Math 120 Exam #4 Key (Spring 2014)

1 We have

$$\begin{array}{r|rrrr|r} -1 & 3 & 0 & -4 & 0 & 2 \\ & & -3 & 3 & 1 & -1 \\ \hline & 3 & -3 & -1 & 1 & 1 \end{array} \quad \longrightarrow \quad 3x^3 - 3x^2 - x + 1 + \frac{1}{x+1}.$$

2 Divide $f(x)$ by $x + 2$:

$$\begin{array}{r|rrrr|r} -2 & 1 & 2 & -7 & -20 & -12 \\ & & -2 & 0 & 14 & 12 \\ \hline & 1 & 0 & -7 & -6 & 0 \end{array}$$

It follows that

$$f(x) = (x + 2)(x^3 - 7x - 6).$$

Now divide $x^3 - 7x - 6$ by $x + 2$:

$$\begin{array}{r|rrr|r} -2 & 1 & 0 & -7 & -6 \\ & & -2 & 4 & 6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

Therefore

$$f(x) = (x + 2)(x + 2)(x^2 - 2x - 3) = (x + 2)^2(x - 3)(x + 1).$$

3a $\dfrac{\text{Factor of } -12}{\text{Factor of } 1} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12.$

3b The division

$$\begin{array}{r|rrrr|r} 1 & 1 & 2 & 1 & 8 & -12 \\ & & 1 & 3 & 4 & 12 \\ \hline & 1 & 3 & 4 & 12 & 0 \end{array}$$

shows that 1 is a zero for f, and we obtain the factorization

$$\begin{aligned} f(x) &= (x - 1)(x^3 + 3x^2 + 4x + 12) = (x - 1)[x^2(x + 3) + 4(x + 3)] \\ &= (x - 1)(x + 3)(x^2 + 1) = (x - 1)(x + 3)[x^2 - (2i)^2] \\ &= (x - 1)(x + 3)(x - 2i)(x + 2i). \end{aligned}$$

So the zeros of f are: 1, -3, $2i$, $-2i$.

3c $f(x) = (x - 1)(x + 3)(x - 2i)(x + 2i).$

4 We must have $f(x) = cx(x + 1)(x - 3)$, with c such that

$$f(2) = c(2)(2 + 1)(2 - 3) = -6c = 10.$$

Clearly $c = -5/3$ is required, so

$$f(x) = -\frac{5}{3}x(x + 1)(x - 3) = -\frac{5}{3}x^3 + \frac{10}{3}x^2 + 5x.$$

5 By the Conjugate Zeros Theorem we must have $2+i$ as a zero also, in order to have rational coefficients. So

$$f(x) = (x+1)[x-(2-i)][x-(2+i)] = x^3 - 3x^2 + x + 5.$$

6a We have

$$\text{Dom}(f) = \{x : x^2 - 4 \neq 0\} = \{x : x \neq -2, 2\}.$$

6b The x-intercepts of f are the points $(x, f(x))$ where $f(x) = 0$:

$$\frac{x^2(x+1)}{(x-2)(x+2)} = 0 \quad \Rightarrow \quad x^2(x+1) = 0 \quad \Rightarrow \quad x = -1, 0$$

so $(-1,0)$ and $(0,0)$ are the x-intercepts. Since $(0,0)$ is also a y-intercept of f and a function can never have more than one y-intercept, we have found all intercepts.

6c The vertical asymptotes of f are $x = -2$ and $x = 2$.

6d The degree of the numerator is 1 greater than the degree of the denominator, so there will be an oblique asymptote. From the division

$$\begin{array}{r|l} & \qquad\qquad x + 1 \\ \hline x^2 - 4 & x^3 + x^2 \\ & -x^3 \qquad + 4x \\ \hline & \qquad x^2 + 4x \\ & \qquad -x^2 \qquad + 4 \\ \hline & \qquad\qquad 4x + 4 \end{array}$$

we find that

$$f(x) = x + 1 + \frac{4x+4}{x^2-4},$$

and therefore $y = x + 1$ is the equation of the oblique asymptote.

6e The graph of f intersects the oblique asymptote $y = x+1$ if there is some $x \in \text{Dom}(f)$ for which $f(x) = x+1$. This results in the equation

$$\frac{x^3+x^2}{x^2-4} = x+1,$$

giving

$$x^3 + x^2 = x^3 + x^2 - 4x - 4 \quad \Rightarrow \quad 4x = -4 \quad \Rightarrow \quad x = -1.$$

Thus the graph of f intersects $y = x+1$ at $(-1, 0)$.

