4.1 - Exponential Functions

Let b>0 with b≠1. A base-b exponential function is a function f for which

$$f(x) = b^x$$

for all real x. So $D_f = \mathbb{R} = (-\infty, \infty)$.

There are variations on this theme, all considered exponential functions. For nonzero constants A, C, and k, we can have

$$f(x) = C \cdot b^x, \quad f(x) = C \cdot b^{kx}, \quad f(x) = C \cdot b^{kx} + A$$

These are all considered to be base-b exponential functions.

The basic model, $f(x) = b^x$, is often given a special symbol: $\exp_b$,

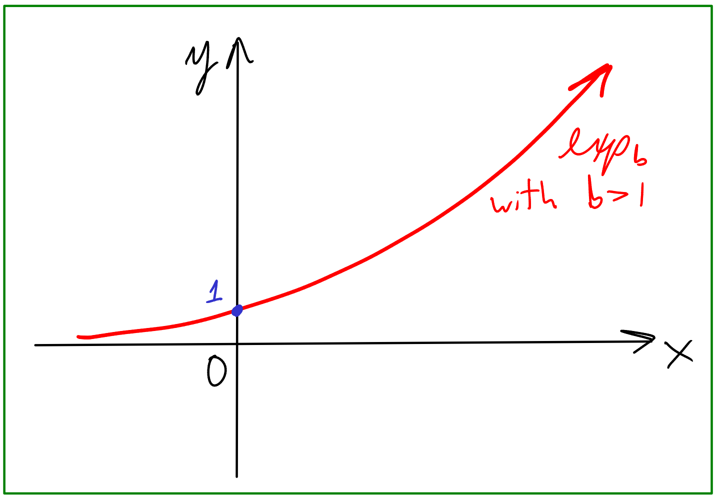
so

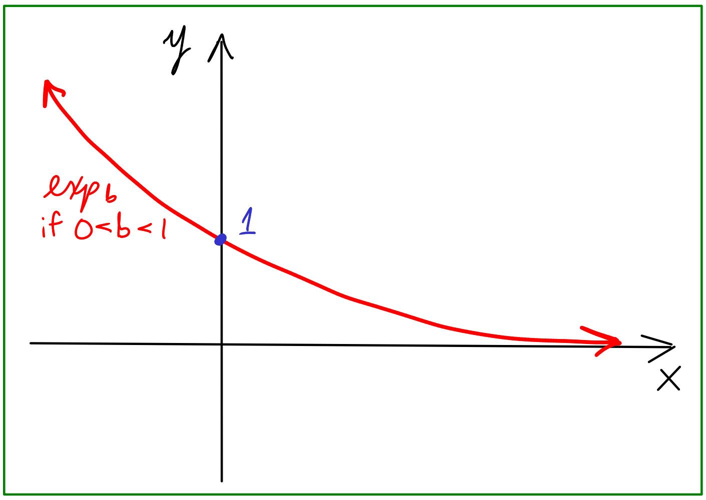
$$\boxed{\exp_b(x) = b^x.}$$

Domain of $\exp_b$ is $(-\infty,\infty)$, range of $\exp_b$ is $(0,\infty)$.

Graph of $\exp_b$ has horizontal asymptote $y=0$ (the x-axis). Because:

$b^x \to 0$ as either $x \to \infty$ (if $0<b<1$) or $x \to -\infty$ (if $b>1$)





One-to-One Theorem: A base-b exponential function is one-to-one.

So if $\exp_b(u) = \exp_b(v)$, then $u = v$.

So $b^u = b^v$ implies $u = v$ if $b>0$ & $b \neq 1$.

An exponential expression is an expression of the form b^x, $C \cdot b^x$, $C \cdot b^{kx}$, $C \cdot b^{kx} + A$, or other variants $(b > 0, b \neq 1)$.

An exponential equation is an equation containing an exponential expression.

base $\rightarrow$ b^{x} $\leftarrow$ exponent

Ex Solve the exponential equation: $2^{3x-4} = 16^{x-4}$.

- First get a common base on each side: since $16 = 2^4$ ...

$$2^{3x-4} = (2^4)^{x-4} \Rightarrow 2^{3x-4} = 2^{4(x-4)}$$

- Use the One-to-One Theorem to equate the exponents:

$$3x - 4 = 4(x-4) \Rightarrow 3x - 4 = 4x - 16 \Rightarrow x = 12.$$

- Solution set is $\boxed{\{12\}}$ ▫

The most common base value b that is used in calculus and beyond is the irrational number e = 2.71828... (Euler's constant)

$\exp_e$ is usually denoted by $\exp$, so:

$$\exp(x) = e^x$$

30 Let $h(x) = 2^{x+2} - 1$. Find D_h, R_h, and horizontal asymptote.

- We know $\exp_2(x) = 2^x$ has domain $(-\infty, \infty)$. So $2^x \in \mathbb{R}$ for all $x \in \mathbb{R}$, implying $2^{x+2} = 2^2 \cdot 2^x \in \mathbb{R}$ for all $x \in \mathbb{R}$, and thus $h(x) = 2^{x+2} - 1 \in \mathbb{R}$ for all $x \in \mathbb{R}$. Hence $D_h = \{x \in \mathbb{R} \mid h(x) \in \mathbb{R}\} = \mathbb{R} = \boxed{(-\infty, \infty)}$.

