Math 120 Exam #4 Key (Summer 2015)

1 We have

$$\begin{array}{r|rrrr|r} -2 & 2 & 0 & -3 & 4 & -9 \\ & & -4 & 8 & -10 & 12 \\ \hline & 2 & -4 & 5 & -6 & 3 \end{array} \quad \longrightarrow \quad 2z^3 - 4z^2 + 5z - 6 + \frac{3}{z+2}.$$

2 Divide $f(x)$ by $x + 2$:

$$\begin{array}{r|rrrr|r} -2 & 1 & 2 & -7 & -20 & -12 \\ & & -2 & 0 & 14 & 12 \\ \hline & 1 & 0 & -7 & -6 & 0 \end{array}$$

It follows that

$$f(x) = (x + 2)(x^3 - 7x - 6).$$

Now divide $x^3 - 7x - 6$ by $x + 2$:

$$\begin{array}{r|rrr|r} -2 & 1 & 0 & -7 & -6 \\ & & -2 & 4 & 6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

Therefore

$$f(x) = (x + 2)(x + 2)(x^2 - 2x - 3) = (x + 2)^2(x - 3)(x + 1).$$

3a $\dfrac{\text{Factor of 30}}{\text{Factor of 2}} = \pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}.$

3b The division

$$\begin{array}{r|rrr|r} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & 0 \end{array}$$

shows that 2 is a zero for f, and we obtain the factorization

$$f(x) = (x - 2)(2x^2 + x - 15) = (x - 2)(2x - 5)(x + 3)$$

So the zeros of f are: 2, $\frac{5}{2}$, -3.

3c $f(x) = (x - 2)(2x - 5)(x + 3)$.

4 We must have $f(x) = cx(x + 2)(x - 3)$, with c such that

$$f(-1) = c(-1)(-1 + 2)(-1 - 3) = 4c = 16.$$

Clearly $c = 4$ is required, so

$$f(x) = 4x(x + 2)(x - 3) = 4x^3 - 4x^2 - 24x.$$

5 By the Conjugate Zeros Theorem we must have $1 - 3i$ as a zero also, in order to have real coefficients. So

$$f(x) = (x + 4)[x - (1 + 3i)][x - (1 - 3i)] = x^3 + 2x^2 + 2x + 40.$$

6a We have

$$\operatorname{Dom}(f) = \{x : x^2 - 4 \neq 0\} = \{x : x \neq -2, 2\}.$$

6b The x-intercepts of f are the points $(x, f(x))$ where $f(x) = 0$:

$$\frac{x^2(x+1)}{(x-2)(x+2)} = 0 \quad \Rightarrow \quad x^2(x+1) = 0 \quad \Rightarrow \quad x = -1, 0$$

so $(-1, 0)$ and $(0, 0)$ are the x-intercepts. Since $(0, 0)$ is also a y-intercept of f and a function can never have more than one y-intercept, we have found all intercepts.

6c The vertical asymptotes of f are $x = -2$ and $x = 2$.

6d The degree of the numerator is 1 greater than the degree of the denominator, so there will be an oblique asymptote. From the division

$$\begin{array}{r|l} & \quad\quad\quad\quad x + 1 \\ x^2 - 4 & x^3 + x^2 \\ & -x^3 \quad\quad + 4x \\ \hline & \quad\quad x^2 + 4x \\ & \quad -x^2 \quad\quad + 4 \\ \hline & \quad\quad\quad 4x + 4 \end{array}$$

we find that

$$f(x) = x + 1 + \frac{4x + 4}{x^2 - 4},$$

and therefore $y = x + 1$ is the equation of the oblique asymptote.

6e The graph of f intersects the oblique asymptote $y = x + 1$ if there is some $x \in \operatorname{Dom}(f)$ for which $f(x) = x + 1$. This results in the equation

$$\frac{x^3 + x^2}{x^2 - 4} = x + 1,$$

giving

$$x^3 + x^2 = x^3 + x^2 - 4x - 4 \quad \Rightarrow \quad 4x = -4 \quad \Rightarrow \quad x = -1.$$

Thus the graph of f intersects $y = x + 1$ at $(-1, 0)$.

