Math 121 Exam #2 Key (Summer 2023)

1 Let $A, B, C \neq \varnothing$, and suppose $A \times B = A \times C$. Let $x \in B$. Then $(a, x) \in A \times B$ for some $a \in A$, so that $(a, x) \in A \times C$ and hence $x \in C$. This shows $B \subseteq C$. The same argument with the roles of B and C reversed (what's often called a "symmetrical argument") leads to the conclusion that $C \subseteq B$, and therefore $B = C$.

2 $A \cup B = \{a, b, c, d, e, f\}$, $A \cap B = \{c, d, e\}$, $A - B = \{b\}$, $B - A = \{a, f\}$.

3a Suppose $x \in B - A$. This immediately implies that $x \notin A$, and hence $x \notin A \cap (B - A)$. Therefore $A \cap (B - A) = \varnothing$.

3b Let $x \in (B - A) \cup (C - A)$. Then either $x \in B - A$ or $x \in C - A$, and so either $x \in B$ and $x \notin A$, or $x \in C$ and $x \notin A$. That is, $x \in B$ or $x \in C$, with $x \notin A$ in either case, and hence $x \in B \cup C$ with $x \notin A$. It follows that $x \in (B \cup C) - A$, and therefore $(B - A) \cup (C - A) \subseteq (B \cup C) - A$.

Now let $x \in (B \cup C) - A$, so that $x \in B \cup C$ and $x \notin A$. Then either $x \in B$ and $x \notin A$, or $x \in C$ and $x \notin A$; that is, either $x \in B - A$ or $x \in C - A$, so that $x \in (B - A) \cup (C - A)$, and therefore $(B \cup C) - A \subseteq (B - A) \cup (C - A)$.

4 Suppose $f(x)$ is strictly increasing on $\mathbb{R}$, and let $x, y \in \mathbb{R}$ with $x < y$. Then $0 < f(x) < f(y)$, whence comes $0 < 1/f(y) < 1/f(x)$ and then $0 < g(y) < g(x)$. Thus g is strictly decreasing.

Conversely, if g is assumed strictly decreasing, then for any $x, y \in \mathbb{R}$ with $x < y$ we find that $0 < g(y) < g(x)$, hence $0 < 1/f(y) < 1/f(x)$, and so $0 < f(x) < f(y)$. Thus f is strictly increasing.

The codomain of f is given to be $(0, \infty)$, so $f(x) \neq 0$ for all $x \in \mathbb{R}$, and hence $g(x)$ is defined for all $x \in \mathbb{R}$.

5a $f(S) = \{f(-1), f(0), f(2), f(4), f(7)\} = \{8, 5, -1, -7, -16\}$

5b $f(S) = \{0, 1, 4, 12\}$

6 With backward substitution,

$$\begin{aligned}
a_n &= a_{n-1} - n = [a_{n-2} - (n-1)] - n = a_{n-2} - 2n + 1 \\
&= [a_{n-3} - (n-2)] - 2n + 1 = a_{n-3} - 3n + (1 + 2) \\
&= [a_{n-4} - (n-3)] - 3n + 3 = a_{n-4} - 4n + (1 + 2 + 3) \\
&\vdots \\
&= a_{n-k} - kn + \sum_{i=1}^{k-1} i = \cdots = a_0 - n^2 + \sum_{i=1}^{n-1} i \\
&= 4 - n^2 + \frac{n(n-1)}{2} = 4 - \frac{n + n^2}{2}.
\end{aligned}$$

7 We have

$$\sum_{i=1}^{3}\left[2i+(2i-3)+(2i-6)+(2i-9)\right]=\sum_{i=1}^{3}(8i-18)=-6.$$

8 A 3×3 matrix results:

$$\begin{bmatrix}-5a & a^2-c & ab-1\\ 0 & bc & b\\ 15 & 2c-3a & 2-3b\end{bmatrix}.$$

9 Just show with long division that $1013 \div 24$ has remainder 5. Thus it will be 5 hours after 17:00, or 22:00.

10 We need $0 \le c \le 16$ such that $c = a - 4b + 17\ell$ for some $\ell \in \mathbb{Z}$, and since $a = 10 + 17j$ and $b = 7 + 17k$ for some $j, k \in \mathbb{Z}$, we have $c = -18 + 17(j - 4k + \ell)$. Now, $0 \le c \le 16$ implies $\frac{18}{17} \le j - 4k + \ell \le 2$, which forces $j - 4k + \ell = 2$. To satisfy this we can choose $j = k = 0$ and $\ell = 2$, and thus $c = 16$.

11a No work required: 1111 1010 1101 0000 1001.

11b With some work: 10 1000 0011.

12 $(504454)_8$

13a $6174 = 2 \cdot 3^2 \cdot 7^3$ and $7800 = 2^3 \cdot 3 \cdot 5^2 \cdot 13$.

13b $\gcd(6174, 7800) = 6$ and $\operatorname{lcm}(6174, 7800) = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7^3 \cdot 13 = 8,026,200$.