- We know $\exp_2(x) = 2^x$ has range $(0, \infty)$, meaning $2^x > 0$ for all $x \in \mathbb{R}$.
Then: $2^x > 0 \Rightarrow 2 \cdot 2^x > 2 \cdot 0 \Rightarrow 2^{x+2} > 0 \Rightarrow 2^{x+2} - 1 > 0 - 1 \Rightarrow h(x) > -1 \Rightarrow \boxed{R_h = (-1, \infty)}$

- $\boxed{\text{h.a. is } y=-1}$ since $h(x) > -1$ for all x, & $2^x \to 0$ as $x \to -\infty$ implies $2^{x+2} = 2 \cdot 2^x \to 0$ as $x \to -\infty$, and therefore $2^{x+2} - 1 \to -1$ as $x \to -\infty$. That is, $h(x) \to -1$ as $x \to -\infty$. ■

4.2 - Logarithmic Functions

Recall that if a function f is one-to-one, then it has an inverse denoted by f^{-1}. Then we have:

$$\boxed{\begin{array}{c} f(x) = y \iff f^{-1}(y) = x \\ \text{for all } x \in D_f \ \& \ y \in R_f \end{array}} \rightarrow \begin{array}{l} \text{So } D_f = R_{f^{-1}} \\ \text{and } R_f = D_{f^{-1}} \end{array}$$

Since a base-b exponential function is one-to-one, it has an inverse called the base-b logarithmic function, denoted by $\log_b$. Thus we see that the inverse of exp_b is $\log_b$:

$$\boxed{\begin{array}{c} exp_b(x) = y \iff \log_b(y) = x \\ \text{for all } x \in D_{exp_b} \ \& \ y \in R_{exp_b} \end{array}}, \qquad (2)$$

$$\boxed{\begin{array}{c} b^x = y \iff \log_b(y) = x \\ \text{for all } x \in (-\infty, \infty) \ \& \ y \in (0, \infty) \end{array}}, \qquad (3)$$

From (3) we have:

$$\boxed{\begin{array}{l} D_{\log_b} = R_{exp_b} = (0, \infty) \\ R_{\log_b} = D_{exp_b} = (-\infty, \infty) \end{array}}, \qquad (4)$$

Note that (4) applies only to the "basic" logarithmic function $\log_b x$.

A logarithmic function is also known as a logarithm. Note that since the domain of a logarithm consists of positive real numbers, negative numbers and 0 cannot go into a logarithm.

The base-e logarithm is used most frequently in calculus and beyond, and so gets a special symbol: $\log_e = \ln$ (read as "lawn"). This is the natural logarithm.

The base-10 logarithm is used a lot as well, and so is called the common logarithm. It's symbol is log. That is, $\log_{10} = \log$.

$\boxed{\text{Ex}}$ Evaluate $\log 10{,}000$ (that is, $\log_{10} 10{,}000$).

- Let $\log 10{,}000 = x$.
- Convert to an exponential equation using (3) on previous page, with b=10, y=10,000, and x=x:

$$\log 10{,}000 = x \Rightarrow 10^x = 10{,}000 \Rightarrow 10^x = 10^4 \Rightarrow x = 4.$$

- $\boxed{\log 10{,}000 = 4}$

$\boxed{\text{Ex}}$ Evaluate $\log_4\left(\frac{1}{\sqrt[3]{16}}\right)$

- Let $\log_4\left(\frac{1}{\sqrt[3]{16}}\right) = x$
- Convert to an exponential equation using (3): $4^x = \frac{1}{\sqrt[3]{16}}$
- Solve: $4^x = \frac{1}{16^{1/3}} = 16^{-1/3} = (4^2)^{-1/3} = 4^{-2/3}$. So $x = -\frac{2}{3}$.
- So $\boxed{\log_4\left(\frac{1}{\sqrt[3]{16}}\right) = -\frac{2}{3}}$

$\boxed{\text{Ex}}$ Find the domain of $f(x) = \ln(4-x^2)$.

- From (4) we have domain of ln is $D_{\ln} = (0,\infty)$.
- So $D_f = \{x \in \mathbb{R} \mid 4 - x^2 > 0\}$
- $4 - x^2 > 0$ could be solved using the technique of section 3.6, but since it's possible to isolate x^2, so we can take a more direct route (without using the intermediate value theorem).

$$4 - x^2 > 0 \Rightarrow x^2 < 4 \Rightarrow \sqrt{x^2} < \sqrt{4} \Rightarrow |x| < 2$$

Now solve the absolute value inequality using method of section 1.7.

$$|x| < 2 \Rightarrow -2 < x < 2.$$

- So, $D_f = \{x \in \mathbb{R} \mid -2 < x < 2\} = \boxed{(-2,2)}$

Some basic properties of logarithms:

B1) $\log_b 1 = 0$

B2) $\log_b b = 1$

B3) $\log_b(b^r) = r$

B4) $b^{\log_b x} = x$

Note: Letting $r=1$ in (B3) yields (B2)
Letting $r=0$ in (B3) yields (B1)

• Proof of (B1): By (3), $\log_b 1 = x \Rightarrow b^x = 1 \Rightarrow b^x = b^0 \Rightarrow x = 0.$

• Proof of (B2): By (3), $\log_b b = x \Rightarrow b^x = b \Rightarrow x = 1.$

• Proof of (B3): By (3), $\log_b(b^r) = x \Rightarrow b^x = b^r \Rightarrow x = r.$

• Proof of (B4): Recall (3) again: $b^x = y \Leftrightarrow \log_b(y) = x$.
So substitute $\log_b y$ for x in $b^x = y$ to get $b^{\log_b y} = y$.
Replace y with x to get (B4). ■

$\boxed{\text{Ex}}$ Evaluate $13^{\log_{13} 9}$.

Use (B4) with $b=13$ & $x=9$ to get $13^{\log_{13} 9} = \boxed{9}$ ■

$\boxed{\text{Ex}}$ Evaluate $e^{\ln 17}$.

Recall $\ln = \log_e$, so $e^{\ln 17} = e^{\log_e 17} = \boxed{17}$ using (B4) with $b=e$ & $x=17$. ■

$\boxed{\text{Ex}}$ Find the domain of $f(x) = \log(18-x)$ $\quad (\log = \log_{10})$

By (4) we know the domain $\log x$ consists of all $x > 0$, and so

$D_f = \{x \mid 18 - x > 0\} = \{x \mid x < 18\} = \boxed{(-\infty, 18)}.$ ■

$\boxed{110}$ Find D_f for $f(x) = \ln(x^2 - 4x - 12)$.

$D_f = \{x \mid x^2 - 4x - 12 > 0\}$. We solve $x^2 - 4x - 12 > 0$, or

$(x-6)(x+2) > 0$.

This could be solved by the method of section 3.6, using the IVT; but here we will take another approach using cases.

