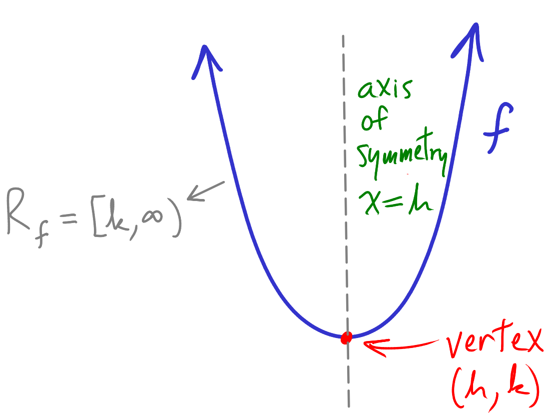
3.1 - Quadratic Functions

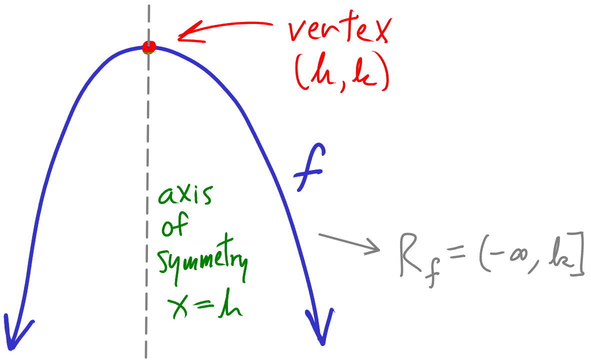
We say f is a quadratic function if, for all real x and some constants a, b, c, with a ≠ 0, we have

$$\boxed{f(x) = ax^2 + bx + c}$$ (Standard form of a quadratic function)

The graph of a quadratic function is always a parabola, a kind of conic section.



$a>0$



$a<0$

Given a quadratic function, we're concerned with finding the coordinates of the associated parabola's vertex. To do this, we get the quadratic function's expression into vertex form:

$$\boxed{f(x) = a(x-h)^2 + k}$$ (Vertex form of quadratic function)

If we suppose for the sake of argument that a>0, then since $(x-h)^2 \geq 0$ for all real x, we have $a(x-h)^2 \geq 0$ for all x, and thus

$$f(x) = a(x-h)^2 + k \geq k$$

for all $x \in D_f = (-\infty, \infty)$. This means that the points on the graph of f have y-coordinates that are no lower than k, and thus the vertex of the parabola has y-coordinate k. Since f(h)=k, we then find that the vertex has x-coordinate h. Note: $R_f = [k, \infty)$ if $a>0$.

Now to find h and k in terms of a, b, and c:

$$
\begin{aligned}
f(x) &= ax^2+bx+c \\
&= a\left(x^2+\frac{b}{a}x\right)+c \\
&= a\left(x^2+\frac{b}{a}x+\underline{\left(\frac{b}{2a}\right)^2}\right)+c-\underline{a\cdot\left(\frac{b}{2a}\right)^2}
\end{aligned}
$$

$$\frac{b}{a} \xrightarrow{\frac{1}{2}\cdot\frac{b}{a}} \frac{b}{2a} \xrightarrow{\square} \left(\frac{b}{2a}\right)^2$$

$$
\begin{aligned}
&= a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right) \\
&= a(x-h)^2+k
\end{aligned}
$$

$$h=-\frac{b}{2a} \qquad k=c-\frac{b^2}{4a}$$

$$\boxed{f(x)=ax^2+bx^2+c \text{ has vertex at } (h,k)=\left(-\frac{b}{2a},\ c-\frac{b^2}{4a}\right).}$$

Since $f(h)=k$, we can say vertex is at $(h,f(h))$, where $h=-\frac{b}{2a}$.

Axis of Symmetry of a parabola with vertex (h,k) is the vertical line x=h.

$\boxed{16}$ Find the vertex & axis of symmetry for $f(x)=-2x^2+8x-1$.

Here we have a=-2, b=8, and c=-1.

The x-coordinate of the vertex is $h = -\frac{b}{2a} = -\frac{8}{2(-2)} = 2$

The y-coordinate of the vertex is $k = f(h) = f(2) = -2(2)^2+8(2)-1 = 7$

Vertex is at (h,k) = $\boxed{(2,7)}$.

Axis of symmetry is $\boxed{x = 2}$. ■

If $f(x)=ax^2+bx+c$, then $D_f=(-\infty,\infty)$ in any case, and

$$R_f=\begin{cases}[k,\infty), & \text{if } a>0\\ (-\infty,k], & \text{if } a<0\end{cases} \qquad \left(k=-\frac{b}{2a}\right)$$

$\boxed{\text{Ex}}$ Find the domain & range for $f(x)=-2x^2+8x-1$.

$\boxed{D_f=(-\infty,\infty)}$ as always. Meanwhile: since $a=-2<0$ and we found that $k=7$ in the previous example, we have $\boxed{R_f=(-\infty,k]=(-\infty,7]}$ ■

$\boxed{36}$ $f(x)=3x^2-2x-4$.

Here we have $a=3$, $b=-2$, $c=-4$, so...

