Math 121 Exam #3 Key (Summer 2023)

1 To prove:

$$\forall n \geq 1\left(\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2\right).$$

Let $P(n)$ denote the equation in parentheses. $P(1)$ is equivalent to $2 = 2$, so the base case is affirmed. Now let $n \geq 1$ and suppose $P(n)$ is true. Using this inductive hypothesis, we find that

$$\sum_{k=1}^{n+1} k2^k = \sum_{k=1}^{n} k2^k + (n+1)2^{n+1} = (n-1)2^{n+1} + 2 + (n+1)2^{n+1} = [(n+1)-1]2^{(n+1)+1} + 2.$$

Thus $P(n+1)$ is true, and the proof by induction is done.

2 To prove: $\forall n \geq 1(3 \mid n^3 + 2n)$. Let $P(n)$ denote $3 \mid n^3 + 2n$. Since $P(1)$ states simply that $3 \mid 3$, which is true, we see the base case holds. Let $n \geq 1$ and suppose $P(n)$, so that $n^3 + 2n = 3k$ for some $k \in \mathbb{Z}$. Now,

$$(n+1)^3 + 2(n+1) = (n^3 + 2n) + (3n^2 + 3n + 3) = 3k + 3(n^2 + n + 1) = 3(n^2 + n + k + 1),$$

which shows $(n+1)^3 + 2(n+1)$ to be a multiple of 3, and hence $P(n+1)$ is true. The proof by induction is done.

3 Letting $P(n)$ be $a_n \leq (\frac{5}{2})^n$, we prove $\forall n \geq 0[P(n)]$ with strong induction. First, $P(0)$ is $1 \leq 1$, which being true affirms the base case.

Let $n \geq 0$, and suppose $P(k)$ (i.e. $a_k \leq (\frac{5}{2})^k$) for $0 \leq k \leq n$. To show is $P(n+1)$. We're given $a_{n+1} = 2a_n + a_{n-1}$, but a_{n-1} becomes the undefined oddity a_{-1} if $n = 0$, so the $n = 0$ case must be investigated separately. When $n = 0$ we have $P(n+1) = P(1)$, which is $a_1 \leq \frac{5}{2}$; and since we're given $a_1 = 2$, it's clear $P(1)$ is true. We henceforth assume $n \geq 1$. Using our inductive hypothesis,

$$a_{n+1} = 2a_n + a_{n-1} \leq 2(\tfrac{5}{2})^n + (\tfrac{5}{2})^{n-1} = 6(\tfrac{5}{2})^{n-1} \leq (\tfrac{5}{2})^2(\tfrac{5}{2})^{n-1} = (\tfrac{5}{2})^{n+1},$$

which shows $P(n+1)$, and the strong induction proof is done.

4 Let $A = \{a, b, \ldots, j\}$ and $B = \{2, 4, 6, 8\}$. There are $\mathbf{4^{10}}$ different functions $f : A \to B$ possible: for each of the 10 values of $x \in A$ any one of 4 values $y \in B$ may be chosen to have $f(x) = y$. Use the product rule.

5 Let $S = \{1000, 1001, \ldots, 9999\}$. We first find the number of integers in S that *are* divisible by 3 or 13. Let

$$D_3 = \{n \in S : 3 \mid n\}, \quad D_{13} = \{n \in S : 13 \mid n\} \quad D_{39} = \{n \in S : 39 \mid n\}.$$

Then

$$D_3 = \{3k \in S : k \in \mathbb{Z}\} = \{3k : 1000 \leq 3k \leq 9999\} = \{3k : 334 \leq k \leq 3333\},$$

since k is an integer, and similarly

$$D_{13} = \{13k : 1000 \leq 13k \leq 9999\} = \{13k : 77 \leq k \leq 769\},$$
$$D_{39} = \{39k : 1000 \leq 39k \leq 9999\} = \{13k : 26 \leq k \leq 256\}.$$

The number of integers in S that are divisible by 3 or 13 is

$$|D_3 \cup D_{13}| = |D_3| + |D_{13}| - |D_3 \cap D_{13}| = |D_3| + |D_{13}| - |D_{39}| = 3000 + 693 - 231 = 3462.$$

Therefore $|S| - |D_3 \cup D_{13}| = 9000 - 3462 = \mathbf{5538}$ integers in S are not divisible by 3 or 13.