To have $(x-6)(x+2) > 0$, we could have:

i) $x - 6 > 0$ & $x + 2 > 0$, $\longrightarrow$ $x > 6$ & $x > -2$ $\longrightarrow$ $\underline{x > 6}$

or

ii) $x - 6 < 0$ & $x + 2 < 0$. $\longrightarrow$ $x < 6$ & $x < -2$ $\longrightarrow$ $\underline{x < -2}$

So must have $x > 6$ or $x < -2$. Solution set: $(-\infty, -2) \cup (6, \infty)$.

$\therefore$ $D_f = \boxed{(-\infty, -2) \cup (6, \infty)}$ ■

4.3 - Properties of Logarithms

A note on notation: $\log_b u^r$ means $\log_b(u^r)$, while $\log_b^r u$ means $(\log_b u)^r$.

From section 4.2 $\rightarrow$ $\boxed{\begin{array}{c} b^x = y \iff \log_b(y) = x \\ \text{for all } x \in (-\infty, \infty) \ \& \ y \in (0, \infty) \end{array}}$, (3)

The "Big Three" properties of logarithms, often called laws of logarithms, are as follows:

L1)	$\log_b(uv) = \log_b u + \log_b v$
L2)	$\log_b(u/v) = \log_b u - \log_b v$
L3)	$\log_b(u^r) = r \log_b u$

Right side of each law is an "expanded form", the left side is a "condensed form".

expand

condense

<u>Proof 1:</u> Let $M = \log_b u$ & $N = \log_b v$. By (3) we then have $b^M = u$ & $b^N = v$. Now:

$$uv = b^M \cdot b^N = b^{M+N}.$$

Using (3) again, we convert $uv = b^{M+N}$ to a logarithmic equation:

$$\log_b(uv) = M + N = \log_b u + \log_b v. \quad \blacksquare$$

$\boxed{20}$ Expand the logarithmic expression $\ln \sqrt[7]{x}$ as much as possible, and where possible, evaluate logarithms in exact terms without a calculator.

$$\ln \sqrt[7]{x} = \ln(x^{1/7}) \overset{L3}{=} \boxed{\frac{1}{7} \ln x} \quad \blacksquare$$

$\boxed{28}$ Same instructions as #20: $\log_b\left(\frac{x^3 y}{z^2}\right)$

$$\log_b\left(\frac{x^3y}{z^2}\right) \overset{L2}{=} \log_b(x^3y) - \log_b(z^2)$$
$$\overset{L1}{=} \log_b(x^3) + \log_b y - \log_b(z^2)$$
$$\overset{L3}{=} \boxed{3\log_b x + \log_b y - 2\log_b z} \quad \blacksquare$$

expanding ↓ ↑ condensing

$\boxed{58}$ Condense the logarithmic expression as much as possible:

$$2\ln x - \frac{1}{2}\ln y.$$

The result should be a single logarithm with a coefficient of 1...

$$2\ln x - \frac{1}{2}\ln y \overset{L3}{=} \ln(x^2) - \ln(y^{1/2}) \overset{L2}{=} \boxed{\ln\left(\frac{x^2}{y^{1/2}}\right)} = \boxed{\ln\left(\frac{x^2}{\sqrt{y}}\right)} \quad \blacksquare$$

$\boxed{64}$ Condense the logarithmic expression as much as possible:

$$\frac{1}{3}\left(\log_4 x - \log_4 y\right).$$

$$\frac{1}{3}\left(\log_4 x - \log_4 y\right) \overset{L2}{=} \frac{1}{3}\log_4\left(\frac{x}{y}\right) \overset{L3}{=} \boxed{\log_4\left(\frac{x}{y}\right)^{1/3}} = \boxed{\log_4 \sqrt[3]{\frac{x}{y}}} \quad \blacksquare$$

Change-of-Base Formula (COB): $\log_b u = \dfrac{\log_a u}{\log_a b}$

Derivation: Let $x = \log_b u$. Then, by (3), $b^x = u$. Now, for any $a > 0$, $a \neq 1$, we have $\log_a(b^x) = \log_a u$. By (L3), $x \log_a b = \log_a u$, and thus $x = \dfrac{\log_a u}{\log_a b}$. Since $x = \log_b u$, this gives us the COB formula. $\blacksquare$

Ex Express $\log_4 100$ in terms of the base-10 (common) logarithm, then simplify if possible.

With the COB formula: $\log_4 100 = \dfrac{\log 100}{\log 4} = \dfrac{\log 10^2}{\log 2^2}$

By (B3) we have $\log(10^2) = 2$. Also $\log(2^2) \overset{L3}{=} 2\log 2$. Now,

$$\log_4 100 = \frac{\log 100}{\log 4} = \frac{\log 10^2}{\log 2^2} = \frac{\cancel{2}}{\cancel{2}\log 2} = \boxed{\frac{1}{\log 2}}$$

4.3.39 $\log\left[\dfrac{10x^2\sqrt[3]{1-x}}{7(x+1)^2}\right] \overset{L2}{=} \log\left(10x^2\cdot\sqrt[3]{1-x}\right) - \log\left(7(x+1)^2\right)$

$\overset{L1}{=} \log(10x^2) + \log\left(7(x+1)^2\right)$

$\overset{L1}{=} \log 10 + \log(x^2) + \log 7 + \log(x+1)^2$

$\overset{L3}{=} \log 10 + 2\log x + \log 7 + 2\log(x+1)$

4.3.73 Find to four decimal places: $\log_{14} 87.5$.

Use $\log_b u = \dfrac{\log_a u}{\log_a b}$ with $b = 14$ & $u = 87.5$ & $a = e$:

$$\log_{14} 87.5 = \frac{\ln 87.5}{\ln 14} = 1.694407\ldots \approx \boxed{1.6944}$$

4.3.87 Let $\log_b 2 = A$ & $\log_b 3 = C$. Write $\log_b\sqrt{\frac{2}{27}}$ in terms of A & C.

- Expand: $\log_b\sqrt{\frac{2}{27}} = \log_b\left(\frac{2}{27}\right)^{1/2} \overset{L3}{=} \frac{1}{2}\log_b\frac{2}{27} \overset{L2}{=} \frac{1}{2}\left(\log_b 2 - \log_b 27\right)$

$= \frac{1}{2}\left(\log_b 2 - \log_b(3^3)\right) \overset{L3}{=} \frac{1}{2}\left(\log_b 2 - 3\log_b 3\right) = \boxed{\frac{1}{2}(A - 3C)}$

$= \boxed{\dfrac{A-3C}{2}}$

4.4 - Exponential & Logarithmic Equations

L1) $\log_b(uv) = \log_b u + \log_b v$

L2) $\log_b(u/v) = \log_b u - \log_b v$

L3) $\log_b(u^r) = r\log_b u$

The laws of logarithms above are only valid when u>0 and v>0!!!

