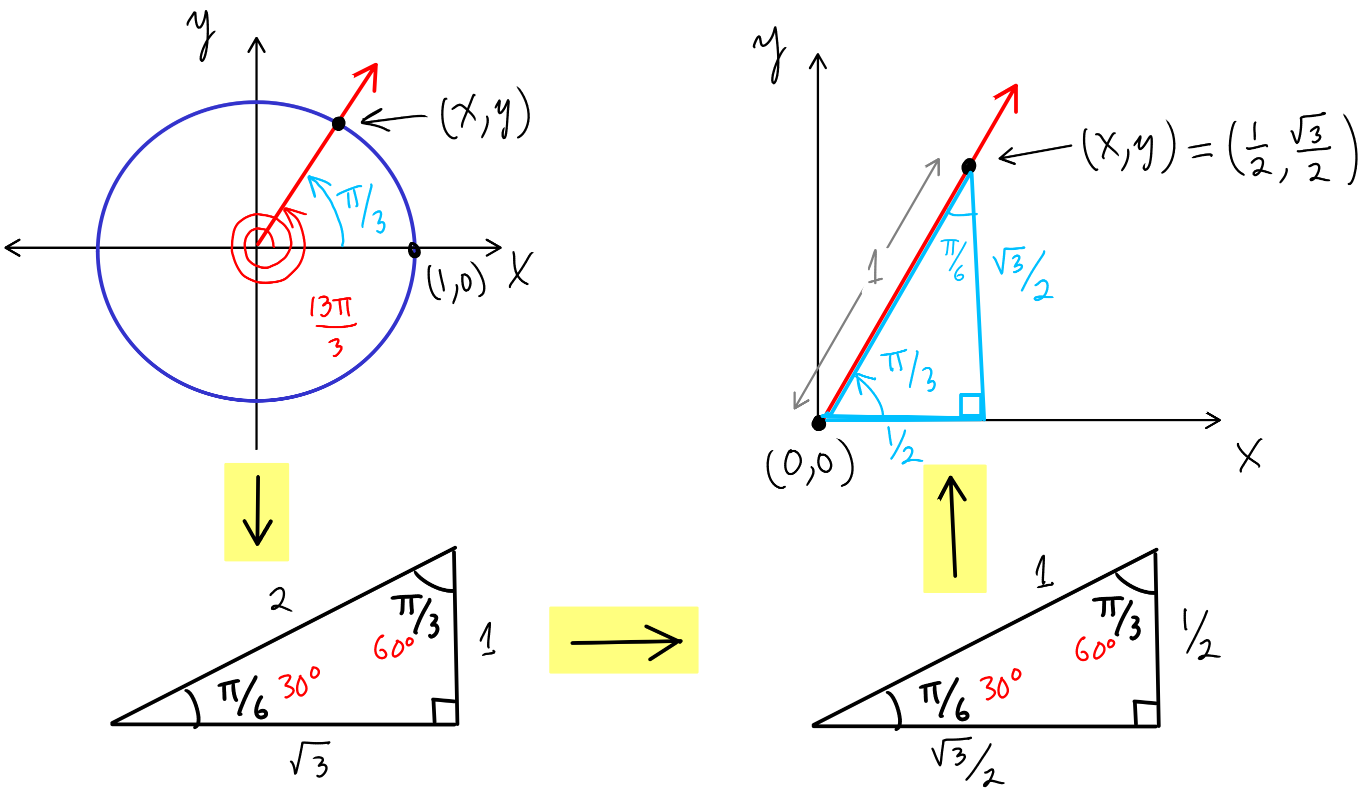


The circular function definitions:	
$\sin s = y$	$\cot s = \frac{x}{y}$
$\cos s = x$	$\sec s = \frac{1}{x}$
$\tan s = \frac{y}{x}$	$\csc s = \frac{1}{y}$

All of this amounts to just another interpretation of the usual six trigonometric functions that is equivalent to the one presented in Section 3.1, which is why we pass through this section quickly.

$\boxed{30}$ Find the exact value of $\csc \frac{13\pi}{3}$.

$\frac{13\pi}{3} = \frac{12\pi}{3} + \frac{\pi}{3} = 4\pi + \frac{\pi}{3}$ (two revolutions plus $\frac{\pi}{3}$).



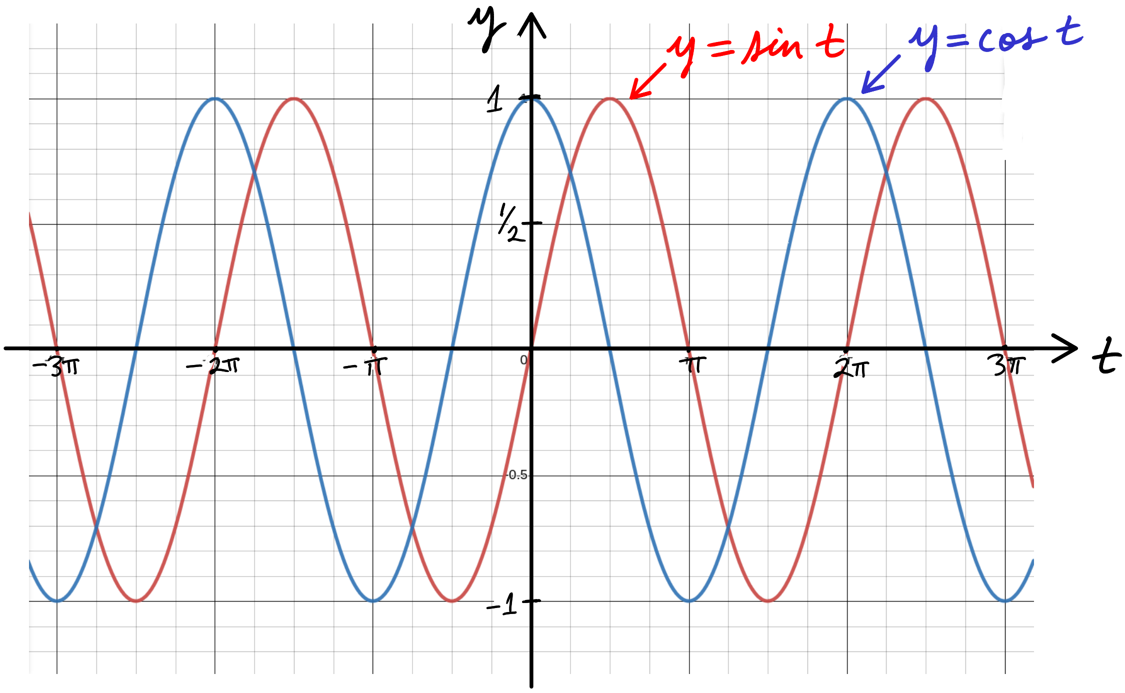
$$\csc\frac{13\pi}{3} = \csc\frac{\pi}{3} = \frac{1}{y} = \frac{1}{\sqrt{3}/2} = \boxed{\frac{2}{\sqrt{3}}} = \boxed{\frac{2\sqrt{3}}{3}}$$

Also: $\tan\frac{13\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \boxed{\sqrt{3}}$

$$\cos\frac{13\pi}{3} = x = \boxed{\frac{1}{2}}$$

4.1 - Graphs of the Sine & Cosine Functions

In this section we take the sine and cosine functions to be functions of a real-valued variable t, which could represent an angle θ in radians or a directed arc length s on the unit circle—pick the interpretation that you like. The textbook uses x instead of t.