6 Let A be a set of any $d+1$ integers. The Division Algorithm implies that for any $a \in A$, the division a/d has remainder $0 \le r \le d-1$, and so only d distinct remainders are possible. Thus we have $d+1$ "objects" (i.e. integers in A), and we may think of each object as being placed into d "boxes" numbered 0 through $d-1$ in the following way: object $a \in A$ is placed into box $0 \le r \le d-1$ if and only if the division a/d has remainder r. According to the Generalized Pigeonhole Principle there is at least one box containing at least $\lceil \frac{d+1}{d} \rceil = 2$ objects. That is, there are two integers in A with the same remainder when divided by d.

7 Leting X=AB and Y=FGI, we find the number of permutations possible for XCDEHY. Since there are 6 symbols here, the answer is $6! = \mathbf{720}$.

8 We assume the ferrets and gerbils are distinguishable! So there are ferrets f_1, f_2, f_3 and gerbils $g_1, g_2, g_3, g_4, g_5, g_6$. Let F denote the three ferrets grouped together (so F is essentially a set). We first find all the ways to permute the seven objects $F, g_1, g_2, g_3, g_4, g_5, g_6$. There are 7! ways to do this. But for each of these 7! ways, the ferrets themselves can be permuted 3! ways. Thus the answer is $7! \cdot 3! = \mathbf{30{,}240}$.

9 Since order does not matter: $C(45,3) \cdot C(57,4) \cdot C(69,5) \approx \mathbf{6.29940 \times 10^{16}}$.

10 There are 13 kinds: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. We figure out how many 5-card hands exist that have 5 different kinds: choose 5 of 13 kinds ($C(13,5)$ ways), and for each of the 5 kinds chosen select 1 of 4 suits (4^5 ways). Total number of hands: $4^5 \cdot C(13,5)$. The total number of 5-card poker hands is $C(52,5)$, so the probability is

$$\frac{4^5 \cdot C(13,5)}{C(52,5)} \approx \mathbf{0.507}.$$

11 Let $S = \{1, 2, \dots, 3500\}$, and define

$$D_7 = \{n \in S : 7 \mid n\}, \quad D_{11} = \{n \in S : 11 \mid n\} \quad D_{77} = \{n \in S : 77 \mid n\}.$$

Then

$$D_7 = \{7k \in S : k \in \mathbb{Z}\} = \{7k : 1 \le 7k \le 3500\} = \{7k : 1 \le k \le 500\},$$

since k is an integer, and similarly

$$D_{11} = \{11k : 1 \le 11k \le 3500\} = \{11k : 1 \le k \le 318\},$$
$$D_{77} = \{77k : 1 \le 77k \le 3500\} = \{77k : 1 \le k \le 45\}.$$

The number of integers in S that are divisible by 7 or 11 is

$$|D_7 \cup D_{11}| = |D_7| + |D_{11}| - |D_7 \cap D_{11}| = |D_7| + |D_{11}| - |D_{77}| = 500 + 318 - 45 = 773.$$

The probability is $\frac{773}{3500} \approx \mathbf{0.221}$.

12 Let P, C, R be the sets of those who like parsnips, carrots, radishes. We have $|P| = 64$, $|C| = 94$, $|R| = 58$, $|P \cap C| = 26$, $|P \cap R| = 28$, $|C \cap R| = 22$, and $|P \cap C \cap R| = 14$. Now,

$$\begin{aligned} |P \cup C \cup R| &= |P| + |C| + |R| - |P \cap C| - |P \cap R| - |C \cap R| + |P \cap C \cap R| \\ &= 64 + 94 + 58 - 26 - 28 - 22 + 14 = 154 \end{aligned}$$

is the number of professors who like at least one of the vegetables, so there are $270 - 154 = \mathbf{116}$ who like none of them.