Math 121 Exam #3 Key (Summer 2024)

1 For any $x \in \mathbb{R}$ we have $|x - x| = 0 \leq 1$, so that xRx, and hence R is **reflexive**.

For $x, y \in \mathbb{R}$ suppose xRy, so that $|x - y| \leq 1$. Since $|x - y| = |y - x|$ we have $|y - x| \leq 1$ also, and thus yRx. R is **symmetric**.

Let $x = 2$, $y = 1$, $z = 0$. Then $|x - y| = 1$ and $|y - z| = 1$, so that xRy and yRz. However, $|x - z| = 2 > 1$, and thus $x \not R z$. R is **not transitive**.

2 For $S \in \mathcal{P}(S)$ we have $S \cap S = S \neq \varnothing$, so that $(S, S) \notin \mathcal{R}$ and $\mathcal{R}$ is **not reflexive**.

Suppose $(X, Y) \in \mathcal{R}$, so that $X \cap Y = \varnothing$. Then $Y \cap X = \varnothing$ also, which implies $(Y, X) \in \mathcal{R}$ and thus $\mathcal{R}$ is **symmetric**.

Let $X = \{a, b\}$, $Y = \{c, d\}$, and $Z = \{a\}$. Then $X \cap Y = \varnothing$ (so $(X, Y) \in \mathcal{R}$) and $Y \cap Z = \varnothing$ (so $(Y, Z) \in \mathcal{R}$), yet $X \cap Z \neq \varnothing$ (so $(X, Z) \notin \mathcal{R}$). Hence $\mathcal{R}$ is **not transitive**.

3a Trouble will be encountered when trying to verify transitivity, and then only one strategy is left: solve the equation $\frac{a+b}{2} = \sqrt{ab}$ for a (or b), and see if what results is simpler to work with. We solve for a:

$$a + b = 2\sqrt{ab} \quad \longrightarrow \quad (a + b)^2 = 4ab \quad \longrightarrow \quad a^2 - 2ab + b^2 = 0 \quad \longrightarrow \quad (a - b)^2 = 0 \quad \longrightarrow \quad a = b.$$

This shows that $\frac{a+b}{2} = \sqrt{ab}$ is equivalent to $a = b$, and so relation R is such that aRb if $a = b$. Thus R is merely the equality relation $=$, which is known to be an equivalence relation and so we are done.

3b For $a \in \mathbb{R}^+$ we have $[a] = \{x \in \mathbb{R}^+ : xRa\} = \{x \in \mathbb{R}^+ : x = a\} = \{a\}$. That is, each positive real number is in an equivalence class by itself.

4 Let $y \in f(A_1 \cup A_2)$. Then $y = f(x)$ for some $x \in A_1 \cup A_2$. If $x \in A_1$, then $y \in f(A_1)$; and if $x \in A_2$, then $y \in f(A_2)$. Therefore $y \in f(A_1) \cup f(A_2)$ and it's established that $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$.

Now suppose $y \in f(A_1) \cup f(A_2)$. If $y \in f(A_1)$, then $y = f(x)$ for some $x \in A_1$, and since $x \in A_1 \cup A_2$ it follows that $y \in f(A_1 \cup A_2)$. If $y \in f(A_2)$, then $y = f(x)$ for some $x \in A_2$, and since $x \in A_1 \cup A_2$ it follows that $y \in f(A_1 \cup A_2)$. Therefore $y \in f(A_1 \cup A_2)$ and it's established that $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.

5 We disprove that f is a function from S to S by noting that $(-1, 1) \in f$ and $(-1, -1) \in f$.

6 For any $n \in \mathbb{Z}$ we find that $f(n) = n$, so f is onto. However, f is not one-to-one since, for instance, $f(0.1) = f(0.2) = 1$.

7 Let $y \in \mathbb{R}$. Solving $y = 4 - x^5$ for x yields $x = \sqrt[5]{4 - y}$, which certainly is a real number, and since

$$f\left(\sqrt[5]{4 - y}\right) = 4 - \left(\sqrt[5]{4 - y}\right)^5 = 4 - (4 - y) = y,$$

it follows that f is onto.

Next, suppose that $f(a) = f(b)$. Then $4 - a^5 = 4 - b^5$, implying $a^5 = b^5$, and finally $a = b$. We conclude that f is one-to-one, and therefore a bijection. For $x \in \mathbb{R}$ we have $f^{-1}(x) = \sqrt[5]{4-x}$.

8 Define $f : E \to O$ by $f(n) = n + 1$ for each $n \in E$. Certainly f is one-to-one, and since for any $k \in O$ we have $k - 1 \in E$ such that $f(k-1) = k$, we see f is also onto and hence a bijection. Therefore $|E| = |O|$.

9 Suppose $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. Then $a - b = kn$ and $b - c = \ell n$ for some $k, \ell \in \mathbb{Z}$. Now, $a - c = (a - b) + (b - c) = kn + \ell n = (k + \ell)n$ for $k + \ell \in \mathbb{Z}$, and therefore $a \equiv c \pmod{n}$.

10 $2^6 + 2^6 = 2^7 = 128$

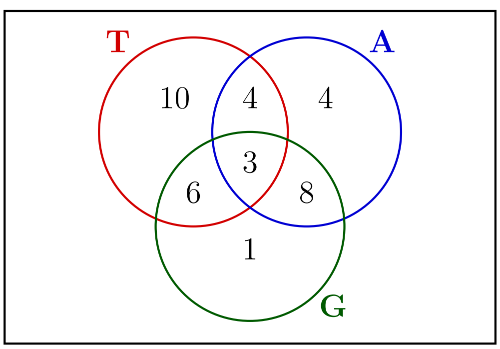
11 $2(26^3 \cdot 10^3) = 35,152,000$

12a Let T, A, and G be the set of those taking topology, abstract algebra, and graph theory, respectively. We have: $|T| = 23$, $|A| = 19$, $|G| = 18$, $|T \cap A| = 7$, $|T \cap G| = 9$, $|A \cap G| = 11$, and $|T \cup A \cup G| = 36$. By the principle of inclusion-exclusion,

$$|T \cap A \cap G| = |T \cup A \cup G| + |T \cap A| + |T \cap G| + |A \cap G| - |T| - |A| - |G| = 3.$$

That is, exactly 3 plan to take all three courses.

12b We can unravel the mystery by filling out a Venn diagram starting by placing a 3 in the $T \cap A \cap G$ region. The answer is $10 + 4 + 1 = 15$.



12c $4 + 6 + 8 = 18$

13 We need to find the minimum number of marbles that must be drawn to be assured of getting 9 red marbles *or* 9 blue marbles *or* 9 yellow marbles. Consider the worst case scenario: drawing marble after marble until 8 of each color is in hand. At this point 24 marbles have been drawn, and so the 25th marble drawn is guaranteed to result in having 9 of one color in hand. The general pigeonhole principle gives precisely this answer: at least **25** marbles must be drawn.