A function f is periodic is there exists some constant p ≠ 0 such that f(x+p) = f(x) for all x in the domain of f. The constant p is called a period of f. The smallest positive value of p that is a period of f is called the function's fundamental period.

In this course we take the term "period" to mean "fundamental period."

As the graphs above illustrate (though we have known this since Chapter 1), we have

$$\sin t = \sin(t + 2\pi) \quad \text{and} \quad \cos t = \cos(t + 2\pi)$$

for all real t, recalling that the domain of both sine and cosine is $(-\infty, \infty)$. Both sine and cosine are periodic functions with period 2π.

- We have:

$\sin(t+4\pi) = \sin((t+2\pi)+2\pi) = \sin(t+2\pi) = \sin t,$

$\sin(t+6\pi) = \sin((t+4\pi)+2\pi) = \sin(t+4\pi) = \sin t,$

and in general, for any integer n,

$$\boxed{\sin(t+2\pi n) = \sin t}$$

- Similary,

$$\boxed{\cos(t+2\pi n) = \cos t}$$

Let $b>0$ be a constant, and let $f(t) = \sin(bt)$. What is the period of f? We seek the smallest $p>0$ such that $f(t+p) = f(t)$ for all $t \in (-\infty, \infty)$.

$f(t) = f(t+p)$ implies that $\sin bt = \sin(b(t+p))$, but we know that, starting at bt, the sine function does not repeat itself until $bt+2\pi$.
(The sine function has period 2π, so in general x must increase to $x+2\pi$ for $\sin x$ to run through its cycle & begin a new cycle.)
So, we need p to be such that $b(t+p) = bt+2\pi$. We solve for p:

$$b(t+p) = bt+2\pi \Rightarrow \cancel{bt}+bp = \cancel{bt}+2\pi \Rightarrow p = \frac{2\pi}{b}$$

The cosine function behaves the same way, so...

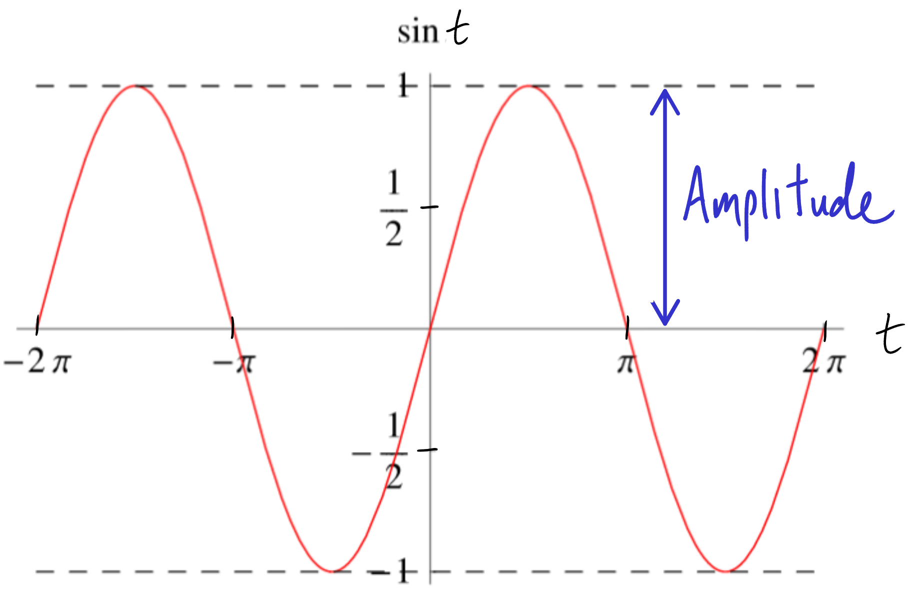
$$\boxed{\sin bt \ \& \ \cos bt \text{ have period } \frac{2\pi}{b}} \qquad \text{(A)}$$

Since a periodic function runs through a repeating pattern, called a cycle, its range is often a bounded closed interval.

> For periodic function f, let M and m be the maximum and minimum values that f(x) attains. The amplitude of f is
> $$\frac{M-m}{2}.$$

The range of the sine and cosine functions are [-1,1]. That is, sin(t) and cos(t) attain a maximum value of 1 and a minimum value of -1. Therefore their amplitudes are

$$\frac{1-(-1)}{2} = 1.$$



Let a ≠ 0 be a constant. Then the graphs of

$$y = a\sin t \quad \text{and} \quad y = a\cos t$$

both have amplitude |a|.

- To see this, we start with the fact that $-1 \le \sin t \le 1$ for all t.
- If $a>0$, then $-a \le a\sin t \le a$. (1)
- If $a<0$, then $-a \ge a\sin t \ge a \Rightarrow a \le a\sin t \le -a$. (2)
- Now, $|a| = a$ if $a>0$, and $|a| = -a$ if $a<0$. So (1) & (2) may both be written as

$$-|a| \le a\sin t \le |a|.$$

- Thus the amplitude of $y = a\sin t$ is $\dfrac{|a|-(-|a|)}{2} = \dfrac{2|a|}{2} = |a|$.
  The same holds for $y = a\cos t$

$$\boxed{y = a\sin t \ \& \ y = a\cos t \text{ have amplitude } |a|.}$$

(B)

(A) and (B) can be combined:

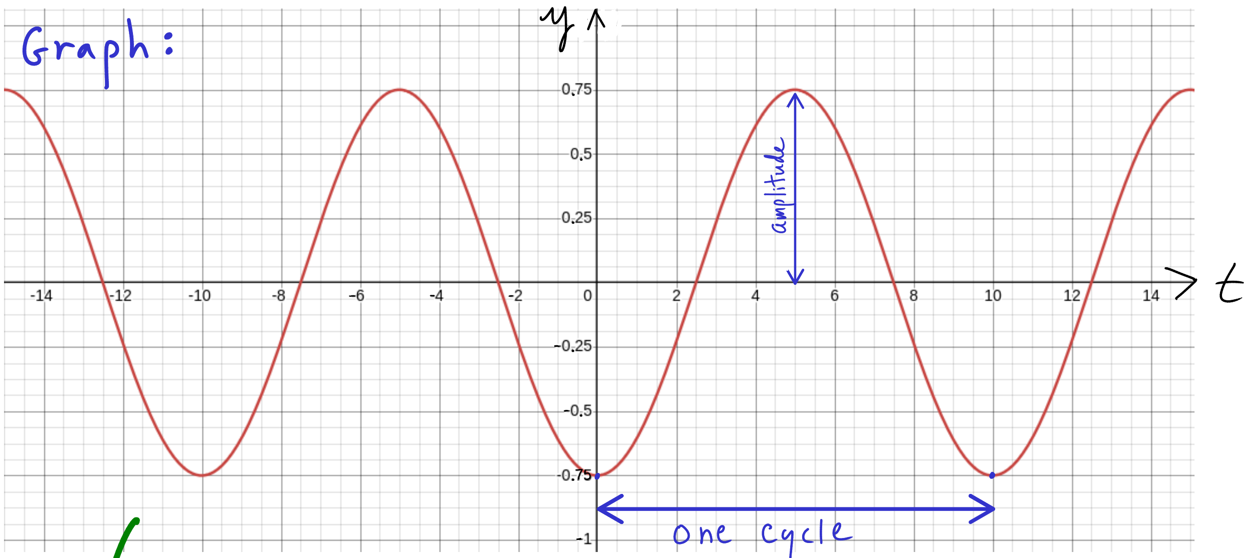
$$\boxed{y = a\sin bt \ \& \ y = a\cos bt \text{ have period } \frac{2\pi}{b} \ \& \text{ amplitude } |a|}$$