Suppose $u = -10$ & $v = -100$ & $b = 10$.

$\log(uv) = \log((-10)(-100)) = \log 1000 = \log(10^3) = 3.$

$\log u + \log v = \log(-10) + \log(-100) = $ undefined.

So $\log_b(uv) \neq \log_b u + \log_b v$ (left side is 3, right side is undefined).

$\boxed{20}$ Solve $8^{1-x} = 4^{x+2}$.

In this case it's possible to get a matching base of 2 on each side:

$8^{1-x} = 4^{x+2} \Rightarrow (2^3)^{1-x} = (2^2)^{x+2} \Rightarrow 2^{3(1-x)} = 2^{2(x+2)}$,

and so $3(1-x) = 2(x+2)$. This yields $\boxed{x = -\frac{1}{5}}$

$\boxed{\text{Ex}}$ Solve $4^{2x+1} = 3^{2-x}$. Give the exact answer and the answer rounded to the nearest hundredth.

The base of 4 at left cannot be reconciled with the base of 3 at right. So we take a logarithm of both sides. What base logarithm? Usually base e (i.e. the natural logarithm ln).

$4^{2x+1} = 3^{2-x} \Rightarrow \ln(4^{2x+1}) = \ln(3^{2-x}) \overset{L3}{\Rightarrow}$

$$(2x+1)\ln 4 = (2-x)\ln 3$$

$$2x\ln 4 + \ln 4 = 2\ln 3 - x\ln 3$$

$$2x\ln 4 + x\ln 3 = 2\ln 3 - \ln 4$$

$$x(2\ln 4 + \ln 3) = 2\ln 3 - \ln 4$$

$$x = \boxed{\frac{2\ln 3 - \ln 4}{2\ln 4 + \ln 3}} \leftarrow \text{Exact answer!}$$

Could write as $x = \dfrac{\ln(3^2) - \ln 4}{\ln(4^2) - \ln 3} = \dfrac{\ln(3^2/4)}{\ln(4^2/3)} = \boxed{\dfrac{\ln(9/4)}{\ln(16/3)}}$

The exact answer should be given. Rounded to the nearest hundredth the answer is $0.4844\ldots \approx \boxed{0.48}$ ■

$\boxed{\text{Ex}}$ Solve $5^{2x} + 5^x - 12 = 0$. Get an exact answer in terms of the natural or common logarithm, and then an approximate answer correct to two decimal places.

- The equation has a quadratic form (section 1.6):

  $(5^x)^2 + 5^x - 12 = 0 \xrightarrow{\text{Let } u = 5^x} u^2 + u - 12 = 0 \rightarrow (u+4)(u-3) = 0,$

  and so $u = -4$ or $u = 3$.

- So $5^x = -4$ or $5^x = 3$

  $5^x = -4$: No solution

  $5^x = 3$: $\ln(5^x) = \ln 3 \xrightarrow{L3} x\ln 5 = \ln 3 \rightarrow x = \dfrac{\ln 3}{\ln 5}$

- Exact answer $\rightarrow \boxed{\dfrac{\ln 3}{\ln 5}} = 0.6826\ldots \approx \boxed{0.68} \leftarrow$ Approximate answer. ■

The following example illustrates a one-logarithm equation:

$\boxed{56}$ Solve exactly & to two decimal places: $\log_2(x+50) = 5$.

- Use $b^x = y \Leftrightarrow \log_b(y) = x$ to write $2^5 = x + 50 \Rightarrow$

  $x = 2^5 - 50 = 32 - 50 = \boxed{-18}$ ■

The following example illustrates a two-logarithm equation:

$\boxed{72}$ Solve exactly & to two decimal places: $\log_4(x+2) - \log_4(x-1) = 1$.

- Consolidate the two logarithms using L2 ... $\log_4\left(\frac{x+2}{x-1}\right) = 1$.
- Now solve like a one-logarithm equation:

$$4^1 = \frac{x+2}{x-1} \Rightarrow 4(x-1) = x+2 \Rightarrow 3x = 6 \Rightarrow \boxed{x=2}$$

Since exact answer is not in terms logarithms a decimal approximation is not necessary.

If a logarithmic equation has three or more logs in it, we consolidate them using laws of logarithms, just like when there are two logs as in #72.

$\boxed{80}$ $\log(2x-1) = \log(x+3) + \log 3$.

$\log(2x-1) = \log[3(x+3)]$ by L1.

one way

$\log(2x-1) - \log(3x+9) = 0$

$\log\left(\frac{2x-1}{3x+9}\right) = 0$

$10^0 = \frac{2x-1}{3x+9}$

$\frac{2x-1}{3x+9} = 1$

$2x-1 = 3x+9$

$x = -10$

We're assuming $2x-1>0$ & $3x+9>0$; that is, $x>\frac{1}{2}$ & $x>-3$, or simply: $x>\frac{1}{2}$.

faster way

$\log_b$ is a one-to-one function for any $b>0, b\neq 1$: $\log_b u = \log_b v$ implies $u=v$.
So here we have
$2x-1 = 3(x+3)$

But x=-10 results in negative numbers appearing in the logarithms in the ORIGINAL equation, and so must be extraneous. Discarding -10, we're left with no solution. Solution set is: $\boxed{\emptyset}$

96 $3|\log x| - 6 = 0.$

This is an absolute value equation; as in section 1.7 the first step is to isolate the absolute value:

$$3|\log x| = 6 \Rightarrow |\log x| = 2 \Rightarrow \log x = 2 \text{ or } \log x = -2.$$

$$10^2 = x \quad \text{or} \quad 10^{-2} = x$$

So $x = 100$ or $x = \frac{1}{100}$.