- Vertex: $h=-\frac{b}{2a}=-\frac{-2}{2(3)}=\frac{1}{3}$,

  $k=f(h)=f(\frac{1}{3})=3(\frac{1}{3})^2-2(\frac{1}{3})-4=-\frac{13}{3}$

  So vertex is at $\boxed{(1/3,-13/3)}$

- Axis of Symmetry: $\boxed{x=\frac{1}{3}}$

- $\boxed{D_f=(-\infty,\infty)}$

- Since $a=3>0$, range is $\boxed{R_f=[k,\infty)=[-\frac{13}{3},\infty)}$

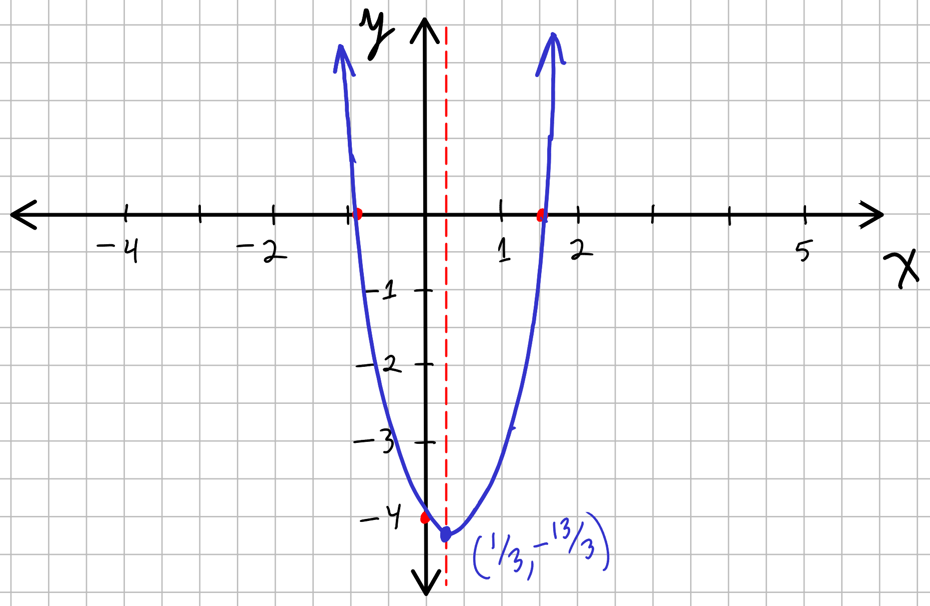
- x-intercepts: Find any real values of x for which $f(x)=0$...

  $f(x)=0 \Rightarrow 3x^2-2x-4=0 \Rightarrow$

  $$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-4)}}{2(3)}=\frac{2\pm\sqrt{4+48}}{6}=\frac{2\pm\sqrt{52}}{6}\Rightarrow$$

  $$x=\frac{2\pm2\sqrt{13}}{6}=\boxed{\frac{1\pm\sqrt{13}}{3}}$$

- y-intercept: $f(0)=\boxed{-4}$ ■



$$\frac{1+\sqrt{13}}{3} \approx 1.54$$

$$\frac{1-\sqrt{13}}{3} \approx -0.87$$

3.2 - Polynomial Functions

We say f is a polynomial function if f(x) equals some polynomial for all real x. More precisely, for n a whole number, f is a polynomial function if

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad \text{(standard form)}$$

for all $x \in (-\infty, \infty)$. Here $a_0, a_1, \ldots, a_n$ are constants called coefficients. If $a_n \neq 0$, then a_n is the leading coefficient of the polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

and we say the degree of the polynomial is n. The degree of a polynomial is thus the highest power of x in the polynomial that has a nonzero coefficient.

Ex

- $f(x) = -2x^4 - 3x + 11$ is a degree 4 polynomial function. We may write $\boxed{\deg(f) = 4.}$ ← If $f(x)$ is a degree 4 polynomial, we also say f is a degree 4 polynomial function.
- $g(x) = 8 - 2x^3 + 4x - 108x^5$ is a degree 5 polynomial function. That is, $\boxed{\deg(g) = 5.}$
- $h(x) = 2$. Since 2 is essentially the same as $2x^0$, so that $h(x) = 2x^0$, we see that h is a degree 0 poly. func.; that is, $\boxed{\deg(h) = 0.}$
- $q(x) = 0$. By definition the zero function has degree $\boxed{-\infty}$. So $\boxed{\deg(q) = -\infty}$. ■

A number c is a zero for a function f if f(c)=0. In this section and in sections 3.3 and 3.4, which are all about polynomial functions, we will occasionally let a complex number to be put into a polynomial function f. A polynomial function is then viewed as having domain consisting of all complex numbers as well as all real numbers.

So, f(c)=0 implies that c is an x-intercept for f only if c is real-valued. (There are no complex numbers on the x-axis, after all.)

Given a polynomial function f, the multiplicity of a zero c is equal to the number of factors x-c that the polynomial f(x) has in its full factorization.

The degree of a polynomial always equals the sum of the multiplicities of its zeros.

$\boxed{\text{Ex}}$ Let $f(x) = 6(x-1)(2x+1)^2(x+3)^5$. Then f has zeros $1, -\frac{1}{2}, -3$:

- When $x=1$, we get $x-1=0$. $\longrightarrow$ Then $f(1)=0$
- When $x=-\frac{1}{2}$, we get $2x+1=0$. $\longrightarrow$ Then $f(-\frac{1}{2})=0$
- When $x=-3$, we get $x+3=0$. $\longrightarrow$ Then $f(-3)=0$

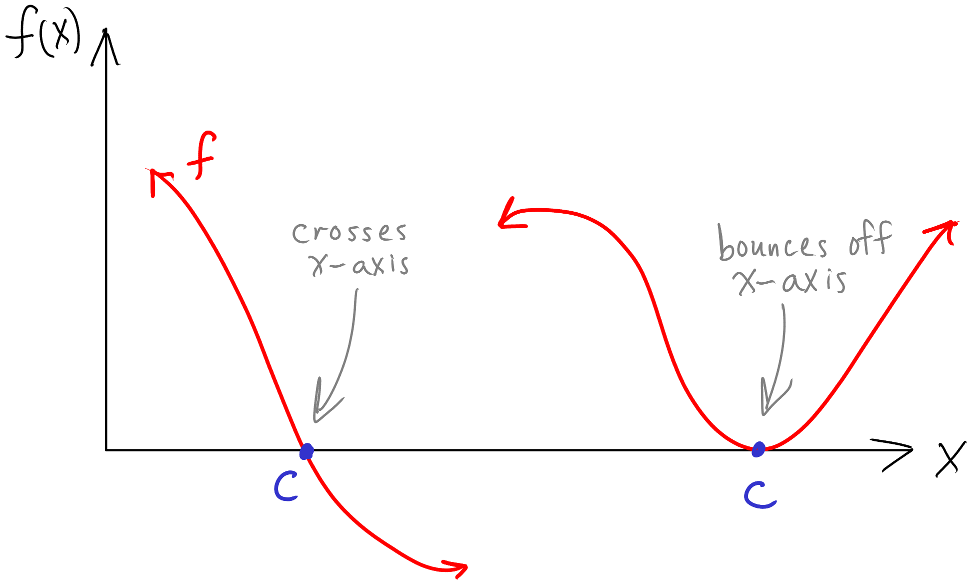
- Multiplicity of the zero 1 is $\boxed{1}$, since the factorization of $f(x)$ has only one $x-1$ factor.