Ex Give the period & amplitude of $y = -\frac{3}{4}\cos\left(\frac{\pi}{5}t\right)$.

We have $y = a\cos bt$ with $a = -\frac{3}{4}$ & $b = \frac{\pi}{5}$.

Period $= \frac{2\pi}{b} = \frac{2\pi}{\pi/5} = 2\pi \cdot \frac{5}{\pi} = \boxed{10}$

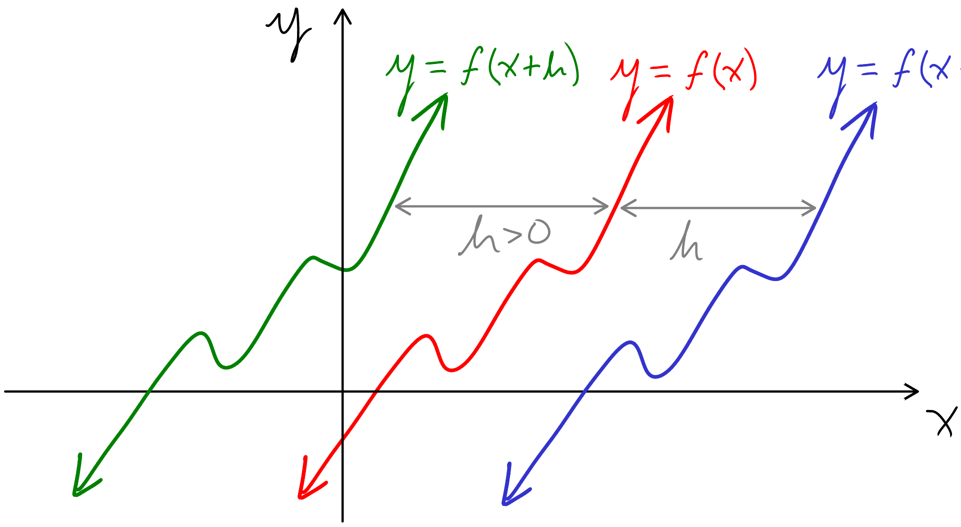
Amplitude $= |a| = \boxed{\frac{3}{4}}$



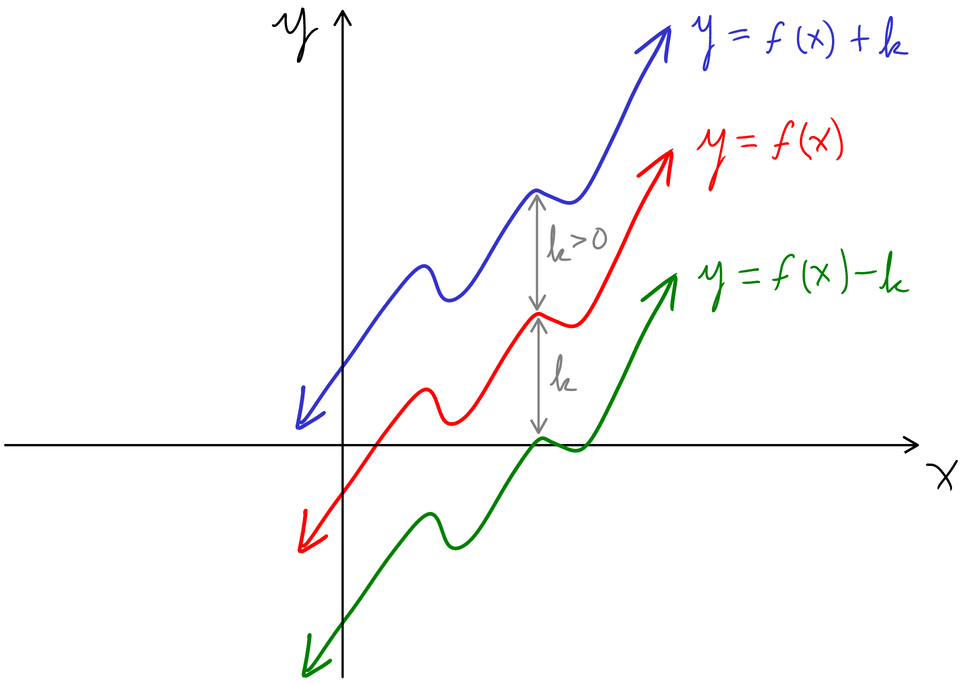
Graph made by Desmos (online for free)

4.2 - Translations of Sine & Cosine

The graph of y=f(x-h) is the graph of y=f(x) shifted horizontally by h units. Generally this is called a horizontal translation, but if f is a periodic function it may be called a phase shift. The case when h>0 is illustrated below.



The graph of y=f(x)+k is the graph of y=f(x) shifted vertically by k units. This is called a vertical translation. The case when k>0 is illustrated below.



The graph of y=f(x-h)+k is the graph of y=f(x) shifted horizontally by h units and vertically by k units.

Thus the graph of y=sin(x-h)+k and y=cos(x-h)+k is the graph of y=sin(x) and y=cos(x) shifted horizontally by h & vertically by k.

Functions $y = a\sin(b(x-h))+k$ & $y = a\cos(b(x-h))+k$ have graphs like $y = a\sin bx$ & $y = a\cos bx$, with period $\frac{2\pi}{b}$ & amplitude $|a|$, only shifted horizontally h & vertically k.

This is the method for graphing the highlighted functions above:

1) Graph $y = a\sin bx$ or $y = a\cos bx$
2) Horizontally and vertically translate the graphs accordingly.

This is all this entire section in the textbook is really about.

