Math 120 Exam #2 Key (Spring 2013)

1a From $9x^2 - 12x + 4 = 0$ we have $(3x-2)(3x-2) = 0$, which leads to $3x - 2 = 0$ and thus $x = 2/3$. Solution set: $\{2/3\}$.

1b We have $x+5 = \pm\sqrt{-3} = \pm i\sqrt{3}$, and so $x = -5\pm\sqrt{3}i$. Solution set: $\{-5-\sqrt{3}i, -5+\sqrt{3}i\}$.

1c Divide by -3 to get $x^2 - 3x = -7/3$. Completing the square yields

$$x^2 - 3x + (-3/2)^2 = -7/3 + (-3/2)^2,$$

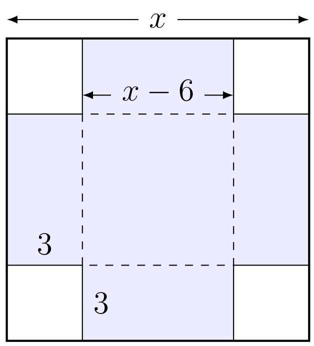
or $(x - 3/2)^2 = -1/12$. From this comes $x - 3/2 = \pm\sqrt{-1/12} = \pm i\sqrt{3}/6$, and so the solution set is

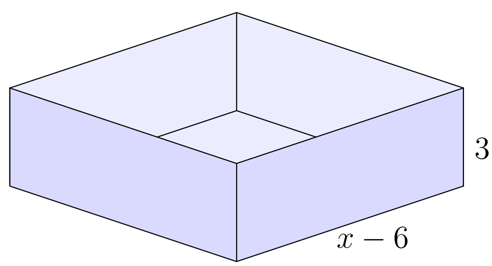
$$\left\{\frac{3}{2} - \frac{\sqrt{3}}{6}i,\ \frac{3}{2} + \frac{\sqrt{3}}{6}i\right\}.$$

2 Write as $-16t^2 + v_0t + (s_0 - h) = 0$, so

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(-16)(s_0 - h)}}{2(-16)} = \frac{v_0 \pm \sqrt{v_0^2 + 64(s_0 - h)}}{32}.$$

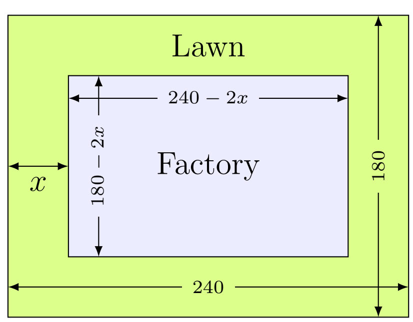
3 Let x be the length of the sides of the square base. Once the corners are cut away and the flaps are folded up (see figure below), a box results with a square base of length $x - 6$.





The volume of the box is $3(x-6)(x-6) = 48$, or $(x-6)^2 = 16$. Solving the equation gives $x - 6 = \pm\sqrt{16} = \pm 4$, or $x = 2, 10$. However, we cannot have $x = 2$ since it results in a flap width of -4! So we must have $x = 10$. That is, the cardboard should be 10 cm × 10 cm.

4 Let x be the width of the lawn. The dimensions involved are as shown in the figure below.



We have

$$\begin{aligned} \text{Area of Lawn} &= \text{Area of Factory} \\ \text{Area of Lot} - \text{Area of Factory} &= \text{Area of Factory} \\ 2(\text{Area of Factory}) &= \text{Area of Lot} \\ 2(240-2x)(180-2x) &= (240)(180) \\ x^2 - 210x + 5400 &= 0, \end{aligned}$$

which becomes $(x-180)(x-30)=0$ and so $x = 30, 180$. We cannot have $x = 180$, however, since that results in a lawn width that equals the width of the lot! So we must have $x = 30$ m as the width of the lawn. The dimensions of the factory are thus 180 m $\times$ 120 m.

5a Multiply by $x^2(x-1)$ to get $x^2 = 2(x-1)$, which becomes $x^2 - 2x + 2 = 0$. Solving using the quadratic equation (or completing the square), we obtain $x = 1 \pm i$. Solution set: $\{1-i, 1+i\}$.

5b Multiply by $(x-2)(x+2)$ to get $(x+5)(x+2) = 5(x-2)+28$, which becomes $x^2+2x-8=0$. Factoring, we get $(x+4)(x-2)=0$, so that $x = -4, 2$. But 2 is extraneous, so the solution set is $\{-4\}$.

5c We have $\sqrt{5-x} = x-3$. Squaring both sides yields $5 - x = (x-3)^2$, or $x^2 - 5x + 4 = 0$. Factoring, we get $(x-4)(x-1) = 0$, so that $x = 1, 4$. But 1 is extraneous, so the solution set is $\{4\}$.

5d Square both sides to get $(3-\sqrt{x})^2 = 2\sqrt{x} - 3$, or

$$9 - 6\sqrt{x} + x = 2\sqrt{x} - 3.$$

This becomes $8\sqrt{x} = x + 12$, and squaring again yields $64x = x^2 + 24x + 144$. So we have $x^2 - 40x + 144 = 0$, and factoring results in $(x-36)(x-4) = 0$ and finally $x = 4, 36$. But 36 is extraneous, so the solution set is $\{4\}$.

5e Letting $u = x^{2/3}$, the equation becomes quadratic: $u^2 - 5u + 6 = 0$. Factoring yields $(u-2)(u-3) = 0$, or $x^{2/3} = u = 2, 3$. Now,

$$x^{2/3} = 2 \quad \Rightarrow \quad x^2 = 8 \quad \Rightarrow \quad x = \pm\sqrt{8} = \pm 2\sqrt{2},$$

and

$$x^{2/3} = 3 \quad \Rightarrow \quad x^2 = 27 \quad \Rightarrow \quad x = \pm\sqrt{27} = \pm 3\sqrt{3}.$$

Solution set: $\{-2\sqrt{2}, 2\sqrt{2}, -3\sqrt{3}, 3\sqrt{3}\}$.

5f We have either $x - 1 = 3x + 2$ or $x - 1 = -(3x+2)$. The first equation solve to give $x = -3/2$, and the second equation solves to give $x = -1/4$. Solution set: $\{-3/2, -1/4\}$.

6a Simplifying yields $8x - 3 \le 3x - 7$, and then $5x \le -4$, and finally $x \le -4/5$. Solution set: $(-\infty, -4/5]$.

6b Multiply by 20 to get $-10 < 4(4-3x) \leq 5$, whence

$$-10 < 16 - 12x \leq 5 \quad \Rightarrow \quad -26 < -12x \leq -11 \quad \Rightarrow \quad 13/6 > x \geq 11/12.$$

Solution set: $\left[\frac{11}{12}, \frac{13}{6}\right)$.

6c Factoring, we get $(x+2)(x+3) > 0$. There are two cases to consider.
Case 1: $x+2 > 0$ & $x+3 > 0$. This gives $x > -2$ & $x > -3$, which is equivalent to $x > -2$.
Case 2: $x+2 < 0$ & $x+3 < 0$. This gives $x < -2$ & $x < -3$, which is equivalent to $x < -3$.
Thus we may have $x < -3$ or $x > -2$. Solution set: $(-\infty, -3) \cup (-2, \infty)$.

6d Manipulate without multiplying by an expression involving x:

$$\frac{2x+1}{x-5} \leq 3 \quad \Leftrightarrow \quad \frac{2x+1}{x-5} - 3 \leq 0 \quad \Leftrightarrow \quad \frac{2x+1-3(x-5)}{x-5} \leq 0 \quad \Leftrightarrow \quad \frac{16-x}{x-5} \leq 0.$$

There are two cases to consider.
Case 1: $16 - x \geq 0$ & $x - 5 < 0$. This gives $x \leq 16$ & $x < 5$, which is equivalent to $x < 5$. (Note that we cannot have $x = 5$ since division by zero would result.)
Case 2: $16 - x \leq 0$ & $x - 5 > 0$. This gives $x \geq 16$ & $x > 5$, which is equivalent to $x \geq 16$. (Letting $x = 16$ results in no division by zero.)
Thus we may have $x < 5$ or $x \geq 16$. Solution set: $(-\infty, 5) \cup [16, \infty)$.

6e We have

$$|4-3x| > 2 \quad \Leftrightarrow \quad 4-3x > 2 \quad \text{or} \quad 4-3x < -2 \quad \Leftrightarrow \quad x < 2/3 \quad \text{or} \quad x > 2.$$

Solution set: $(-\infty, 2/3) \cup (2, \infty)$.

6f There is no solution (i.e. solution set is $\varnothing$) since absolute value is never negative.

6g We have

$$|5-x| \leq 12 \quad \Leftrightarrow \quad -12 \leq 5-x \leq 12 \quad \Leftrightarrow \quad -17 \leq -x \leq 7 \quad \Leftrightarrow \quad -7 \leq x \leq 17.$$

Solution set: $[-7, 17]$.