- Multiplicity of the zero $-\frac{1}{2}$ is $\boxed{2}$, since the factorization of $f(x)$ has 2 factors of $2x+1$.

- Multiplicity of the zero -3 is $\boxed{5}$, since the factorization of $f(x)$ has 5 factors of $x+3$.

- $\deg(f) = 1+2+5 = \boxed{8}$ (the sum of the multiplicities) ■

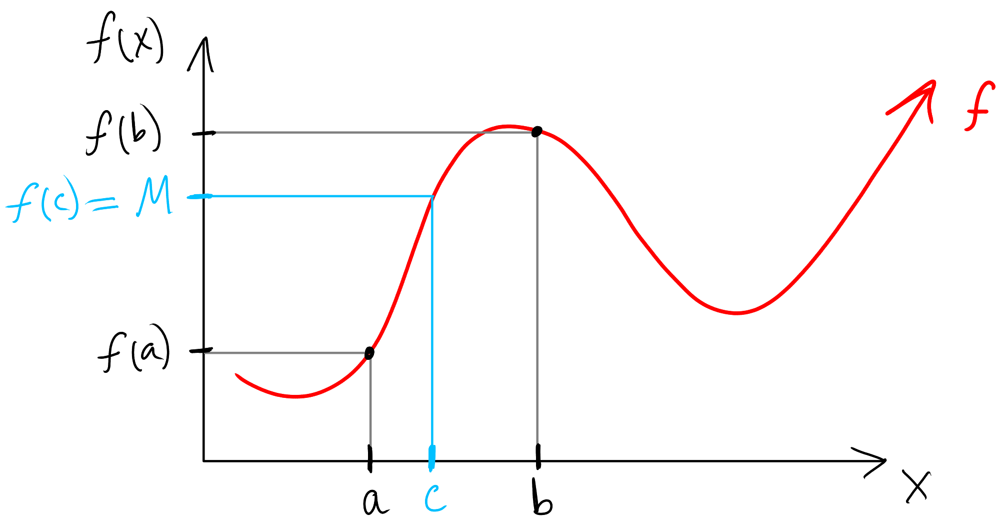
If a zero c for f has odd multiplicity, then the graph of f will cross the x-axis at c. If zero c has even multiplicity, then the graph of f will touch but not cross the x-axis (we'll say the graph "bounces off" the x-axis).



$\boxed{\text{Ex}}$ We found that $f(x) = 6(x-1)(2x+1)^2(x+3)^5$ has zeros $1, -\frac{1}{2}, -3$. Multiplicity of 1 is 1, which is odd, so graph of f crosses the x-axis at $x=1$. Multiplicity of $-\frac{1}{2}$ is 2, which is even, so graph bounces off x-axis at $x=-\frac{1}{2}$. Multiplicity of -3 is 5, which is odd, so graph crosses x-axis at $x=-3$. □

Intermediate Value Theorem for Polynomial Functions (IVT):

Let f be a polynomial function, let a<b, and suppose f(a) ≠ f(b). If M is a value that lies between f(a) and f(b), then there exists some a<c<b such that f(c)=M.



$\boxed{40}$ Use the IVT to show that the polynomial function has a zero between 2 and 3: $f(x) = 3x^3 - 8x^2 + x + 2$.

- We have $f(2) = 3\cdot 2^3 - 8\cdot 2^2 + 2 + 2 = -4 < 0$

  $f(3) = 3\cdot 3^3 - 8\cdot 3^2 + 3 + 2 = 14 > 0$

- So $M = 0$ is a value that lies between $f(2) < 0$ & $f(3) > 0$.

  By the IVT there exists some c between 2 and 3 (i.e. 2<c<3) such that f(c)=0. □ (Note: we're not supposed to find c exactly.)

$\boxed{\text{Ex}}$ Consider $f(x) = 3x^3 - 8x^2 + x + 2$ again.

- $f(2.5) = 3(2.5)^3 - 8(2.5)^2 + 2.5 + 2 = 1.375 \Rightarrow f(2.5) > 0$.

  In #40 we found $f(2) < 0$, so IVT implies there is some $2 < c < 2.5$ such that $f(c) = 0$.

- $f(2.25) \approx -2.08 < 0$, while $f(2.5) > 0$. So IVT implies $f(c) = 0$ for some $2.25 < c < 2.5$.

- Continuing this process, we can hone in on the zero c to an arbitrary degree of accuracy. This is the "bisection method," which is not as efficient as Newton's method, but Newton's method requires calculus. □

3.3 - Dividing Polynomials

It is important to know how to do polynomial long division, so an example or two will be done by way of review.

$\boxed{6}$ Divide using long division: $(6x^3+17x^2+27x+20)\div(3x+4)$

divisor

$$
\begin{array}{r l l}
\downarrow & & \\
 & 2x^2+3x+5 & \leftarrow \text{quotient}\\
3x+4\,\big) & \overline{6x^3+17x^2+27x+20} & \leftarrow \text{dividend}\\
 & \underline{6x^3+8x^2}\quad\downarrow\quad\ \ \downarrow & \\
 & \quad\ \ 9x^2+27x & \\
 & \quad\ \ \underline{9x^2+12x}\quad\ \ \downarrow & \\
 & \qquad\qquad 15x+20 & \\
 & \qquad\qquad \underline{15x+20} & \\
 & \qquad\qquad\qquad 0 & \leftarrow \text{remainder}
\end{array}
$$

So $$\frac{6x^3+17x^2+27x+20}{3x+4} = \boxed{2x^2+3x+5} \quad ■$$

$\downarrow$

Note that we thus have $6x^3+17x^2+27x+20=(3x+4)(2x^2+3x+5)$, so $6x^3+17x^2+27x+20$ has been partially factored!