$\boxed{38}$ Find the amplitude, period, vertical translation, and phase shift for the function

$$y = -1 + \frac{1}{2}\cos(2x - 3\pi).$$

- We recall that $y = a\cos(b(x-h))+k$ has: period $\frac{2\pi}{b}$, amplitude $|a|$, vertical translation k, phase shift h.
- Here we have $y = \frac{1}{2}\cos(2x-3\pi) + (-1) = \frac{1}{2}\cos\left(2\left(x - \frac{3\pi}{2}\right)\right) + (-1)$.
- So: $a = \frac{1}{2}$, $b = 2$, $h = \frac{3\pi}{2}$, $k = -1$.
- Answer:

$$\boxed{\begin{aligned} &\text{amplitude} = \left|\tfrac{1}{2}\right| = \tfrac{1}{2} \\ &\text{period} = \tfrac{2\pi}{2} = \pi \\ &\text{vertical translation} = -1 \\ &\text{phase shift} = 3\pi/2 \end{aligned}}$$

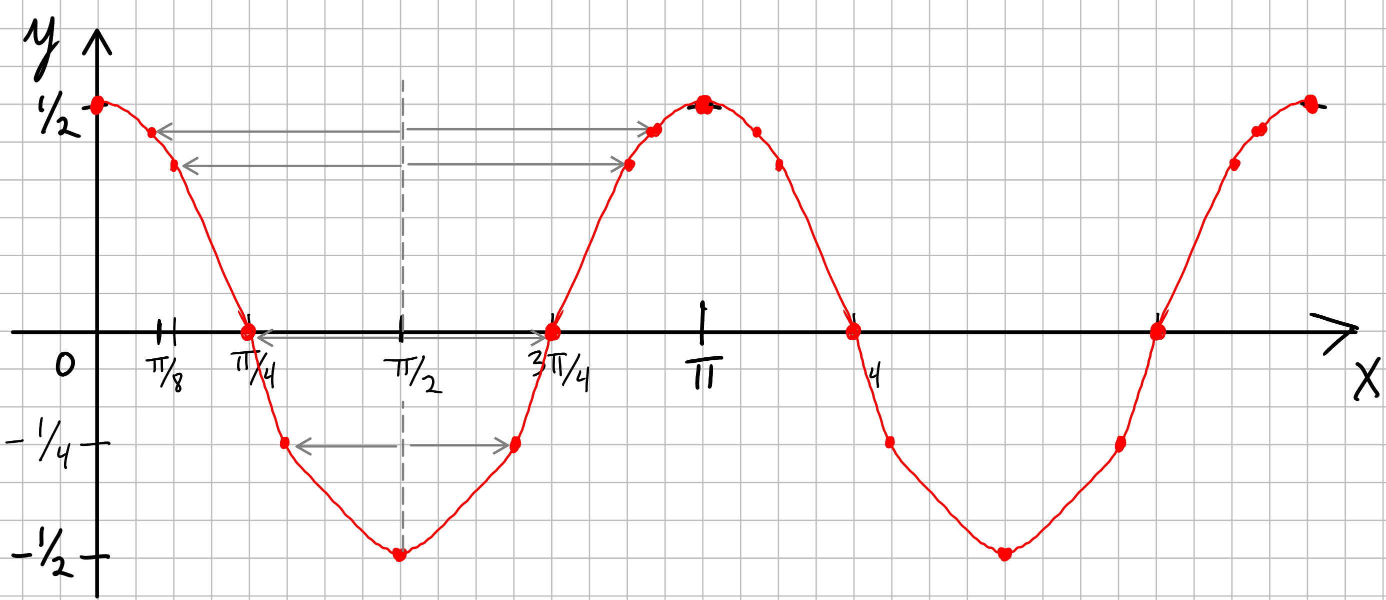
$\boxed{\text{Ex}}$ Graph $y = -1 + \frac{1}{2}\cos(2x - 3\pi)$.

- This is the same as $y = \frac{1}{2}\cos\left(2\left(x - \frac{3\pi}{2}\right)\right) + (-1)$.
- We use Method 2 in the book: graph $y = \frac{1}{2}\cos 2x$.
- We want values of x such that $2x$ is a "special angle":

  $2x = 0\ (0°),\ \frac{\pi}{6}\ (30°),\ \frac{\pi}{4}\ (45°),\ \frac{\pi}{3}\ (60°),\ \frac{\pi}{2}\ (90°)$, etc.

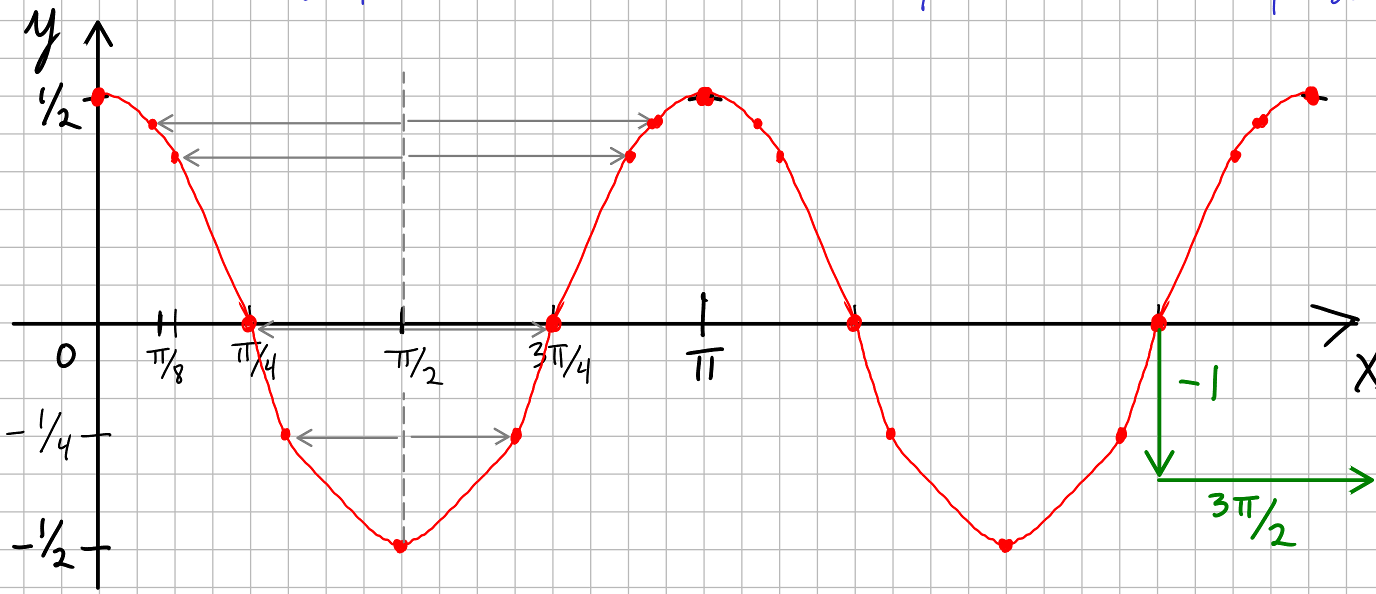
  So: $x = 0, \frac{\pi}{12}, \frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{4}$, etc. up to $x = \pi$ (completing one period)
- Make a table:

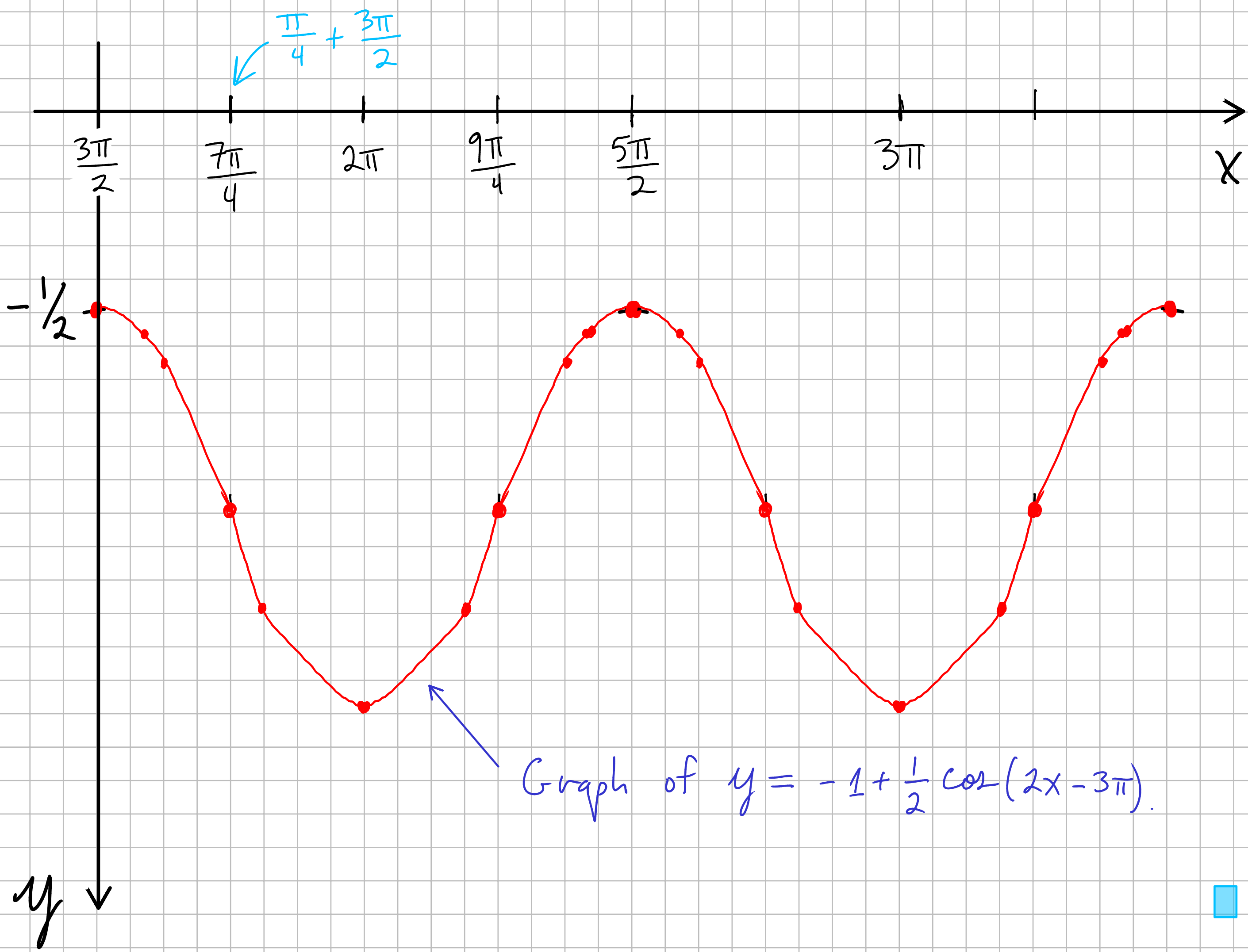
x	calculation of $\frac{1}{2}\cos 2x$	y	(x,y)
0	$\frac{1}{2}\cos(2\cdot 0) = \frac{1}{2}\cos 0 = \frac{1}{2}$	$\frac{1}{2}$	$(0, \frac{1}{2})$
$\pi/12$	$\frac{1}{2}\cos\frac{\pi}{6} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$(\frac{\pi}{12}, \frac{\sqrt{3}}{4})$
$\pi/8$	$\frac{1}{2}\cos\frac{\pi}{4} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$(\frac{\pi}{8}, \frac{\sqrt{2}}{4})$
$\pi/6$	$\frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{4}$	$(\frac{\pi}{6}, \frac{1}{4})$
$\pi/4$	$\frac{1}{2}\cos\frac{\pi}{2} = \frac{1}{2}\cdot 0$	0	$(\pi/4, 0)$
$\pi/3$	$\frac{1}{2}\cos\frac{2\pi}{3} = \frac{1}{2}\cdot\frac{-1}{2}$	$-\frac{1}{4}$	$(\pi/3, -1/4)$
$\pi/2$	$\frac{1}{2}\cos\pi = \frac{1}{2}\cdot(-1)$	$-\frac{1}{2}$	$(\pi/2, -1/2)$
π	$\frac{1}{2}\cos 2\pi = \frac{1}{2}\cdot 1$	$\frac{1}{2}$	$(\pi, 1/2)$



- Now we graph the given function: $y = \frac{1}{2}\cos\left(2\left(x - \frac{3\pi}{2}\right)\right) + (-1)$.

  This is the graph above, but shifted vertically -1 & horizontally $\frac{3\pi}{2}$.





Graph of $y = -1 + \frac{1}{2}\cos(2x - 3\pi)$.