7 We find the distance between the points:

$$d(P,Q) = \sqrt{4^2+8^2} = \sqrt{80} = 4\sqrt{5}$$
$$d(P,R) = \sqrt{6^2+12^2} = \sqrt{180} = 6\sqrt{5}$$
$$d(Q,R) = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$$

Since $d(P,Q) + d(Q,R) = d(P,R)$ (i.e. the two smaller distances equal the largest distance), the points are collinear.

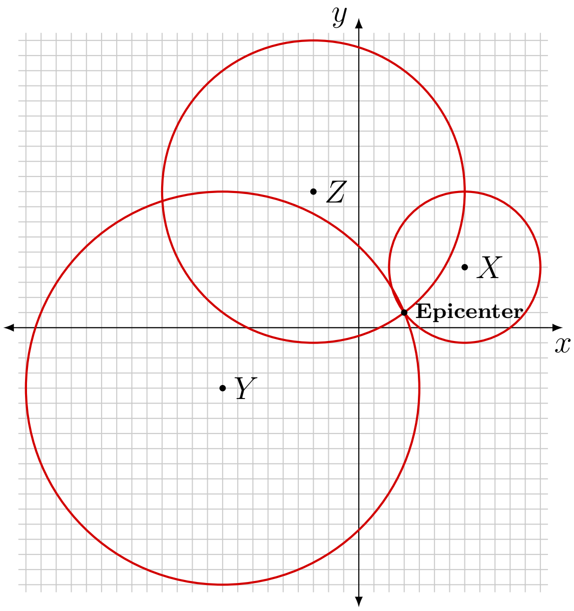
8 Several solutions are: $(3,0)$, $(4,1)$, $(5,\sqrt{2})$, $(7,2)$, $(12,3)$, $(19,4)$.

9 Complete the square for each variable as follows:

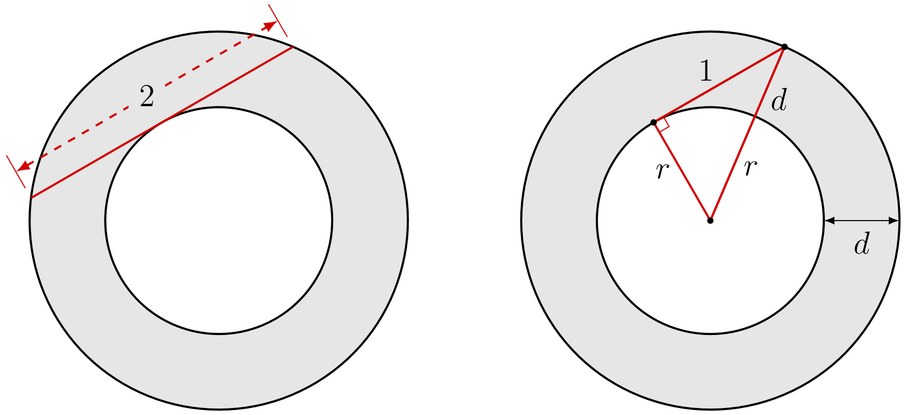
$$(x^2 - 12x + 36) + (y^2 + 10y + 25) = -25 + 36 + 25.$$

Thus we have $(x-6)^2 + (y+5)^2 = 36$, which is a circle with center at $(6, -5)$ and radius 6.

10 Employ a graphical approach as in the example in the textbook: at coordinates $(7, 4)$ graph a circle of radius 5, at $(-9, -4)$ graph a circle of radius 13, and at $(-3, 9)$ graph a circle of radius 10. Looking at the graph below, only the one point $(3, 1)$ lies on all three circles, and so the epicenter of the earthquake must be at $(3, 1)$.



EC Let r be the radius of the smaller circle and $r + d$ the radius of the larger circle. See the figure at right below.



We have a right triangle with hypotenuse of length $r + d$, and so by the Pythagorean Theorem

$$r^2 + 1^2 = (r + d)^2,$$

or $d^2 + 2rd - 1 = 0$. Use the quadratic formula to solve for d:

$$d = \frac{-2r \pm \sqrt{(2r)^2 - 4(1)(-1)}}{2(1)} = -r \pm \sqrt{r^2 + 1}.$$

Since d cannot be negative we obtain $d = \sqrt{r^2+1} - r$, and hence $r + d = \sqrt{r^2+1}$ is the radius of the larger circle. If A is the area of the shaded region, then

$$\begin{aligned} A &= \text{Area of Large Circle} - \text{Area of Small Circle} \\ &= \pi\left(\sqrt{r^2+1}\right)^2 - \pi r^2 \\ &= \pi(r^2+1) - \pi r^2 \\ &= \pi \end{aligned}$$