$\boxed{16}$ Divide $\dfrac{2x^5-8x^4+2x^3+x^2}{2x^3+1}$

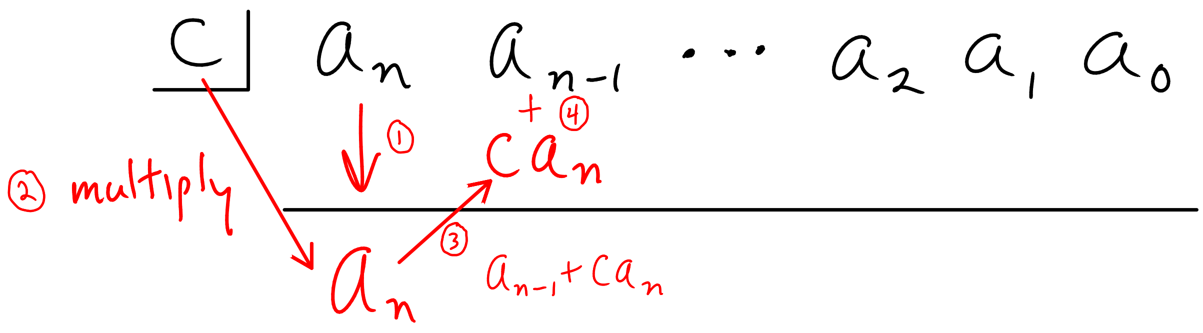
$$\begin{array}{r|rrrrrr}
 & & & x^2 & -4x & +1 & \\
\hline
2x^3+1 & 2x^5 & -8x^4 & +2x^3 & +x^2 & & \\
 & 2x^5 & & & +x^2 & & \\
\cline{2-6}
 & & -8x^4 & +2x^3 & +0x^2 & +0x & \\
 & & -8x^4 & & & -4x & \\
\cline{3-6}
 & & & 2x^3 & & +4x & \xleftarrow{\quad} 0x-(-4x) \\
 & & & 2x^3 & & +1 & \\
\cline{4-6}
 & & & & & 4x-1 &
\end{array}$$

$$\frac{2x^5-8x^4+2x^3+x^2}{2x^3+1} = \boxed{x^2-4x+1+\frac{4x-1}{2x^3+1}}$$

Synthetic division is a faster way to divide polynomials, but it is more specialized: the divisor must be of the form x-c for some constant c.

Say we want to do the division $\dfrac{f(x)}{x-c}$, where f is a polynomial function and so $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$.

The set-up & procedure:



$\boxed{\text{Ex}}$ Divide using synthetic division:

$$(5x^4+2x^3+x-4) \div (x-2).$$

This is $\dfrac{f(x)}{x-c}$ with $f(x) = 5x^4+2x^3+x-4$ & $x-c = x-2$, so that $c=2$.

$$\begin{array}{r|rrrrr} 2 & 5 & 2 & 0 & 1 & -4 \\ & \downarrow & 10 & 24 & 48 & 98 \\ \hline & 5 & 12 & 24 & 49 & 94 \\ & \multicolumn{4}{c}{\underbrace{\hphantom{5 \quad 12 \quad 24 \quad 49}}_{\text{quotient}}} & \underbrace{\hphantom{94}}_{\text{remainder}} \end{array} \quad \longleftarrow f(x) = 5x^4+2x^3+0x^2+1x+(-4)$$

$$\frac{5x^4+2x^3+x-4}{x-2} = \boxed{5x^3+12x^2+24x+49+\frac{94}{x-2}}$$ ■

Contrast with the long division:

$$\begin{array}{rl} & \quad 5x^3+12x^2+24x+49 \\ x-2 \big) & 5x^4+2x^3+0x^2+x-4 \\ & \underline{5x^4-10x^3} \qquad \downarrow \quad \downarrow \quad \downarrow \\ & \qquad 12x^3+0x^2 \\ & \qquad \underline{12x^3-24x^2} \\ & \qquad\qquad 24x^2+x \\ & \qquad\qquad \underline{24x^2-48x} \\ & \qquad\qquad\qquad 49x-4 \\ & \qquad\qquad\qquad \underline{49x-98} \\ & \qquad\qquad\qquad\qquad 94 \end{array}$$

$\left(2x^3-(-10x^3) = 2x^3+10x^3 = 12x^3\right)$

$\boxed{\text{Ex}}$ Divide by synthetic division: $\dfrac{x^5+32}{x+2}$

This is $\dfrac{x^5+32}{x-(-2)}$, so $c=-2$

$x^5+32 = 1x^5+0x^4+0x^3+0x^2+0x+32$

$$\begin{array}{r|rrrrrr} -2 & 1 & 0 & 0 & 0 & 0 & 32 \\ & \downarrow & \overset{+}{-2} & 4 & -8 & 16 & -32 \\ \hline & 1 & -2 & 4 & -8 & 16 & 0 \end{array}$$

$$\frac{x^5+32}{x+2} = 1x^4-2x^3+4x^2-8x+16+\frac{0}{x+2}$$

$$\frac{x^5+32}{x+2} = \boxed{x^4-2x^3+4x^2-8x+16}$$ ■

Note this means $x^5+32 = (x+2)(x^4-2x^3+4x^2-8x+16)$

Fact: when dividing a polynomial $f(x)$ by $x-c$, the quotient $q(x)$ is a polynomial with degree one less than $f(x)$, and the remainder is a constant.

Remainder Theorem (RT) Let f be a polynomial function. The remainder of the division

$$\frac{f(x)}{x-c}$$

is equal to f(c).

Proof: By the fact above, $\dfrac{f(x)}{x-c} = q(x) + \dfrac{r}{x-c}$. Multiply both sides by $x-c$:

$$(x-c)\frac{f(x)}{x-c} = \left(q(x)+\frac{r}{x-c}\right)(x-c)$$

$$f(x) = q(x)(x-c)+r.$$

Now we note that

$$f(c) = q(c)(c-c)+r = q(c)\cdot 0+r = r.$$ QED. ■

$\boxed{\text{Ex}}$ Let $f(x) = 5x^4+2x^3+x-4$. Find $f(2)$.

- By RT, $f(2)$ equals the remainder of $\frac{f(x)}{x-2}$. In an earlier example we found the remainder to be 94, so: $\boxed{f(2)=94}$
- Direct verification:

$$f(2) = 5(2)^4+2(2)^3+2-4 = 5\cdot 16+2\cdot 8-2$$

$$= 80+16-2 = 96-2 = 94. \checkmark$$

■

Let P and Q be two statements. To write "P if and only if Q" means: "If P, then Q; and if Q, then P."

Factor Theorem (FT) Let f be a polynomial function. Then x-c is a factor of f(x) if and only if f(c)=0.