Solution set: $\boxed{\left\{\frac{1}{100}, 100\right\}}$.

---

100 Solve $\ln 3 - \ln(x+5) - \ln x = 0.$

$$\ln 3 = \ln(x+5) + \ln x$$

$$\ln 3 = \ln x(x+5)$$

$$3 = x(x+5),$$ since $\ln$ is one-to-one

$$x^2 + 5x - 3 = 0$$

This is not factorable over obvious values, so use the quadratic formula to solve the quadratic equation:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(-3)}}{2(1)} = \frac{-5 \pm \sqrt{37}}{2}.$$

With a calculator we find that $\frac{-5+\sqrt{37}}{2} > 0$, so if $x = \frac{-5+\sqrt{37}}{2}$, then $x > 0$ & $x+5 > 0$, and the original equation is satisfied. Thus $\frac{-5+\sqrt{37}}{2}$ is a solution!

But: $\frac{-5-\sqrt{37}}{2} < 0$, so if $x = \frac{-5-\sqrt{37}}{2}$ then $\ln x$ is undefined. Hence $\frac{-5-\sqrt{37}}{2}$ is extraneous.

Solution set: $\boxed{\left\{\frac{-5+\sqrt{37}}{2}\right\}}$

$\boxed{47}$ Solve $3^{2x}+3^x-2=0$.

- This is $(3^x)^2+3^x-2=0$.
- Let $u=3^x$
- Have $u^2+u-2=0$.
- Solve: $(u+2)(u-1)=0 \Rightarrow$

$u+2=0$ or $u-1=0$.

$\downarrow$ $\qquad\qquad$ $\downarrow$

$u=-2$ $\qquad\qquad$ $u=1$

$3^x=-2$ $\qquad\qquad$ $3^x=1$

No solution $\qquad\qquad$ $3^x=3^0$

$x=0$ ✓

- Solution set: $\boxed{\{0\}}$ ■

$\boxed{46}$ Solve $e^{4x}-3e^{2x}-18=0$.

- This is $(e^{2x})^2-3e^{2x}-18=0$
- Let $u=e^{2x}$
- So $u^2-3u-18=0 \Rightarrow (u-6)(u+3)=0 \Rightarrow$

$u-6=0$ or $u+3=0$

$u=6$ or $u=-3$

$e^{2x}=6$ or $e^{2x}=-3$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$\ln e^{2x}=\ln 6$ $\qquad\qquad$ No solution

$2x=\ln 6$

$x=\boxed{\frac{1}{2}\ln 6}=0.895\ldots\approx\boxed{0.90}$ ■

$\boxed{49}$ Solve $\log_3 x = 4$

- Use $\boxed{b^x = y \iff \log_b(y) = x}$
- We get $3^4 = x$, or $x = \boxed{81}$ ∎

4.5 - Exponential Growth & Decay

An example of exponential decay would be the decay of a radioactive isotope. The amount A of an isotope is a function of time t, and according to the Law of Uninhibited Decay the function is

$$A(t) = A_0 e^{-kt},$$

where k>0 is a constant and A_0 is the amount of isotope at time t=0. That is, $A_0 = A(0)$.

The half-life of a radioactive isotope is the time h required for half of the isotope to decay. (By decay is meant the nucleus fissions into simpler particles.)

For population growth there is the Law of Uninhibited Growth:

$$A(t) = A_0 e^{kt},$$

where k>0 is a constant, A(t) is the population at time t, and A_0= A(0).

$\boxed{20}$ Carbon-14 (C14) is a radioactive isotope of carbon that decays according to the model:

$$A(t) = A_0 e^{-0.000121t},$$

where t is time in years.

A skeleton was found at a construction site in 1989. It contained 88% of the C14 that would be found in a living specimen. When did the individual whose skeleton was found die?

Let t=0 be the time of death. We must find what the value of t is in the year 1989 (in years). The time t must be that time it takes for 12% of the C14 in a skeleton to decay away. If there was A_0 grams of C14 in the skeleton at time t=0, then in 1989 there is $0.88A_0$ g left. So... $A(t) = 0.88A_0$, and with the model above

$$\cancel{A_0}e^{-0.000121t} = 0.88\cancel{A_0}.$$

$$e^{-0.000121t} = 0.88$$

$$\ln e^{-0.000121t} = \ln 0.88$$

$$-0.000121t = \ln 0.88 \quad (\text{use } \log_b(b^r) = r)$$

$$t = \frac{\ln 0.88}{-0.000121} \leftarrow \text{LN(0.88)/-0.000121 in TI-30X}$$

For a problem like this is makes sense to round to the nearest year...

$$t = 1056.47 \approx \boxed{1056 \text{ years ago}}$$

---

28 The half-life of thorium-229 is 7340 years. How long will it take for a sample of this substance to decay to 20% of its original amount?

- As with all radioactive decay problems we assume A(t), the amount of thorium-229 present at time t, is given by the Law of Uninhibited Decay: $A(t) = A_0 e^{-kt}$, $k > 0$.

- We're given the half-life is 7340 years. So t will be in years, and we know that A(7340) = $\frac{1}{2}$A(0).

$$A(7340) = \tfrac{1}{2}A(0) \Rightarrow \cancel{A_0}e^{-7340k} = \tfrac{1}{2}\cancel{A_0}\underbrace{e^{-0\cdot k}}_{e^0 = 1} \Rightarrow e^{-7340k} = \tfrac{1}{2} \Rightarrow$$

$$\ln e^{-7340k} = \ln\tfrac{1}{2} \Rightarrow -7340k = \ln\tfrac{1}{2} \Rightarrow k = \frac{\ln 0.5}{-7340},$$

or $k = 9.443 \times 10^{-5}$. **So:**

$$A(t) = A_0 e^{-9.443\times10^{-5}t} = A_0 e^{-0.00009443t}, \qquad (4.5.1)$$

- We now use our model to find the time t for which A(t) = 0.2A_0 :

Using (4.5.1) we get :

$$0.2\cancel{A}_0 = \cancel{A}_0 e^{-0.00009443t} \Rightarrow e^{-0.00009443t} = 0.2 \Rightarrow$$

$$\ln e^{-0.00009443t} = \ln 0.2 \Rightarrow -0.00009443t = \ln 0.2,$$

and so $t = \dfrac{\ln 0.2}{-0.00009443} = 17{,}043.7 \approx \boxed{17{,}044 \text{ years}}$

$\boxed{\text{Ex}}$ At time t=0 there is 100 g of a radioactive isotope. Then at time t=10 hours only 7.6 g remains. For the following, round to the nearest tenth.

a) How much isotope will be present after 15 hours?
b) What is the half-life of the isotope?
c) When will there be just 1 gram of isotope left?

Before starting part (a) we need to find k for our model. We already know that $A_0 = 100$, so that

$$A(t) = 100e^{-kt} \qquad (4.5.2)$$

We are given that A(10) = 7.6, which with (4.5.2) we can use to find k:

$$7.6 = A(10) = 100e^{-k\cdot 10} \longrightarrow 100e^{-10k} = 7.6 \longrightarrow$$

$$e^{-10k} = 0.076 \longrightarrow \ln e^{-10k} = \ln 0.076 \longrightarrow$$

$$-10k = \ln 0.076 \longrightarrow k = -\frac{\ln 0.076}{10} \approx 0.2577$$

Our model is therefore

$$A(t) = 100e^{-0.2577t}. \qquad (4.5.3)$$

a) At time t=15 hours we have

$$A(15) = 100e^{-0.2577(15)} = 2.095 \approx \boxed{2.1 \text{ g}}$$

b) To find the half-life we find the time t for which A(t) = 50 g (i.e. half of the initial 100 g).

$$A(t)=50 \xrightarrow{(4.5.3)} 100e^{-0.2577t}=50 \longrightarrow e^{-0.2577t}=0.5$$

$$\longrightarrow \ln e^{-0.2577t}=\ln 0.5 \longrightarrow -0.2577t=\ln 0.5 \longrightarrow$$

$$t=\frac{\ln 0.5}{-0.2577}=2.689\approx \boxed{2.7 \text{ hours.}}$$

c) Here we find the time t when just 1 gram of isotope is left; that is, find t such that A(t)=1.

$$A(t)=1 \xrightarrow{(4.5.3)} 100e^{-0.2577t}=1 \longrightarrow e^{-0.2577t}=0.01$$

$$\longrightarrow \ln e^{-0.2577t}=\ln 0.01 \longrightarrow -0.2577t=\ln 0.01 \longrightarrow$$

$$t=\frac{\ln 0.01}{-0.2577}=17.87\approx \boxed{17.9 \text{ hours}}$$

$\boxed{4.5.27a}$ Potassium-40 (K40) decays with a half-life of 1.31 billion years. Find the decay model for K40.

- If we start with A_0 grams of K40 at time t=0, then at time $t=1.31\times10^9$ we have $\frac{1}{2}A_0$ grams left. By the Law of Uninhibited Decay

$$A(t)=A_0e^{-kt}.$$

So we have $A(1.31\times10^9)=\frac{1}{2}A_0$ & $A(1.31\times10^9)=A_0e^{-k(1.31\times10^9)}$,

implying $\frac{1}{2}\cancel{A_0}=\cancel{A_0}e^{-k(1.31\times10^9)}$. Then...

$$e^{-k(1.31\times10^9)}=\frac{1}{2} \Rightarrow \ln e^{-k(1.31\times10^9)}=\ln\frac{1}{2} \Rightarrow$$

$$-k(1.31\times10^9)=\ln\frac{1}{2}=-\ln 2 \Rightarrow k=\frac{\ln 2}{1.31\times10^9}=5.2912\times10^{-10}$$

So $A(t)=A_0e^{-5.2912\times10^{-10}t}$

To get what the textbook indicates we should, note that the book has time measured in "billions of years" rather than "years." If we let time be measured in billions of years, then t = 1.31, so that

$$k = \frac{\ln 2}{1.31} = 0.52912,$$

and then $A(t) = A_0 e^{-0.52912t}$.

$\boxed{4.5.27b}$ Analysis of the rocks surrounding the dinosaur bones indicated that 94.5% of the original amount of K40 was still present. How old are the bones?

- Let A_0 be the original amount of K40 (amount at time $t=0$)

  Today is time t, with $A(t) = 0.945 A_0$.

  We want to find the value of t. We have:

$$0.945 \cancel{A_0} = \cancel{A_0} e^{-0.52912t} \Rightarrow e^{-0.52912t} = 0.945 \Rightarrow$$

$$\ln e^{-0.52912t} = \ln 0.945 \Rightarrow -0.52912t = \ln 0.945 \Rightarrow$$

$$t = \frac{\ln 0.945}{-0.52912} = 0.1069 \text{ billion years.}$$

So the bones are $0.1069 \times 10^9 = \boxed{106{,}900{,}000 \text{ years}}$ old ■

5.1 - Systems of Linear Equations in Two Variables

A linear equation in two variables x and y has standard form $Ax+By=C$, where A, B, and C are constants (A and B not both 0).

A system of linear equations in two variables is a set of two or more linear equations in two variables, which, taken together, feature variables x and y. The form is this:

$$\underbrace{\begin{cases} A_1x+B_1y=C_1 & , \quad (E1) \\ A_2x+B_2y=C_2 & , \quad (E2) \end{cases}}_{\text{Brace indicates a set of equations.}}$$

Named for convenience.

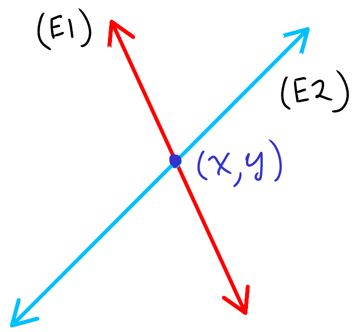
The solution set of the system above consists of all ordered pairs (x,y) that are in the solution set of both (E1) and (E2).