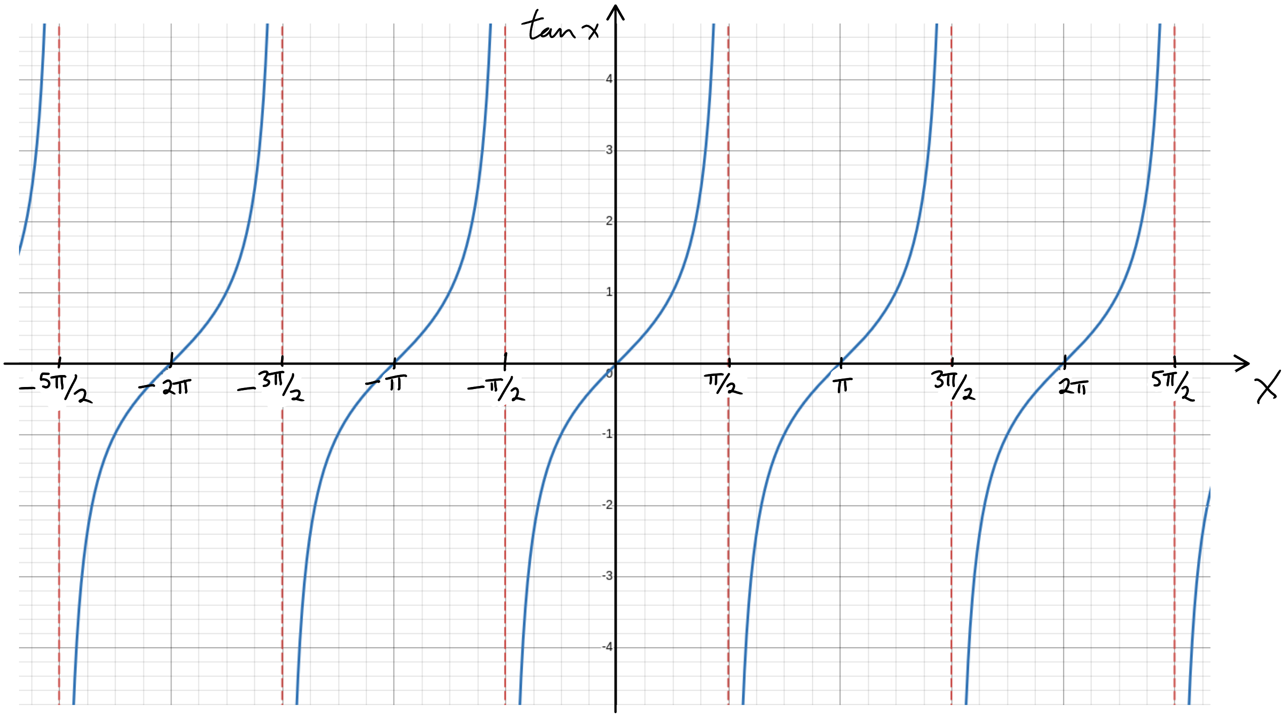
4.3 - Graphs of Tangent & Cotangent

- Since $\tan x = \dfrac{\sin x}{\cos x}$, $\tan x$ is undefined whenever $\cos x = 0$.

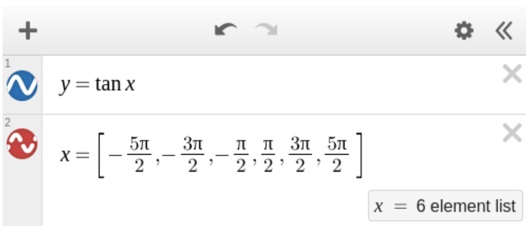
Since $\cos x = 0$ & $\sin x \neq 0$ for $x = \frac{\pi}{2} + n\pi$, where n is any integer, we find the graph of $y = \tan x$ has vertical asymptotes at

$x_n = \frac{\pi}{2} + n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$

- We observe that $x_{n+1} - x_n = \left[\frac{\pi}{2} + (n+1)\pi\right] - \left(\frac{\pi}{2} + n\pi\right) = \pi$, so the asymptotes are spaced π units apart. Since $\tan x$ has period π, this means $\tan x$ runs through 1 cycle between any two neighboring asymptotes.



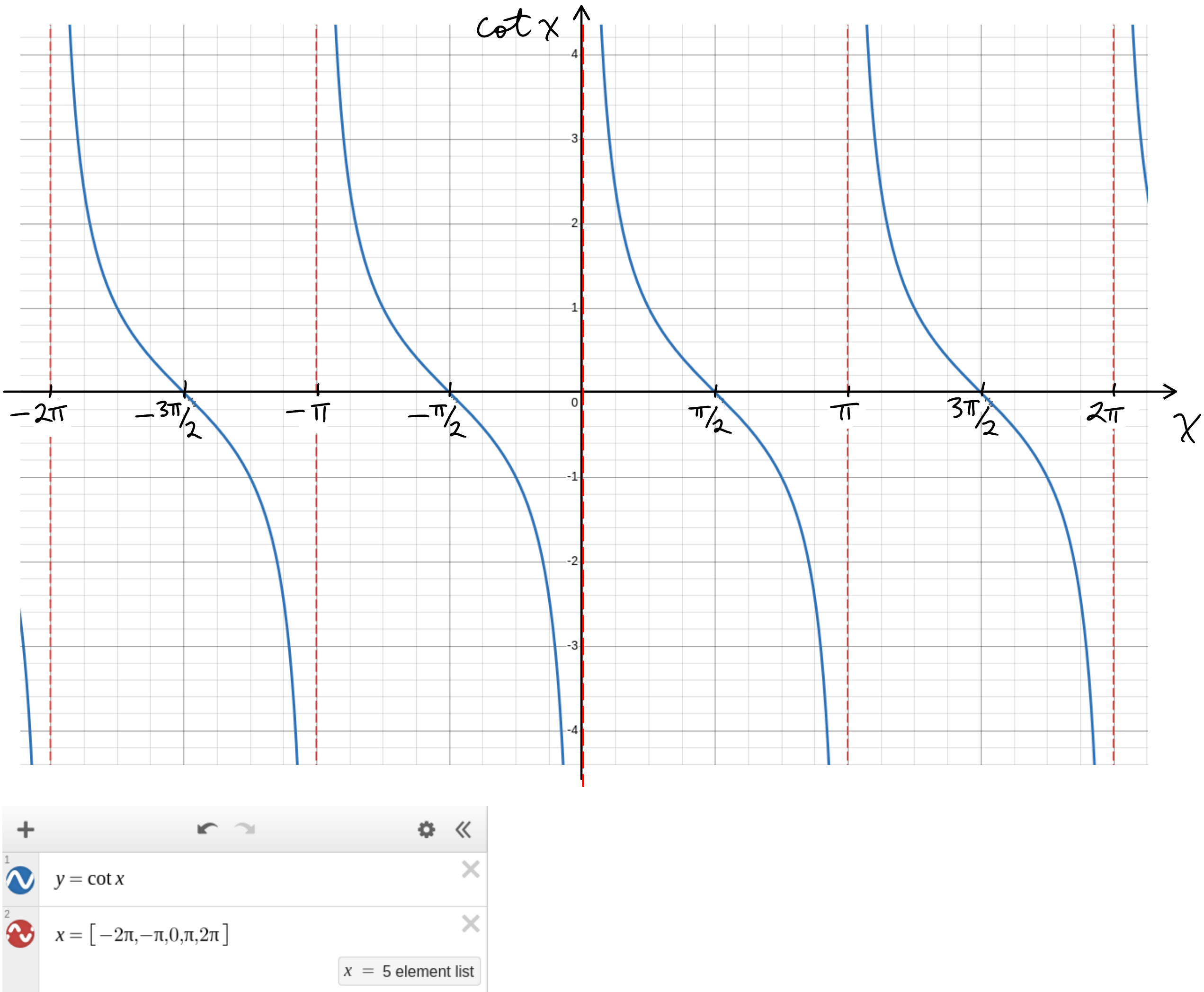
The graph of y=tan(x) above was done using Desmos.



- Since $\cot x = \dfrac{\cos x}{\sin x}$, $\cot x$ is undefined whenever $\sin x = 0$.