Proof: This is in two parts:

- Proof that "If $x-c$ is a factor of $f(x)$, then $f(c)=0$":

  Suppose $x-c$ is a factor of $f(x)$. This means there exists a polynomial $p(x)$ such that $f(x) = (x-c)p(x)$. Then $f(c) = (c-c)p(c) = 0\cdot p(c) = 0$.

- Proof that "If $f(c)=0$, then $x-c$ is a factor of $f(x)$."

  Suppose $f(c)=0$. By RT, $f(c)$ is the remainder of

$\dfrac{f(x)}{x-c}$. Thus $\dfrac{f(x)}{x-c} = q(x) + \dfrac{0}{x-c} = q(x)$, where the polynomial $q(x)$ is the quotient. Now,

$\dfrac{f(x)}{x-c} = q(x) \implies f(x) = (x-c)q(x)$, which shows that $x-c$ is a factor of $f(x)$. $\blacksquare$

$\boxed{74}$ Find k so that $4x+3$ is a factor of $20x^3+23x^2-10x+k$.

- Let $f(x) = 20x^3+23x^2-10x+k$. To have $4x+3$ be a factor of $f(x)$ means $f(x) = (4x+3)p(x)$ for some polynomial $p(x)$. Then $f(x) = 4(x+3/4)p(x)$, so $x+3/4$ is a factor of $f(x)$.
- Factor Theorem states: $x+3/4$ is a factor of $f(x)$ if & only if $f(-3/4)=0$.
- By the Remainder Theorem $f(-3/4)$ is the remainder of $\dfrac{f(x)}{x+3/4}$.
- $$\begin{array}{r|rrrr} -3/4 & 20 & 23 & -10 & k \\ & & -15 & -6 & 12 \\ \hline & 20 & 8 & -16 & k+12 \end{array} \longrightarrow f(-3/4) = k+12.$$
- We want $f(-3/4)=0$, or $k+12=0$, or $\boxed{k=-12}$ $\blacksquare$

3.4 - Zeros of Polynomial Functions

Rational Zero Theorem (RZT)

Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ has integer coefficients. If r is a rational zero for f, the r has the form

$$\frac{\text{factor of } a_0}{|\text{factor of } a_n|}$$

$\boxed{22}$ Solve $2x^3-5x^2-6x+4=0$. Let $f(x)=2x^3-5x^2-6x+4$, so the problem is to find all x such that f(x)=0.

a) By the RZT, if r is a rational number and f(r)=0, then r must have the form

$4=(1)(4)=(-1)(-4)=(2)(2)=(-2)(-2)$

$$\frac{\text{factor of } 4}{|\text{factor of } 2|}=\frac{\pm1,\pm2,\pm4}{|\pm1,\pm2|}=\frac{\pm1,\pm2,\pm4}{1,2}$$

$$=\boxed{\pm1,\pm2,\pm4,\pm\tfrac{1}{2}}$$

The eight numbers in this list represent the only rational numbers that MIGHT be a zero for f.

b) Try rational numbers in the list until either a zero for f is found, or we find that the list contains no zeros for f.

We will try 1 to start. Is f(1)=0? We will determine this by dividing f(x) by x-1. By the RT, if the remainder is 0, then we will know that f(1)=0.

We do the division $\dfrac{f(x)}{x-1}$ to start: $\dfrac{2x^3-5x^2-6x+4}{x-1}$

$$\begin{array}{r|rrrr} 1 & 2 & -5 & -6 & 4 \\ & & 2 & -3 & -9 \\ \hline & 2 & -3 & -9 & -5 \end{array} \longrightarrow \text{So: } f(1)=-5\neq0$$

Try -1: $\begin{array}{r|rrrr} -1 & 2 & -5 & -6 & 4 \\ & & -2 & 7 & -1 \\ \hline & 2 & -7 & 1 & 3 \end{array} \rightarrow$ So $f(-1) = 3 \neq 0$

Try 2: $\begin{array}{r|rrrr} 2 & 2 & -5 & -6 & 4 \\ & & 4 & -2 & -16 \\ \hline & 2 & -1 & -8 & -12 \end{array} \rightarrow$ So $f(2) = -12 \neq 0$

Try -2: $\begin{array}{r|rrrr} -2 & 2 & -5 & -6 & 4 \\ & & -4 & 18 & -24 \\ \hline & 2 & -9 & 12 & -20 \end{array} \rightarrow$ So $f(-2) = -20 \neq 0$

Try $\frac{1}{2}$: $\begin{array}{r|rrrr} 1/2 & 2 & -5 & -6 & 4 \\ & & 1 & -2 & -4 \\ \hline & 2 & -4 & -8 & 0 \end{array} \rightarrow$ So $f(\frac{1}{2}) = 0$

So we have $\frac{f(x)}{x - 1/2} = 2x^2 - 4x - 8$, and thus:

$$f(x) = (x - \tfrac{1}{2})(2x^2 - 4x - 8) = (2x-1)(x^2 - 2x - 4)$$

Factor 2 out of the trinomial for a nicer looking presentation.

c) Solve $f(x) = 0$.

We have $(2x-1)(x^2-2x-4) = 0$.

So $2x - 1 = 0$ or $x^2 - 2x - 4 = 0$

$x = 1/2$ or $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$

$$= \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

The are irrational zeros of f

Solution set is $\boxed{\{\frac{1}{2}, 1 \pm \sqrt{5}\}.}$

Conjugate Zeros Theorem Let f be a polynomial function with real coefficients. If the complex number a+bi is a zero for f, then the conjugate a-bi is also a zero. (Note: conjugate of bi is $-bi$)

Theorem If f is a polynomial function of degree n>0, then f has precisely n zeros if each zero is counted according to its multiplicity.

$\boxed{28}$ Find a degree 3 polynomial function f having 6 and -5+2i among its zeros, real coefficients, and f(2)=-636.

By the Conjugate Zeros Theorem -5-2i must also be a zero for f.

By the Factor Theorem we must have :

$$f(x) = C(x-6)[x-(-5+2i)][x-(-5-2i)]$$

Here C is some constant that we'll determine to make f(2)=-636.