Recall that linear equations graph as lines on the xy-plane. A point with coordinates (x,y) satisfies a linear equation if and only if the point lies on the line. So, (E1) and (E2) both graph as lines, and for an ordered pair (x,y) to satisfy both equations, it must be a point that lies on BOTH lines.

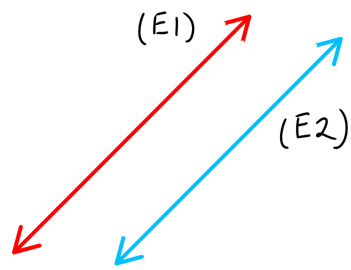
There are three cases: the lines (E1) and (E2) intersect at a single point (skew lines); or the lines are parallel but distinct, and so intersect at no point; or the lines are identical and so intersect at every one of their points. Algebraically this means the system above must have either:

1) Precisely one solution.
2) No solution.
3) An infinite number of solutions.

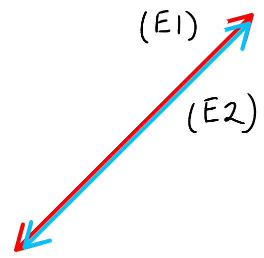
A system is consistent if it has at least one solution (cases 1 & 3), and inconsistent if it has no solution (case 2). A system is dependent if it has infinitely many solutions (case 3), otherwise it's independent (cases 1 & 2).



One solution



No solution



Infinite solutions

$\boxed{24}$ Solve the system $\begin{cases} 2x - 7y = 2 & , \ (E1) \\ 3x + y = -20 & , \ (E2) \end{cases}$

Substitution method: use one equation to express one of the variables in terms of the other variable, then substitute the expression into the other equation to solve for the other variable.

- Let convenience guide you: in this case it would be easiest to solve (E2) for y: $y = -3x - 20$, (E3)
- Substitute $-3x-20$ for y in (E1): $2x - 7(-3x - 20) = 2$
- Now solve for x: $2x + 21x + 140 = 2 \rightarrow 23x = -138 \rightarrow x = -6$
- Now use $x = -6$ & (E3) to solve for y:

  $y = -3(-6) - 20 = -2$.
- State the solution set for the system: $\boxed{\{(-6,-2)\}}$

Elimination method (textbook: "Addition method"): Potentially faster, but best used when the system features only integers. The approach: multiply each equation in the system by constants chosen so that, when the equations are added or subtracted, one of the variables is eliminated. The rest of the procedure is much like the substitution method.

- $\begin{cases} 2x - 7y = 2 & , (E1) \\ 3x + y = -20 & , (E2) \end{cases} \xrightarrow{7\cdot(E2)} \begin{cases} 2x - 7y = 2 \\ 21x + 7y = -140 \end{cases}$

$$\begin{aligned} + \quad & \\ 23x + 0y &= -138 \\ 23x &= -138 \\ x &= -6 \end{aligned}$$

← Add the equations

- Now substitute -6 for x in either (E1) or (E2) and solve for y. In this case I'd say (E2) is more convenient to work with:

$3x + y = -20 \rightarrow 3(-6) + y = -20 \rightarrow y = -2.$

- Solution set: $\boxed{\{(-6,-2)\}}$

Ex Solve $\begin{cases} 4x - 8y = 16 & , (E1) \\ 3x - 6y = 12 & , (E2) \end{cases}$

We'll use the elimination/addition method: multiply (E1) by 3/4 to get

$\begin{cases} \frac{3}{4}(4x - 8y) = \frac{3}{4}(16) & , (E1) \\ 3x - 6y = 12 & , (E2) \end{cases} \longrightarrow \begin{cases} 3x - 6y = 12 & , (E1') \\ 3x - 6y = 12 & , (E2') \end{cases}$

The two equations (E1') and (E2') are identical, so (E1) and (E2) in the original system represent the same line, and the equations have the same solution set. Any solution to (E1) is also a solution to (E2), and vice-versa. The solution set to the system will be the solution set of either (E1) or (E2).

Using (E1) we can express the solution set to the system as follows:

$$\boxed{\{(x,y) \mid 4x - 8y = 16\}} \quad (1)$$

It's understood that x and y can only be real numbers here.

We could write the solution set other ways. For instance, from 4x-8y=16 we find that x=2y+4. So we could replace x with 2y+4 in (1):

$$\boxed{\{(2y+4, y) \mid y \in \mathbb{R}\}}$$

(Note: $y \in \mathbb{R}$ means y is any real number)

Some solutions:

letting $y=0$ we get $(2y+4,y)=(4,0)$

letting $y=-\frac{1}{2}$ we get $(2y+4,y)=(2(-\frac{1}{2})+4,-\frac{1}{2})=(-1+4,-\frac{1}{2})=(3,-\frac{1}{2})$

letting $y=50$ we get $(2y+4,y)=(2\cdot 50+4,50)=(104,50)$

etc.

$\boxed{\text{Ex}}$ A restaurant is to have two-seat tables and four-seat tables. Fire codes limit the restaurant's occupancy to 56. If the owners have hired enough servers to handle 17 tables, how many of each kind of table should they get?

- Define the variables involved:
  x = # of 2-seat tables
  y = # of 4-seat tables

- The total number of tables, ideally, will be exactly 17: x+y=17.

- Number of customers should not exceed 56, so: 2x+4y=56.

- System is: $\begin{cases} x+y=17 \xrightarrow{\text{isolate } y} y=17-x \\ 2x+4y=56 \end{cases}$

substitution

$$2x+4(17-x)=56 \xrightarrow[x]{\text{solve for}} x=6$$

$$y=17-x=17-6=11 \xleftarrow{\text{solve for } y}$$

Answer: $\boxed{\text{The restaurant needs 6 two-seat and 11 four-seat tables.}}$

We could solve by the Addition Method: multiply, say, the top equation by 4:

$$\left.\begin{array}{r} 4x+4y=68 \\ -\quad 2x+4y=56 \\ \hline 2x+0=12 \end{array}\right\} \text{subtract bottom from top!}$$

$\rightarrow 2x=12 \rightarrow x=6$. From this we can get that $y=11$.