Since $\cos x \neq 0$ & $\sin x = 0$ for $x = n\pi$, where n is any integer, we find the graph of $y = \cot x$ has vertical asymptotes at $x_n = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$

Again, asymptotes are spaced π units apart. Since $\cot x$ has period π, this means $\cot x$ runs through 1 cycle between any two neighboring asymptotes.



- The range of both tan(x) and cot(x) is $(-\infty, \infty)$, an unbounded interval, and hence neither function has an amplitude.
- The period of both $y = a\tan x$ & $y = a\cot x$ is π/b for any $a \neq 0$.

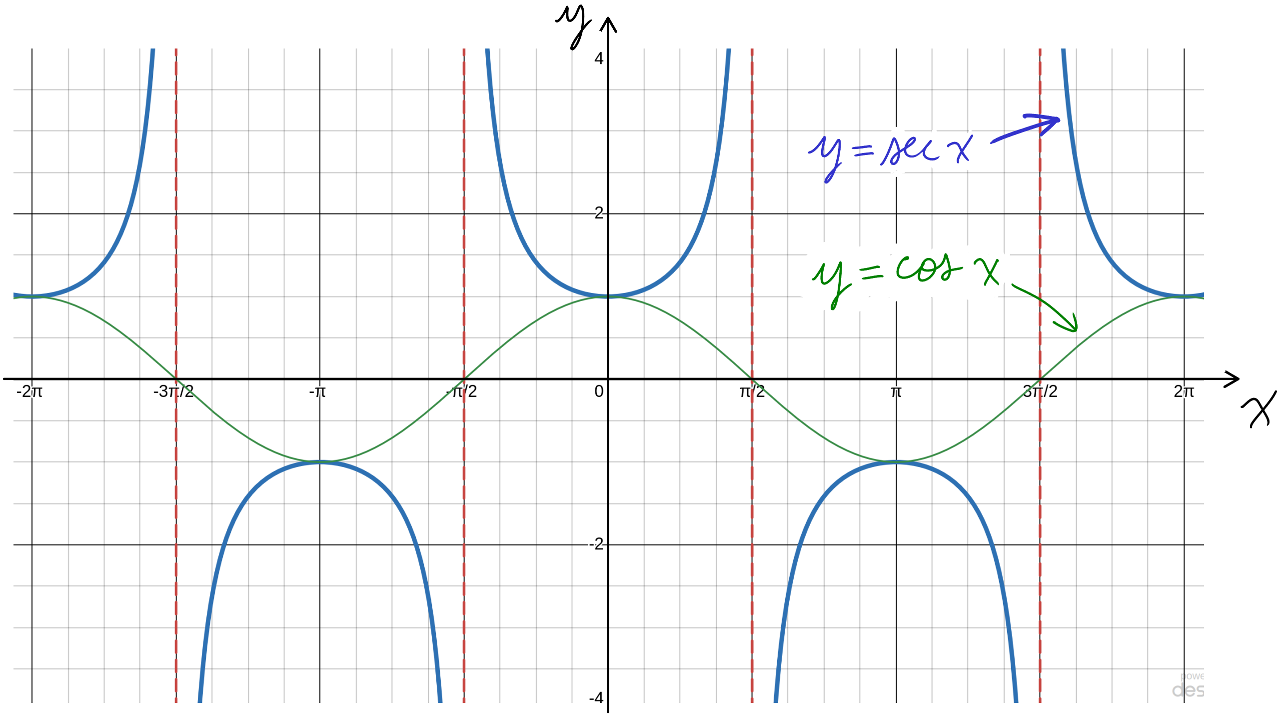
4.4 - Graphs of Secant & Cosecant

- Since $\sec x = \dfrac{1}{\cos x}$, $\sec x$ is undefined whenever $\cos x = 0$. Since $\cos x = 0$ for $x = \frac{\pi}{2} + n\pi$, where n is any integer, we find the graph of $y = \sec x$ has vertical asymptotes at $x_n = \frac{\pi}{2} + n\pi$ for $n = 0, \pm 1, \pm 2, \dots$

- So $y = \sec x$ has the same vertical asymptotes as $y = \tan x$, spaced π units apart.

- But the range of $\sec x$ is not all reals: since $-1 \le \cos x \le 1$, we find that either $\sec x \ge 1$ or $\sec x \le -1$. That is, the range of $\sec x$ is $(-\infty, -1] \cup [1, \infty)$. The range being unbounded, $y = \sec x$ has no amplitude!

- For nonzero constants a & b, the function $y = a\sec(bx)$ has range $(-\infty, -|a|] \cup [|a|, \infty)$ and period $\frac{2\pi}{b}$. The range being unbounded, there is no amplitude.



$\boxed{\text{Ex}}$ Graph $y = \sec\left(2x + \frac{\pi}{2}\right) + 1$ over one period.

- We write $y = \sec\left(2\left(x + \frac{\pi}{4}\right)\right) + 1$ to see that there is a phase shift of $-\frac{\pi}{4}$ & a vertical translation of 1.

- We graph $y = \sec(2x)$ to start, which has period $\frac{2\pi}{2} = \pi$.

- Start by finding vertical asymptotes of $y = \sec(2x)$, which exist wherever $\cos(2x) = 0$. We have $2x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$, etc. (and so $x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$, etc.)

- To graph one period, we can graph $y = \sec(2x)$ for x between $-\frac{\pi}{4}$ & $\frac{3\pi}{4}$ (Note: $3\pi/4 - \pi/4 = \pi$, the period of $\sec(2x)$).

- "Nice" values of x to work with between asymptotes $-\frac{\pi}{4}$ & $\frac{\pi}{4}$:
  We have $-\frac{\pi}{2} < 2x < \frac{\pi}{2}$, so try $2x = -\frac{\pi}{3}, -\frac{\pi}{4}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.
  That is: $x = -\frac{\pi}{6}, -\frac{\pi}{8}, -\frac{\pi}{12}, 0, \frac{\pi}{12}, \frac{\pi}{8}, \frac{\pi}{6}$.