Now we get f into the standard form (recall that $i=\sqrt{-1}$, so $i^2=-1$)

$$f(x) = C(x-6)[x^2-(-5-2i)x-(-5+2i)x+(-5+2i)(-5-2i)]$$

$$= C(x-6)(x^2+\underline{5x}+\cancel{2i}x+\underline{5x}-\cancel{2i}x+\underline{25}+\cancel{10}i-\cancel{10}i-\underline{4i^2}), \quad 4i^2=-4$$

$$= C(x-6)(x^2+10x+29)$$

$$= C(x^3+10x^2+29x-6x^2-60x-174)$$

$$f(x) = C(x^3+4x^2-31x-174)$$

Next, $-636 = f(2) = C(2^3+4\cdot 2^2-31\cdot 2-174) \Rightarrow -212C = -636 \Rightarrow$ $C=3$.

Finally: $f(x) = 3(x^3+4x^2-31x-174) \Rightarrow$

$$\boxed{f(x) = 3x^3+12x^2-93x-522}$$

(We always want the polynomial in standard form) □

$\boxed{46}$ Solve the equation $x^4-x^3+2x^2-4x-8=0$.

- Let $f(x) = x^4-x^3+2x^2-4x-8$ (a degree 4 polynomial function)
- Any rational zero of f must be expressible as

$$\frac{\text{Factor of } -8}{|\text{Factor of } 1|} = \frac{\pm1, \pm2, \pm4, \pm8}{|\pm1|} = \frac{\pm1, \pm2, \pm4, \pm8}{1}$$

$= \pm1, \pm2, \pm4, \pm8.$

- To see if $f(1)=0$, we see if the remainder of $\frac{f(x)}{x-1}$ is 0.

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & 2 & -4 & -8 \\ & \downarrow & 1 & 0 & 2 & -2 \\ \hline & 1 & 0 & 2 & -2 & -10 \end{array} \longrightarrow \text{So } f(1) = -10 \neq 0$$

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 2 & -4 & -8 \\ & & -1 & 2 & -4 & 8 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array} \longrightarrow \text{So } f(-1)=0. \text{ Then } x+1 \text{ is a factor of } f(x).$$

$$f(x) = (x+1)(x^3-2x^2+4x-8)$$
$$= (x+1)\left[x^2(x-2)+4(x-2)\right]$$

$$f(x) = \underbrace{(x+1)}_{-1 \text{ is a zero of } f}\underbrace{(x-2)}_{2 \text{ is a zero of } f}(x^2+4)$$

So $f(x)=0$ if $x+1=0$ or $x-2=0$ or $x^2+4=0$

$x=-1$ | $x=2$ | $x^2=-4$

$x = \pm\sqrt{-4}$

$x = \pm i\sqrt{4} = \pm 2i$

Zeros of f are $-1, 2, -2i, 2i$

So equation has solution set $\boxed{\{-1, 2, -2i, 2i\}}$ ■

$\boxed{25}$ Find a degree 3 polynomial function f having 1 and 5i among its zeros, real coefficients, and f(-1)=-104.

- By the Conjugate Zeros Theorem -5i must also be a zero for f.
- By the Factor Theorem f(x) has factors x-1, x-5i, and x-(-5i)=x+5i.
- So $f(x) = C(x-1)(x-5i)(x+5i)$, C some nonzero constant.
- Multiply: $f(x) = C(x-1)(x^2+\cancel{5ix}-\cancel{5ix}-25i^2)$

$f(x) = C(x-1)(x^2+25)$, since $-25i^2 = -25(-1) = 25$

$$f(x) = C(x^3 - x^2 + 25x - 25)$$

- $-104 = f(-1) = C[(-1)^3 - (-1)^2 + 25(-1) - 25] = C(-52)$, so we have $-52C = -104$, or $C = 2$.
- So $f(x) = 2(x^3 - x^2 + 25x - 25)$, or $\boxed{f(x) = 2x^3 - 2x^2 + 50x - 50}$ ■

3.5 - Rational Functions

A function R is a rational function if R = f/g, where f and g are both polynomial functions, g not the zero function. (So $R(x)$ is a rational expression)

$\boxed{4}$ Find the domain of $R(x) = \dfrac{2x^2}{(x-2)(x+6)}$.

Domain of $R = D_R = \{x \in \mathbb{R} \mid R(x) \in \mathbb{R}\} = \{x \in \mathbb{R} \mid (x-2)(x+6) \neq 0\}$

$= \boxed{\{x \in \mathbb{R} \mid x \neq -6, 2\}} = \boxed{(-\infty,-6) \cup (-6,2) \cup (2,\infty)}$ ■

A vertical asymptote (v.a.) for a rational function R is any x=c, where $|R(x)| \to \infty$ as $x \to c$. That is, $|R(x)|$ can be made arbitrarily large by bringing x sufficiently close to c. So the graph of R appears to "blow up" on both sides of the vertical line x=c.

Let $R(x) = \dfrac{f(x)}{g(x)}$ be a rational function.

Theorem 1 Let $R(x) = f(x)/g(x)$ be a rational function.

If c is a zero for f and g with the same multiplicity, then the graph of R will have a hole at x=c. Canceling all x-c factors out of the fraction f(x)/g(x) will yield a reduced fraction $\hat{f}(x)/\hat{g}(x)$. The coordinates of the hole will be: $(c, \hat{f}(c)/\hat{g}(c))$.

Fully reduce f(x)/g(x) to get $\hat{f}(x)/\hat{g}(x)$. If c is such that $\hat{g}(c)=0$, then the vertical line x=c is a vertical asymptote for R.

$\boxed{34}$ Find all vertical asymptotes and holes for $h(x) = \dfrac{x+6}{x^2+2x-24}$.

Factor wherever possible (the denominator in this case):

$$\underbrace{h(x) = \frac{x+6}{(x+6)(x-4)}} \Rightarrow \underbrace{h(x) = \frac{1}{x-4},\quad x \neq -6}$$

Equivalent ways of defining function h

$D_h = \{x \mid x \neq -6\ 4\}$

For $h(x) = \dfrac{x+6}{(x+6)(x-4)}$ we have -6 is a zero of the numerator with multiplicity 1, and also -6 is a zero of the denominator with multiplicity 1. Therefore h has a hole as x=-6.

Reducing the fraction to get $h(x) = \dfrac{1}{x-4}$ (with $x \neq -6$), we we find that the only zero in the denominator of the (reduced) fraction is 4. Therefore h has a vertical asymptote x=4. ■

Rational function R has a horizontal asymptote (h.a.) y=b if $R(x) \to b$ as $|x| \to \infty$ (i.e. as $x \to \infty$ or $x \to -\infty$)

So the graph of R can be made arbitrarily close to the line y=b by making |x| sufficiently large.