$\boxed{\text{Ex}}$ Solve $\begin{cases} 6x + 2y = 7 \\ 3x + y = 2 \end{cases}$

- From the 2nd equation we easily get y=2-3x.
- Substitute 2-3x for y in the 1st equation:

$$6x + 2(2 - 3x) = 7$$

- Solve for x, if possible...

$$\cancel{6x} + 4 - \cancel{6x} = 7 \implies 4 = 7$$

- Since 4=7 is false, the system has no solution. Solution set: $\boxed{\emptyset}$

5.2 - Systems of Linear Equations in Three Variables

A linear equation in three variables x, y, z has the form

$$Ax + By + Cz = D$$

for constants A, B, C, D, where A=B=C=0 is disallowed.

A system of three linear equations in three variables has the form

$$\begin{cases} A_1x + B_1y + C_1z = D_1 \\ A_2x + B_2y + C_2z = D_2 \\ A_3x + B_3y + C_3z = D_3 \end{cases}$$

This kind of system can have no solution, or exactly one solution, or an infinite number of solutions, but we will only consider systems with exactly one solution in this section (i.e. consistent & independent systems). The solution will be an ordered triple of the form (x,y,z).

$\boxed{\text{Ex}}$ Solve $\begin{cases} 2x - y + z = 1 & , \ (E1) \\ 3x - 3y + 4z = 5 & , \ (E2) \\ 4x - 2y + 3z = 4 & , \ (E3) \end{cases}$

Most convenient would be to use (E1) to get z in terms of x & y:

$$z = 1 + y - 2x, \quad (E4)$$

- Now we put (E4) into (E2) and (E3) to get a system of two equations in the two unknowns x & y:

$$\begin{cases} 3x - 3y + 4(1 + y - 2x) = 5 \\ 4x - 2y + 3(1 + y - 2x) = 4 \end{cases}$$

$$\downarrow$$

$$\begin{cases} -5x + y = 1 \\ -2x + y = 1 \end{cases}$$

- We now solve this system by using either substitution method or elimination/addition method...

$$\begin{cases} -5x + y = 1, & (E2') \\ -2x + y = 1, & (E3') \end{cases}$$
$$-$$
$$-3x + 0 = 0 \Rightarrow x = 0.$$

- Now put $x=0$ into, say, $(E2')$ to find y...

$$-5x+y=1 \Rightarrow -5(0)+y=1 \Rightarrow y=1.$$

- Finally, put $x=0$ & $y=1$ into $(E4)$ to get z...

$$z = 1+y-2x = 1+1-2(0) = 2.$$

- State the solution in the form (x,y,z). Solution set is: $\boxed{\{(0,1,2)\}}$

---

38 A brand of razor blades comes in packages of 6, 12, and 24, costing \$2, \$3, and \$4, respectively. A store sold twelve packages containing a total of 162 blades for \$35. How many packages of each type were sold?

- Define the variables:
  x = # of 6-packs sold
  y = # of 12-packs sold
  z = # of 24-packs sold

- Given: twelve packages were sold, so $x+y+z=12$.
- Given: 162 blades were sold, so $6x+12y+24z=162$
- Given: \$35 was raked in, so $2x+3y+4z=35$
- Resultant system:

$$\begin{cases} x + y + z = 12 \xrightarrow{1} z = 12 - x - y \\ 6x + 12y + 24z = 162 \\ 2x + 3y + 4z = 35 \end{cases}$$

$$\downarrow$$

$$\begin{cases} 6x+12y+24(12-x-y) = 162 \\ 2x+3y+4(12-x-y) = 35 \end{cases}$$

$$\begin{cases} -18x - 12y = -126 \\ -2x - y = -13 \end{cases} , \quad (E)$$

- We could use elimination/addition here: multiply the 2nd by -12 to set up for an elimination of y...

$$\begin{array}{r} \begin{cases} -18x - 12y = -126 \\ 24x + 12y = 156 \end{cases} \\ + \hline 6x + 0 = 30 \end{array} \Rightarrow x = 5.$$

- From (E): $y = 13 - 2x = 13 - 2(5) = 3$.
- Now get z: $z = 12 - x - y = 12 - 5 - 3 = 4$.

State the solution to the problem in a plain English sentence, as with any word problem...

- $\boxed{\text{The store sold 5 six-packs, 3 twelve-packs, and 4 24-packs.}}$

$\boxed{20}$ Find the quadratic function $y = ax^2 + bx + c$ whose graph passes through the points (-2,7), (1,-2), (2,3).

- Each of the three given ordered pairs are solutions to the equation $y = ax^2 + bx + c$.

So for (-2,7) we have $x = -2$ & $y = 7$, and the equation $y = ax^2 + bx + c$ is satisfied when $x = -2$ & $y = 7$. Thus:

$$7 = a(-2)^2 + b(-2) + c \Rightarrow 4a - 2b + c = 7$$

Also $y = ax^2 + bx + c$ is satisfied when $x = 1$ & $y = -2$:

$$-2 = a(1)^2 + b(1) + c \Rightarrow a + b + c = -2$$

Also $y = ax^2 + bx + c$ is satisfied when $x = 2$ & $y = 3$:

$$3 = a(2)^2 + b(2) + c \Rightarrow 4a + 2b + c = 3$$

$$\begin{cases} 4a-2b+c=7 \\ a+b+c=-2 \\ 4a+2b+c=3 \end{cases} \qquad a+b+c=-2 \xrightarrow{1} c=-2-a-b$$

$$\begin{cases} 4a-2b+(-2-a-b)=7 \\ 4a+2b+(-2-a-b)=3 \end{cases}$$

$$\begin{cases} 3a-3b=9 \\ 3a+b=5 \quad (E) \end{cases}$$

$$0-4b=4 \Rightarrow \boxed{b=-1}$$

Substitute -1 for b in (E): $3a+(-1)=5 \Rightarrow 3a=6 \Rightarrow \boxed{a=2}$.

Finally: $c=-2-a-b=-2-2-(-1) \Rightarrow \boxed{c=-3}$

Therefore $y=ax^2+bx+c$ must be $\boxed{y=2x^2-x-3}$