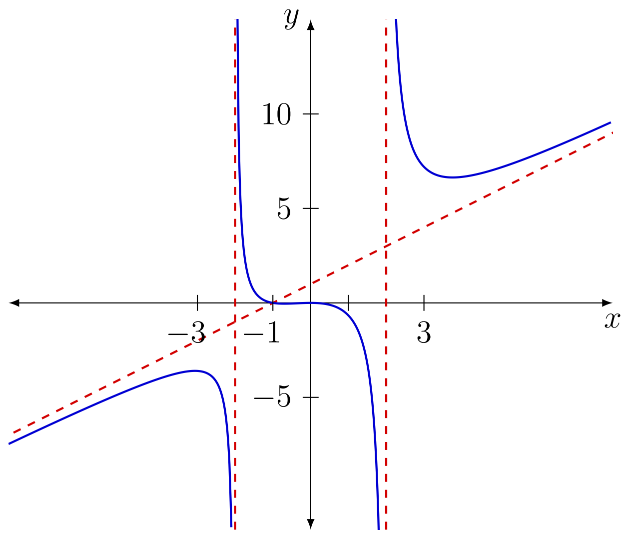
6f The vertical asymptotes partition the plane into three regions:

$$R_1 = \{x : x < -2\}, \quad R_2 = \{x : -2 < x < 2\}, \quad \text{and} \quad R_3 = \{x : x > 2\}.$$

We will want at least one point that lies on the graph of f in each region. We calculate

$$f(-3) = -3\tfrac{3}{5}, \quad f(3) = 7\tfrac{1}{5},$$

we obtain the points $\left(-3, -3\tfrac{3}{5}\right)$ and $\left(3, 7\tfrac{1}{5}\right)$. We sketch the graph:



7a Using the formula provided,

$$A(t) = 1000\left(1 + \frac{0.0875}{12}\right)^{12t} = 1000(1.00729)^{12t}.$$

7b $1000(1.00729)^{12(7)} = \$1840.94$ and $1000(1.00729)^{12(14)} = \$3389.07$.

8 Using a law of logarithms,

$$2\log(a+1) - 3\log(b^2) + \log(4c) = \log(a+1)^2 - \log(b^2)^3 + \log(4c) = \log\left[\frac{4c(a+1)^2}{b^6}\right].$$

9a We have

$$5^{4x+1} = 625 \quad \Rightarrow \quad 5^{4x+1} = 5^4 \quad \Rightarrow \quad 4x + 1 = 4 \quad \Rightarrow \quad x = \tfrac{3}{4}.$$

9b Take the logarithm of each side:

$$\ln(3^x) = \ln(8^{x+2}) \quad \Rightarrow \quad x\ln 3 = (x+2)\ln 8 \quad \Rightarrow \quad x = \frac{2\ln 8}{\ln 3 - \ln 8} = \frac{6\ln 2}{\ln 3 - 3\ln 2}.$$

9c Convert to an exponential equation:

$$\log_6(8 - 3x) = 2 \quad \Rightarrow \quad 6^2 = 8 - 3x \quad \Rightarrow \quad x = -\tfrac{28}{3}.$$

9d Consolidate logarithms:

$$\log_2(x+1)+\log_2(x-1)=3 \;\Rightarrow\; \log_2(x+1)(x-1)=3 \;\Rightarrow\; 2^3=x^2-1 \;\Rightarrow\; x=\pm 3.$$

But -3 is an extraneous solution (it results in the logarithm of a negative number in the original equation), so $x=3$ is the only solution.

10 Using the formula $A=Pe^{rt}$, we have

$$5750=4000e^{3r} \;\Rightarrow\; e^{3r}=\tfrac{23}{16} \;\Rightarrow\; r=\tfrac{1}{3}\ln\tfrac{23}{16}\approx 0.121,$$

or 12.1%.

11 Whatever the starting amount P is, we want to find the time t at which $A=3P$. Using the appropriate formula,

$$3P=P\left(1+\frac{0.055}{4}\right)^{4t} \;\Rightarrow\; 1.013325^{4t}=3 \;\Rightarrow\; 4t\ln 1.013325=\ln 3 \;\Rightarrow\; t=\frac{\ln 3}{4\ln 1.013325},$$

and so $t\approx 20.75$, or about 20 years and 9 months.

12 The basic model, starting with 230 grams, is:

$$A(t)=230e^{-kt}$$

We're given that $A(1)=198$, which is to say

$$198=230e^{-k}.$$

Solving:

$$e^{-k}=\tfrac{198}{230} \;\Rightarrow\; \ln(e^{-k})=\ln\left(\tfrac{198}{230}\right) \;\Rightarrow\; -k=\ln\left(\tfrac{99}{115}\right) \;\Rightarrow\; k\approx 0.1498.$$

Thus we have

$$A(t)=230e^{-0.1498t}.$$

Now we find the time t when $A(t)=9$ grams:

$$9=230e^{-0.1498t} \;\Rightarrow\; e^{-0.1498t}=\tfrac{9}{230} \;\Rightarrow\; -0.1498t=\ln\left(\tfrac{9}{230}\right) \;\Rightarrow\; t=\frac{\ln\left(\frac{9}{230}\right)}{-0.1498}\approx 21.6.$$

That is, after about 21.6 hours there will be 9 grams of narfzortium remaining.