Theorem 2 Let $R(x) = f(x)/g(x)$ be a rational function, so $f(x)$ and $g(x)$ are polynomials.

i) If $\deg(f) > \deg(g)$, then R will have no h.a.

ii) If $\deg(f) = \deg(g)$, then R will have h.a. $y = \dfrac{\text{leading coefficient of } f(x)}{\text{leading coefficient of } g(x)}$

iii) If $\deg(f) < \deg(g)$, then R will have h.a. $y=0$.

Ex Find the h.a. of $R(x) = \dfrac{x+6}{x^2+2x-24}$.

Here $R(x) = \dfrac{f(x)}{g(x)}$ with $f(x) = x+6$ & $g(x) = x^2+2x-24$, so $\deg(f) = 1 < 2 = \deg(g)$, and thus R has h.a. $\boxed{y=0.}$ ■

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40 Find the h.a. for $R(x) = \dfrac{15x^2}{3x^2+1}$.

Here $f(x) = 15x^2$ & $g(x) = 3x^2+1$, so $\deg(f) = 2 = \deg(g)$, and thus $y = \dfrac{15}{3}$, or $\boxed{y=5}$, is the h.a. for R. ■

A line y=mx+b is a slant asymptote (s.a.) for rational function R if

$$\left| R(x) - (mx+b) \right| \to 0 \quad \text{as} \quad |x| \to \infty$$

So the graph of R can be made arbitrarily close to the line y=mx+b by making |x| sufficiently large.

Theorem 3

$R(x) = f(x)/g(x)$ has a slant asymptote if $\deg(f) = 1 + \deg(g)$. Then the s.a. is $y = (\text{Quotient of the division } f(x) \div g(x))$.

Ex Find the s.a. for $R(x) = \dfrac{3x^4 - 2x^2 + x - 4}{x^3 - x + 1}$

Since the degree of the numerator is 1 greater than the degree of the denominator, there will be a slant asymptote. Find the quotient of the division...

$$\begin{array}{r|l} & \qquad\qquad\qquad\quad 3x \longleftarrow \text{Quotient} \\ \hline x^3-x+1 & 3x^4+0x^3-2x^2+x-4 \; \downarrow \\ & 3x^4 \qquad\quad -3x^2+3x \\ \cline{2-2} & \qquad\qquad\quad x^2-2x-4 \longleftarrow \text{Remainder} \end{array}$$

$$-2x^2 - (-3x^2) = -2x^2 + 3x^2 = x^2$$

Slant asymptote is $\boxed{y = 3x}$ □

Note: In the example above, the division tells us that

$$R(x) = 3x + \frac{x^2 - 2x - 4}{x^3 - x + 1} \cdot \frac{1/x^3}{1/x^3} = 3x + \frac{\frac{1}{x} - \frac{2}{x^2} - \frac{4}{x^3}}{1 - \frac{1}{x^2} + \frac{1}{x^3}}$$

Now, as $|x| \to \infty$, we find that $\frac{1}{x} \to 0$, $\frac{2}{x^2} \to 0$, $\frac{4}{x^3} \to 0$, etc.

So $\dfrac{\frac{1}{x} - \frac{2}{x^2} - \frac{4}{x^3}}{1 - \frac{1}{x^2} + \frac{1}{x^3}} \to \dfrac{0}{1} = 0$ as $|x| \to \infty$. Then $R(x) \to 3x$ as $|x| \to \infty$. This is why the theorem above works.

Procedure for sketching the graph of a rational function

$$R(x) = \frac{f(x)}{g(x)}$$

1) Find the domain of R.

2) Check for symmetry: is R an even or odd function?

3) Find the intercepts for R.

4) Find any holes and vertical asymptotes (use Theorem 1).

5) Find any horizontal or slant asymptote (use Theorems 2 & 3).
Find any points where the graph crosses the asymptote, if any.

6) Use the x-intercepts, vertical asymptotes, and any holes on the x-axis to partition the x-axis into subintervals, then find a point on the graph of R in each subinterval.

7) Sketch the graph of y=R(x).

$\boxed{88}$ Sketch the graph of $R(x) = \dfrac{x^3-1}{x^2-9}$

1) $D_R = \{x \mid x^2-9 \neq 0\} = \{x \mid x^2 \neq 9\} = \{x \mid x \neq \pm 3\}$

$= \boxed{(-\infty,-3)\cup(-3,3)\cup(3,\infty)}$

2) $R(-x) = \dfrac{(-x)^3-1}{(-x)^2-9} = \dfrac{-x^3-1}{x^2-9} = -\dfrac{x^3+1}{x^2-9} \neq \pm R(x)$, so

R is neither even nor odd. $\boxed{\text{No symmetry!}}$

3) $R(x)=0 \Rightarrow \dfrac{x^3-1}{x^2-9} = 0 \Rightarrow x^3-1=0 \Rightarrow x^3=1 \Rightarrow x=1$

So $\boxed{1 \text{ is an } x\text{-intercept}}$

$R(0) = \dfrac{-1}{-9} = \dfrac{1}{9}$, so $\boxed{\frac{1}{9} \text{ is the } y\text{-intercept}}$

4) Find holes and vertical asymptotes:

We have $\frac{x^3-1}{x^2-9} = \frac{(x-1)(x^2+x+1)}{(x-3)(x+3)}$, so the fraction is reduced already! By Theorem 1, the vertical asymptotes are located at the zeros of the denominator of the reduced fraction. Setting $x^2-9=0$ yields $x=\pm 3$. The vertical asymptotes are $\boxed{x=-3 \ \ \& \ \ x=3.}$ There are $\boxed{\text{no holes,}}$ since the only zero for the numerator of R(x) is 1, zeros of the denominator are -3 and 3, and so there is no common zero for numerator and denominator.

