Math 121 Exam #4 Key (Summer 2023)

1a $R_2 - R_1 = \{(a,b) \in \mathbb{R}^2 : a = b\} = R_5$.

1b $R_4 \oplus R_6 = (R_4 - R_6) \cup (R_6 - R_4) = \{(a,b) : a = b\} \cup \{(a,b) : a > b\} = R_5 \cup R_1$.

1c All relations R_k are subsets of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. So

$$\begin{aligned} R_1 \circ R_5 &= \{(a,c) \in \mathbb{R}^2 : \exists b \in \mathbb{R}[(a,b) \in R_5 \wedge (b,c) \in R_1]\} \\ &= \{(a,c) \in \mathbb{R}^2 : \exists b \in \mathbb{R}[(a = b) \wedge (b > c)]\} \\ &= \{(a,c) \in \mathbb{R}^2 : a > c\} = R_1. \end{aligned}$$

1d Whenever $a < c$ there exists some $b \in \mathbb{R}$ such that $a < b < c$, so

$$\begin{aligned} R_3 \circ R_3 &= \{(a,c) \in \mathbb{R}^2 : \exists b \in \mathbb{R}[(a,b) \in R_3 \wedge (b,c) \in R_3]\} \\ &= \{(a,c) \in \mathbb{R}^2 : \exists b \in \mathbb{R}[(a < b) \wedge (b < c)]\} \\ &= \{(a,c) \in \mathbb{R}^2 : a < c\} = R_3. \end{aligned}$$

2 This will be done with induction. In the case when $n = 1$ we find $R^n = R$ becomes $R = R$, which is trivially true.

Let $n \in \mathbb{Z}^+$ be arbitrary, and suppose that $R^n = R$. Then $R^{n+1} = R^n \circ R = R \circ R$. We show that $R \circ R = R$. Suppose $(x,y) \in R$. Since R is reflexive we have $(x,x) \in R$ as well, so there exists some z (namely $z = x$) such that $(x,z) \in R$ and $(z,y) \in R$, and thus $(x,y) \in R \circ R$. This establishes that $R \subseteq R \circ R$.

Next suppose $(x,y) \in R \circ R$. Then there exists z such that $(x,z) \in R$ and $(z,y) \in R$, and hence $(x,y) \in R$ since R is transitive. This establishes that $R \circ R \subseteq R$. Therefore $R \circ R = R$, and with our inductive hypothesis we obtain $R^{n+1} = R^n \circ R = R \circ R = R$.

3a We have

$$\mathbf{M}_{R_2 \circ R_1} = \mathbf{M}_{R_1} \odot \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

3b $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1) = (R_1 \cap \overline{R}_2) \cup (R_2 \cap \overline{R}_1)$, so with

$$\mathbf{M}_{\overline{R}_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{\overline{R}_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we obtain

$$\mathbf{M}_{R_1 \oplus R_2} = (\mathbf{M}_{R_1} \wedge \mathbf{M}_{\overline{R}_2}) \vee (\mathbf{M}_{R_2} \wedge \mathbf{M}_{\overline{R}_1}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

4 Let A/R (read as "A modulo R") be the set of all equivalence classes with respect to R; that is, $A/R = \{[x]_R : x \in A\}$. We now define a function $f : A \to A/R$ by $f(x) = [x]_R$. Observe that $f(x) = f(y)$ implies $[x]_R = [y]_R$, which in turn by a theorem in §9.5 of the text implies $(x, y) \in R$. Conversely, by the same theorem, $(x, y) \in R$ implies $[x]_R = [y]_R$ implies $f(x) = f(y)$.

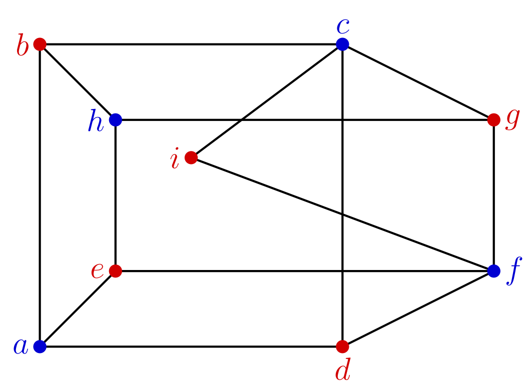
5a $[4]_6 = \{4 + 6n : n \in \mathbb{Z}\} = \{\dots, -14, -8, -2, 4, 10, 16, 22, \dots\}$.