Add π

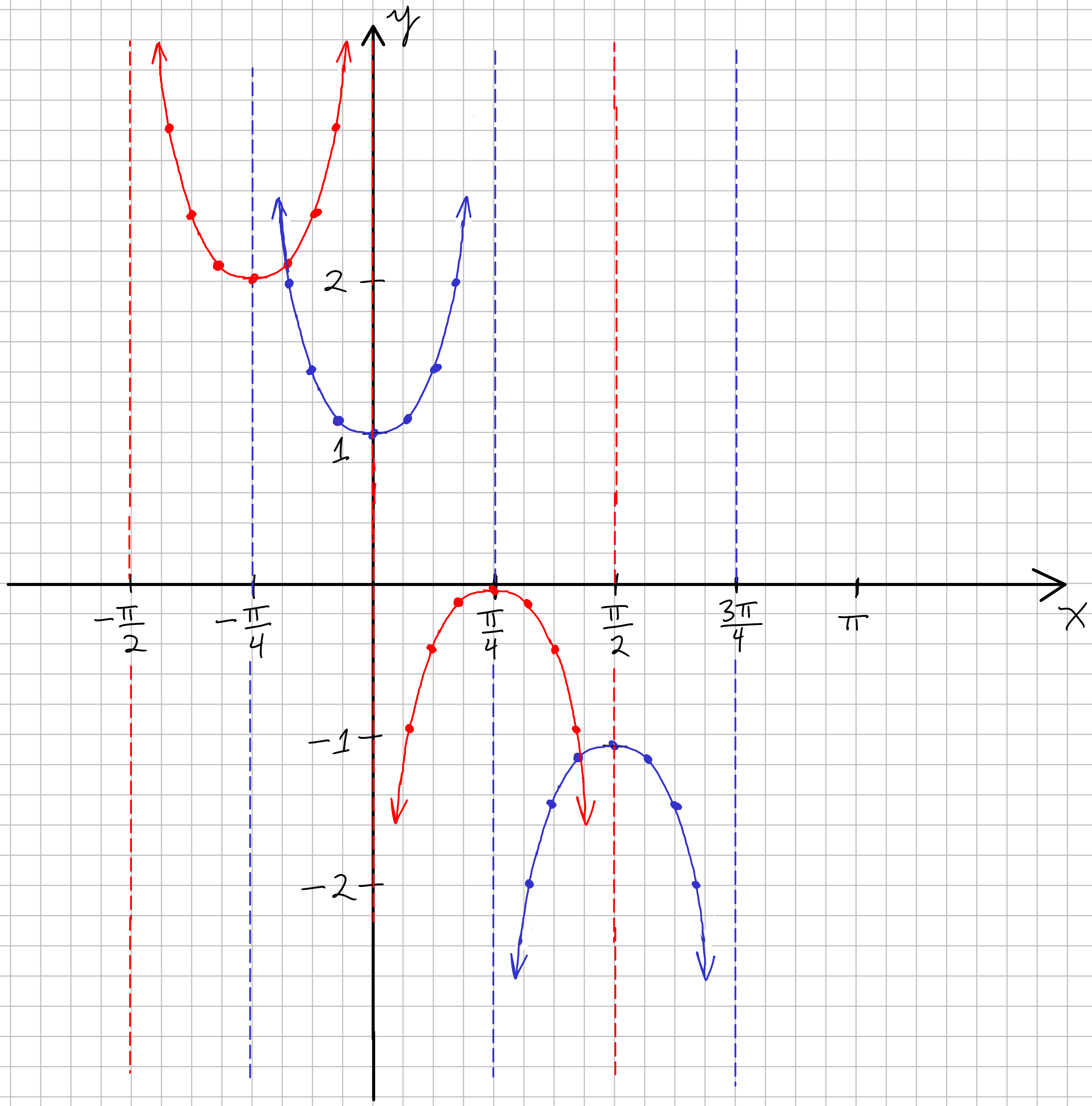
- "Nice" values of x to work with between asymptotes $\frac{\pi}{4}$ & $\frac{3\pi}{4}$:
  We have $\frac{\pi}{2} < 2x < \frac{3\pi}{2}$, so try $2x = \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{9\pi}{4}, \frac{4\pi}{3}$
  That is: $x = \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{7\pi}{12}, \frac{9\pi}{8}, \frac{2\pi}{3}$.

Meaning x s.t. $2x$ is a special angle

x	$\sec(2x)$	y	(x,y)
$-\frac{\pi}{6}$	$\sec\left(-\frac{\pi}{3}\right)$	2	$\left(-\frac{\pi}{6}, 2\right)$
$-\frac{\pi}{8}$	$\sec\left(-\frac{\pi}{4}\right)$	$\sqrt{2}$	$\left(-\frac{\pi}{8}, 1.41\right)$
$-\frac{\pi}{12}$	$\sec\left(-\frac{\pi}{6}\right)$	$\frac{2}{\sqrt{3}}$	$\left(-\frac{\pi}{12}, 1.15\right)$
0	$\sec(0)$	1	$(0, 1)$
$\frac{\pi}{12}$	$\sec\left(\frac{\pi}{6}\right)$	$\frac{2}{\sqrt{3}}$	$\left(\frac{\pi}{12}, 1.15\right)$
$\frac{\pi}{8}$	$\sec\left(\frac{\pi}{4}\right)$	$\sqrt{2}$	$\left(\frac{\pi}{8}, 1.41\right)$
$\frac{\pi}{6}$	$\sec\left(\frac{\pi}{3}\right)$	2	$\left(\frac{\pi}{6}, 2\right)$

x	$\sec 2x$	y	(x,y)
$\frac{\pi}{3}$	$\sec\left(\frac{2\pi}{3}\right)$	-2	$\left(-\frac{\pi}{6}, -2\right)$
$\frac{3\pi}{8}$	$\sec\left(\frac{3\pi}{4}\right)$	$-\sqrt{2}$	$\left(-\frac{\pi}{8}, -1.41\right)$
$\frac{5\pi}{12}$	$\sec\left(\frac{5\pi}{6}\right)$	$-\frac{2}{\sqrt{3}}$	$\left(-\frac{\pi}{12}, -1.15\right)$
$\frac{\pi}{2}$	$\sec(\pi)$	-1	$(0, -1)$
$\frac{7\pi}{12}$	$\sec\left(\frac{7\pi}{6}\right)$	$-\frac{2}{\sqrt{3}}$	$\left(\frac{\pi}{12}, -1.15\right)$
$\frac{9\pi}{8}$	$\sec\left(\frac{9\pi}{4}\right)$	$-\sqrt{2}$	$\left(\frac{\pi}{8}, -1.41\right)$
$\frac{2\pi}{3}$	$\sec\left(\frac{4\pi}{3}\right)$	-2	$\left(\frac{\pi}{6}, -2\right)$

The graph of $y = \sec(2x)$ is in blue below. We shift this graph left by $\pi/4$ & up by 1 to get the graph of $y = \sec\left(2x + \frac{\pi}{2}\right) + 1$ in red. The asymptotes likewise shift.



- Since $\csc x = \dfrac{1}{\sin x}$, $\csc x$ is undefined whenever $\sin x = 0$. Since $\sin x = 0$ for $x = n\pi$, where n is any integer, we find the graph of $y = \csc x$ has vertical asymptotes at $x_n = n\pi$ for $n = 0, \pm 1, \pm 2, \dots$

- So $y = \csc x$ has the same vertical asymptotes as $y = \cot x$, spaced π units apart.

- But the range of $\csc x$ is not all reals: since $-1 \le \sin x \le 1$, we find that either $\csc x \ge 1$ or $\csc x \le -1$. That is, the range of $\csc x$ is $(-\infty, -1] \cup [1, \infty)$. The range being unbounded, $y = \csc x$ has no amplitude!

- For nonzero constants a & b, the function $y = a\csc(bx)$ has range $(-\infty, -|a|] \cup [|a|, \infty)$ and period $\dfrac{2\pi}{b}$. The range being unbounded, there is no amplitude.