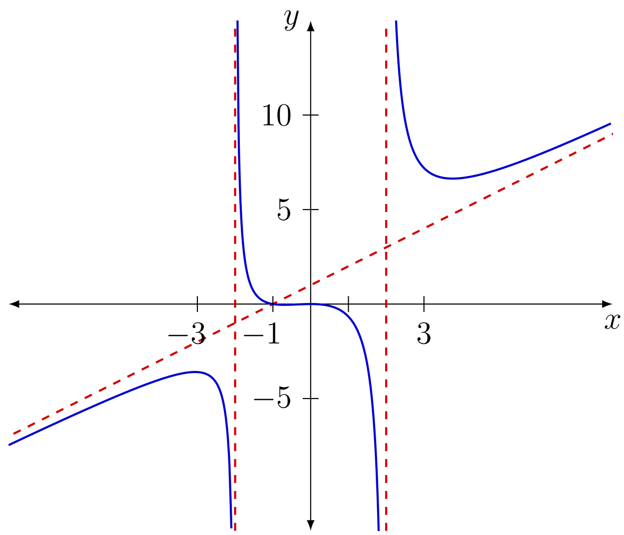
6f The vertical asymptotes partition the plane into three regions:

$$R_1 = \{x : x < -2\}, \quad R_2 = \{x : -2 < x < 2\}, \quad \text{and} \quad R_3 = \{x : x > 2\}.$$

We will want at least one point that lies on the graph of f in each region. We calculate

$$f(-3) = -3\tfrac{3}{5}, \quad f(3) = 7\tfrac{1}{5},$$

we obtain the points $\left(-3, -3\tfrac{3}{5}\right)$ and $\left(3, 7\tfrac{1}{5}\right)$. We sketch the graph:



7a Using the formula provided,

$$A(t) = 1000\left(1 + \frac{0.092}{4}\right)^{4t} = 1000(1.023)^{4t}.$$

7b $1000(1.023)^{4(5)} = \$1575.84$ and $1000(1.023)^{4(10)} = \$2483.28$.

8 Using a law of logarithms,

$$2\log_5 a - 3\log_5 b^2 = \log_5 a^2 - \log_5 (b^2)^3 = \log_5\left(\frac{a^2}{b^6}\right).$$

9a We have

$$5^{4x-7} = 125 \quad\Rightarrow\quad 5^{4x-7} = 5^3 \quad\Rightarrow\quad 4x - 7 = 3 \quad\Rightarrow\quad x = 5/2.$$

9b Take the logarithm of each side:

$$\ln(3^x) = \ln(6^{x-1}) \quad\Rightarrow\quad x\ln 3 = (x-1)\ln 6 \quad\Rightarrow\quad x = \frac{\ln 6}{\ln 6 - \ln 3} = \frac{\ln 6}{\ln 2}.$$

9c Convert to an exponential equation:

$$\log_2(10 + 3x) = 5 \quad\Rightarrow\quad 2^5 = 10 + 3x \quad\Rightarrow\quad 3x = 22 \quad\Rightarrow\quad x = 22/3.$$

9d Consolidate logarithms:

$$\log_2(x+1) + \log_2(x-1) = 3 \quad \Rightarrow \quad \log_2(x+1)(x-1) = 3 \quad \Rightarrow \quad 2^3 = x^2 - 1 \quad \Rightarrow \quad x = \pm 3.$$

But -3 is an extraneous solution (it results in the logarithm of a negative number in the original equation), so $x = 3$ is the only solution.

10 Using the appropriate formula, we have

$$7500 = 5000\left(1 + \frac{0.09}{12}\right)^{12t} \quad \Rightarrow \quad 1.0075^{12t} = 1.5 \quad \Rightarrow \quad \ln(1.0075^{12t}) = \ln 1.5,$$

and so

$$12t \ln 1.0075 = \ln 1.5 \quad \Rightarrow \quad t = \frac{\ln 1.5}{12 \ln 1.0075} \approx 4.522.$$

It will take about 4.5 years.

11 Whatever the starting amount P is, we want to find the time t at which $A = 2P$. Using the appropriate formula,

$$2P = Pe^{0.036t} \quad \Rightarrow \quad e^{0.036t} = 2 \quad \Rightarrow \quad 0.036t = \ln 2 \quad \Rightarrow \quad t = \frac{\ln 2}{0.036} \approx 19.254.$$

It will take about 19.3 years.

12 The basic model, starting with 150 grams, is:

$$A(t) = 150e^{-kt}$$

We're given that $A(1) = 148$, which is to say

$$148 = 150e^{-k}.$$

Solving:

$$e^{-k} = \tfrac{148}{150} \quad \Rightarrow \quad \ln(e^{-k}) = \ln\left(\tfrac{74}{75}\right) \quad \Rightarrow \quad -k = \ln\left(\tfrac{74}{75}\right) \quad \Rightarrow \quad k \approx 0.0134.$$

Thus we have

$$A(t) = 150e^{-0.0134t}.$$

Now we find the time t when $A(t) = 100$ grams:

$$100 = 150e^{-0.0134t} \quad \Rightarrow \quad e^{-0.0134t} = \tfrac{2}{3} \quad \Rightarrow \quad -0.0134t = \ln(2/3) \quad \Rightarrow \quad t = \frac{\ln(2/3)}{-0.013} \approx 31.2.$$

That is, after about 31.2 hours there will be 100 grams of narfzortium remaining.