5b $[-5]_6 = \{-5 + 6n : n \in \mathbb{Z}\} = \{\dots, -23, -17, -11, -5, 1, 7, 13, \dots\}$.

6 So relation R on V is such that $(u, v) \in R$ iff $\{u, v\} \in E$. Let $(u, v) \in R$. Then $\{v, u\} = \{u, v\} \in R$, which immediately implies that $(v, u) \in R$ and hence R is symmetric. We also find that $\{v, v\} \notin E$ for all $v \in V$ since G is a simple graph and so has no loops, which implies that $(v, v) \notin R$ for all $v \in V$ and therefore R irreflexive.

7 Let $G = (V, E)$ be a simple graph with $|V| = n \geq 2$. Each vertex $v \in V$ may be adjacent to any combination of the other $n - 1$ vertices, including all of them or none of them. Since G is a simple graph, this means that each vertex $v \in V$ must have $0 \leq \deg(v) \leq n - 1$. If no vertex is isolated, then in fact we have $1 \leq \deg(v) \leq n - 1$ for each of the n choices for $v \in V$, and the Pigeonhole Principle implies that there must exist two vertices with the same degree. If at least two vertices are isolated, then there are at least two vertices with degree zero and hence have the same degree. Finally, to have precisely one isolated vertex v_0 requires $n \geq 3$ (why?), in which case we may carry out the same argument for the subgraph $H = (V - \{v_0\}, E)$ that we earlier made for G on the assumption that G has no isolated vertices; then, because the degrees of the vertices of H equal the degrees of the corresponding (nonisolated) vertices of G, we again conclude that G has at least two vertices of the same degree.

8 Coloring the vertices with two colors, we obtain a bipartition (V_1, V_2) with $V_1 = \{a, c, f, h\}$ and $V_2 = \{b, d, e, g, i\}$ (or vice-versa).

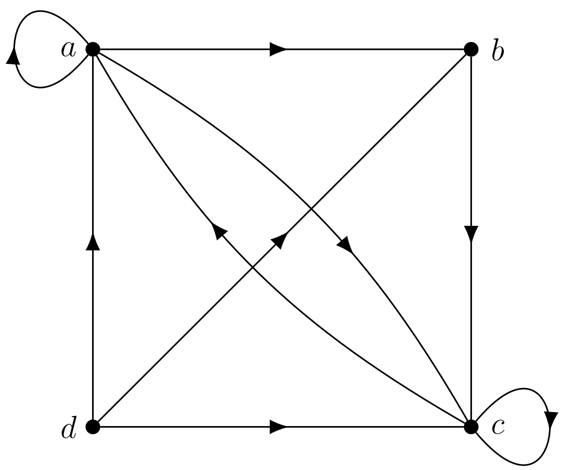


9 Since $G = (V, E)$ is directed and without multiple edges, the relation associated with G is simply the set E, which is a relation on V with $(u, v) \in E$ iff u is adjacent to v.

Suppose $G^c = (V, F)$ is such that $G^c = G$, so that $E = F$. Let $(u, v) \in E$. Then $(v, u) \in F$, and since $E = F$ it follows that $(v, u) \in E$. Thus for all $u, v \in V$, $(v, u) \in E$ whenever $(u, v) \in E$, and therefore E (the relation associated with G) is symmetric.

Now suppose that E, the relation associated with G, is symmetric. Suppose $u, v \in V$ are such that $(u, v) \in E$. Then $(v, u) \in E$ also, which implies that $(u, v) \in F$ and hence $E \subseteq F$. Similarly, if $(u, v) \in F$, then $(v, u) \in E$, implying that $(u, v) \in E$ and hence $F \subseteq E$. Therefore $E = F$ and we conclude that $G^c = G$.

10 Because the matrix is not symmetric the graph must be directed:



11 Arranging the vertices into a triangle is not necessary (they could be collinear), but it renders the graph easier to look at: