Math 120 Exam #3 Key (Spring 2011)

1. Solve $3x - 5y = 1$ for y to get $y = \frac{3}{5}x - \frac{1}{5}$, which shows the slope for the line given by $3x - 5y = 1$ to be $\frac{3}{5}$. Since the line L through $(-1, 5)$ is parallel to this line, its slope must also be $\frac{3}{5}$. So the equation for L is $y - 5 = \frac{3}{5}(x + 1)$, or $y = \frac{3}{5}x + \frac{28}{5}$.

2. Solve $8x + 3y = 6$ for y to get $y = -\frac{8}{3}x + 2$, which shows the slope for the line given by $8x + 3y = 6$ to be $-\frac{8}{3}$. Since the line L through $(-2, 3)$ is perpendicular to this line, its slope must also be $\frac{3}{8}$. So the equation for L is $y - 3 = \frac{3}{8}(x + 2)$, or $y = \frac{3}{8}x + \frac{15}{4}$.

3. The ordered pairs $(1, 1)$ and $(1, -1)$ belong to the relation R, so R is not a function since, by definition, a function is a set of ordered pairs in which no two distinct pairs have the same first-coordinate value. To find the domain and range it helps to solve for y: $y = \pm\sqrt[6]{x}$, or equivalently "$y = \sqrt[6]{x}$ or $y = -\sqrt[6]{x}$" (the ordered pairs that belong to R are the ones that make this statement true). We have $\text{Dom}(R) = [0, \infty)$ and $\text{Ran}(R) = (-\infty, \infty)$.

4. $f(-4) = (-4)^2 - 3(-4) = 16 + 12 = 28$ & $f(c + 2) = (c + 2)^2 - 3(c + 2) = c^2 + c - 2$.

5a. We have $y = \dfrac{2}{x}$, so domain and range are both $(-\infty, 0) \cup (0, \infty)$.

5b. Domain is $(-\infty, \infty)$ and range is $[-8, \infty)$.

6a. $\text{Dom}(\alpha) = \left\{x \mid x \neq \frac{2}{3}\right\} = \left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$.

6b. $\text{Dom}(\beta) = \{x \mid 9x - 5 \geq 0\} = \left\{x \mid x \geq \frac{5}{9}\right\} = \left[\frac{5}{9}, \infty\right)$.

6c. $\text{Dom}(\gamma) = \{x \mid 16 - x^2 \geq 0\} = \left\{x \mid x^2 \leq 16\right\} = \{x \mid -4 \leq x \leq 4\} = [-4, 4]$.

7a. We find $(\alpha - \gamma)(x) = \alpha(x) - \gamma(x) = \dfrac{x + 1}{3x - 2} - \sqrt{16 - x^2}$, and $\text{Dom}(\alpha - \gamma) = \text{Dom}(\alpha) \cap \text{Dom}(\gamma) = \left[\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)\right] \cap [-4, 4] = \left[-4, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 4\right]$.

7b. We find $(\alpha\beta)(x) = \alpha(x) \cdot \beta(x) = \dfrac{x + 1}{3x - 2} \cdot \sqrt{9x - 5} = \dfrac{(x + 1)\sqrt{9x - 5}}{3x - 2}$, and $\text{Dom}(\alpha\beta) = \text{Dom}(\alpha) \cap \text{Dom}(\beta) = \left[\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)\right] \cap \left[\frac{5}{9}, \infty\right) = \left[\frac{5}{9}, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$.

7c. We find $(\beta \circ \beta)(x) = \beta(\beta(x)) = \beta(\sqrt{9x - 5}) = \sqrt{9\sqrt{9x - 5} - 5}$, and the domain is given by $\text{Dom}(\beta \circ \beta) = \{x \mid x \in \text{Dom}(\beta)\ \&\ \beta(x) \in \text{Dom}(\beta)\} = \left\{x \mid x \geq \frac{5}{9}\ \&\ \sqrt{9x - 5} \geq \frac{5}{9}\right\} = \left\{x \mid x \geq \frac{5}{9}\ \&\ x \geq \frac{430}{729}\right\} = \left\{x \mid x \geq \frac{430}{729}\right\} = \left[\frac{430}{729}, \infty\right)$ (since $\frac{430}{729} \approx 0.59$ is slightly larger than $\frac{5}{9} \approx 0.56$).

7d. We find $(\beta \circ \gamma)(x) = \beta(\gamma(x)) = \beta(\sqrt{16 - x^2}) = \sqrt{9\sqrt{16 - x^2} - 5}$, and the domain is given by $\text{Dom}(\beta \circ \gamma) = \{x \mid x \in \text{Dom}(\gamma)\ \&\ \gamma(x) \in \text{Dom}(\beta)\} = \left\{x \,\middle|\, -4 \leq x \leq 4\ \&\ \sqrt{16 - x^2} \geq \frac{5}{9}\right\}$, which then leads to $\text{Dom}(\beta \circ \gamma) = \left\{x \,\middle|\, -4 \leq x \leq 4\ \ \&\ -\sqrt{461/81} \leq x \leq \sqrt{461/81}\right\} = \left\{x \,\middle|\, -\sqrt{461/81} \leq x \leq \sqrt{461/81}\right\} = \left[-\sqrt{461/81}, \sqrt{461/81}\right]$ (since $\sqrt{461/81} \approx 2.39 < 4$).

8. Let $g(x) = 2x - 3$ and $f(x) = x^8$, so $(f \circ g)(x) = f(g(x)) = f(2x-3) = (2x-3)^8 = \Omega(x)$.

9. Suppose that $f(a) = f(b)$. Then $2a^3 - 1 = 2b^3 - 1 \;\Rightarrow\; 2a^3 = 2b^3 \;\Rightarrow\; a^3 = b^3 \;\Rightarrow\; a = b$. Therefore f is one-to-one.

10. $g(-2) = 3(-2)^2 - 4 = 8 = 3(2)^2 - 4 = g(2)$.

11a. Suppose that $f(x) = y$. Then $y = \dfrac{x+1}{x-3} \;\Rightarrow\; xy - 3y = x + 1 \;\Rightarrow\; xy - x = 3y + 1 \;\Rightarrow\; x = \dfrac{3y+1}{y-1}$, and since $f^{-1}(y) = x$ by definition, it follows that $f^{-1}(y) = \dfrac{3y+1}{y-1}$.

11b. $\mathrm{Ran}(f) = \mathrm{Dom}(f^{-1}) = (-\infty, 1) \cup (1, \infty)$.

11c. $\mathrm{Ran}(f^{-1}) = \mathrm{Dom}(f) = (-\infty, 3) \cup (3, \infty)$.

E.C. $\mathrm{Dom}(F) = \left\{ x \;\middle|\; \dfrac{x^2+2x}{x-3} \geq 0 \right\}$, so it all comes down to solving $\dfrac{x(x+2)}{x-3} \geq 0$, which is something we've done before. One case is: $x \geq 0$, $x + 2 \geq 0$, $x - 3 > 0$, which gives $x \geq 0$, $x \geq -2$, $x > 3$ and thus the interval $(3, \infty)$. Another workable case is: $x \leq 0$, $x + 2 \geq 0$, $x - 3 < 0$, which gives $x \leq 0$, $x \geq -2$, $x < 3$ and thus the interval $[-2, 0]$. Two other cases lead to contradiction. So the solution set of the inequality is $[-2, 0] \cup (3, \infty)$, and therefore $\mathrm{Dom}(F) = [-2, 0] \cup (3, \infty)$.