Math 121 Exam #2 Key (Summer 2024)

1a There exists an irrational number x such that, for any rational number y, $xy \neq 0$.

1b For every odd integer m there exists an integer n such that mn is even.

2a Let E be the set of even integers, and O the set of odd integers. Then:

$$\exists p \in E\, \exists q \in O[(p+8)^2 + (q-5)^2 = 0].$$

2b $\forall p \in E\, \forall q \in O[(p+8)^2 + (q-5)^2 \neq 0]$.

2c For every even integer p and odd integer q, $(p+8)^2 + (q-5)^2 \neq 0$.

3 Let $a, b \in \mathbb{Z}$ be arbitrary. Suppose a and b are odd. Then there exist $k, \ell \in \mathbb{Z}$ such that $a = 2k+1$ and $b = 2\ell + 1$. Now,

$$ab + a + b = (2k+1)(2\ell+1) + (2k+1) + (2\ell+1) = 2(2k\ell + 2k + 2\ell + 1) + 1,$$

and since $2k\ell + 2k + 2\ell + 1$ is an integer, we conclude that $ab + a + b$ is odd.

4 The contrapositive is "If $k < 3n+1$ and $m < 2n+1$, then $2k + 3m < 12n + 1$. Proof follows.

Suppose $k < 3n+1$ and $m < 2n+1$. Since k and m are integers, it follows that $k \leq 3n$ and $m \leq 2n$, and thus $2k \leq 6n$ and $3m \leq 6n$. Now we find that $2k + 3m \leq 6n + 6n = 12n$, and therefore $2k + 3m < 12n + 1$.

5 Let $x \in A \cap (B \cup C)$, so that $x \in A$ and $x \in B \cup C$. There are two cases: either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. *Case 1:* Suppose $x \in A$ and $x \in B$. Then $x \in A \cap B$, and hence $x \in (A \cap B) \cup (A \cap C)$. *Case 2:* Suppose $x \in A$ and $x \in C$. Then $x \in A \cap C$, and hence $x \in (A \cap B) \cup (A \cap C)$. In either case we conclude that $x \in (A \cap B) \cup (A \cap C)$, and therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. There are two cases: either $x \in A \cap B$ or $x \in A \cap C$. *Case 1:* Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$, and since the latter implies that $x \in B \cup C$, we find $x \in A \cap (B \cup C)$. *Case 2:* Suppose $x \in A \cap C$. Then $x \in A$ and $x \in C$, and since the latter implies that $x \in B \cup C$, we find $x \in A \cap (B \cup C)$. In either case we conclude that $x \in A \cap (B \cup C)$, and therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

6a Disproof: The integer 1 cannot be expressed as the sum of two positive integers.

6b Disproof: Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$ and $C = \{2, 3\}$. Then $A \cup B = A \cup C$ (both equal A), and yet $B \neq C$.

7 Let $m = 1$ and $n = 3$. Then m and n have the same parity (they are both odd), and the given equation is satisfied.

8 Suppose u and v are positive real numbers. Suppose $\sqrt{u}+\sqrt{v} = \sqrt{u+v}$. Then $(\sqrt{u}+\sqrt{v})^2 = (\sqrt{u+v})^2$, which implies $u + 2\sqrt{uv} + v = u + v$, and hence $2\sqrt{uv} = 0$. From this we obtain $uv = 0$, so that either $u = 0$ or $v = 0$. Since either option contradicts the assumption that u and v are positive, we conclude that $\sqrt{u} + \sqrt{v} \neq \sqrt{u+v}$.

9 Define $P(n):\ \sum_{k=1}^{n}(3k-1) = n(3n+1)/2$. We show $\forall n \in \mathbb{N}\,[P(n)]$ by induction.

$P(1)$ states $2 = 2$, which is true, and so the basis step is established.

Let $n \in \mathbb{N}$ be arbitrary, and suppose $P(n)$ is true. Now,

$$\sum_{k=1}^{n+1}(3k-1) = \sum_{k=1}^{n}(3k-1) + [3(n+1)-1] = \frac{n(3n+1)}{2} + (3n+2) = \frac{(n+1)(3n+4)}{2},$$

which shows $P(n+1)$ to be true. Therefore $\forall n \in \mathbb{N}\,[P(n)]$.

10 Define $P(n):\ n^2 > n+1$. We show $\forall n \geq 2\,[P(n)]$ by induction.

$P(2)$ states that $2^2 > 2+1$, which is true, and establishes the basis step.

Let $n \geq 2$ be arbitrary, and suppose $P(n)$ is true. Then

$$(n+1)^2 = n^2 + 2n + 1 > (n+1) + 2n + 1 = 3n + 2 > n + 2 = (n+1) + 1,$$

which shows that $P(n+1)$ is true. Therefore $\forall n \geq 2\,[P(n)]$.

11a $a_2 = 7$, $a_3 = 15$, $a_4 = 31$, $a_5 = 63$.

11b It appears $a_n = 2^{n+1} - 1$ for $n \geq 1$. Define $P(n):\ a_n = 2^{n+1} - 1$. We show by induction that $\forall n \geq 1[P(n)]$.

$P(1)$ states that $a_1 = 2^{1+1} - 1 = 3$, which is true, and establishes the basis step.

Let $n \geq 1$, and suppose $P(n)$ is true. Now, using the given recurrence relation as well as $P(n)$, we have

$$a_{n+1} = 2a_n + 1 = 2(2^{n+1} - 1) + 1 = (2^{n+2} - 2) + 1 = 2^{n+2} - 1,$$

which shows $P(n+1)$ to be true. Therefore $\forall n \geq 1(a_n = 2^{n+1} - 1)$.