5) Since $\deg(x^3-1)=3$ and $\deg(x^2-9)=2$, so that the degree of the numerator is 1 greater than the degree of the denominator, there will be a slant asymptote.

$$\begin{array}{rl} & \qquad\qquad\qquad\quad x \\ x^2-9 \,\big) & \overline{x^3+0x^2+0x-1} \\ & x^3 \qquad\quad -9x \quad \downarrow \\ \cline{2-2} & \qquad\qquad 9x-1 \end{array}$$

$\longrightarrow$ quotient is x, so slant asymptote is $\boxed{y=x}$

Note: it is possible for the graph of a rational function to cross a horizontal or slant asymptote. We can check if R crosses the slant asymptote y=x by seeing if there exists some x value for this y=R(x) and y=x yield the same y value. That is, see if there is an x for which R(x)=x.

$R(x)=x \Rightarrow \frac{x^3-1}{x^2-9}=x \Rightarrow x^3-1=x(x^2-9) \Rightarrow$ $x^3-1=x^3-9x \Rightarrow -1=-9x \Rightarrow x=1/9$. So it looks like the graph of $y=R(x)$ crosses the slant asymptote at the point $\left(\frac{1}{9}, R\left(\frac{1}{9}\right)\right)$ $= \boxed{\left(\frac{1}{9}, \frac{1}{9}\right)}$.

6) With x-intercept 1 & V.a.'s $x = \pm 3$, we partition the x-axis into subintervals $(-\infty,-3), (-3,1), (1,3), (3,\infty)$. We want a point $(x, R(x))$ plotted for at least one x value in each of the 4 subintervals.

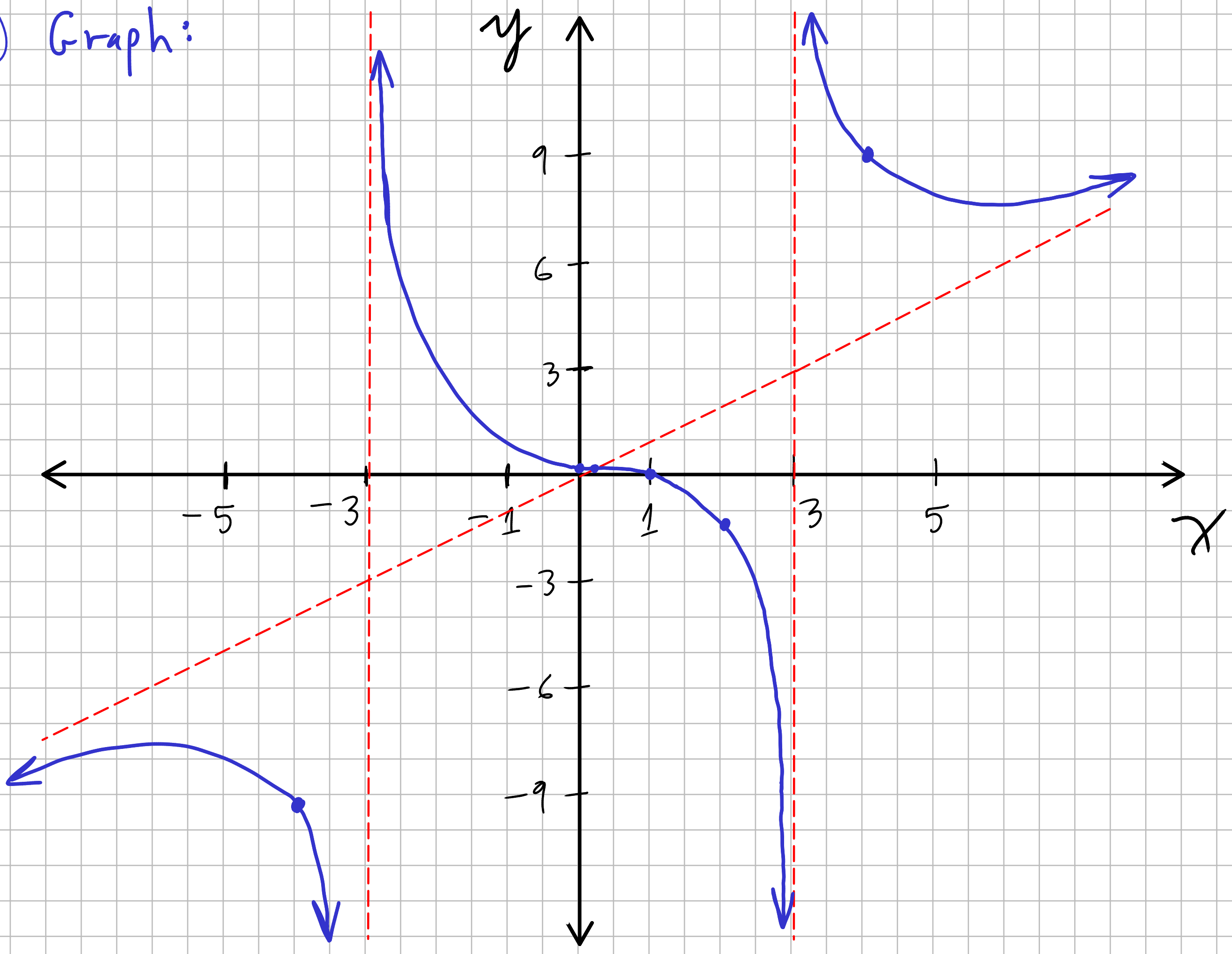
We already found $R(0) = \frac{1}{9}$, so $(0, R(0)) = (0, 1/9)$. This takes care of the interval $(-3,1)$.

$(-\infty,-3)$: pick -4, so $(-4, R(-4)) = \left(-4, \frac{(-4)^3-1}{(-4)^2-9}\right) = (-4, -9\frac{2}{7})$

$(1,3)$: pick 2, so $(2, R(2)) = \left(2, \frac{(2)^3-1}{(2)^2-9}\right) = (2, -1\frac{2}{5})$

$(3,\infty)$: pick 4, so $(4, R(4)) = \left(4, \frac{4^3-1}{4^2-9}\right) = (4, 9)$

7) Graph:



3.6 - Polynomial & Rational Inequalities

A polynomial inequality is an inequality with a polynomial on each side; a rational inequality is an inequality with a rational expression (i.e. a ratio of polynomials) on each side. Remember that 0 is considered to be a polynomial as well as a rational expression.

Intermediate Value Theorem for Rational Functions (IVT) Let R be a rational function, let a<b, and suppose [a,b] is in the domain of R. If M is a value between f(a) and f(b), then there exists some a<c<b such that R(c)=M.

$\boxed{Ex}$ Solve $x^2 \le 2x+2$

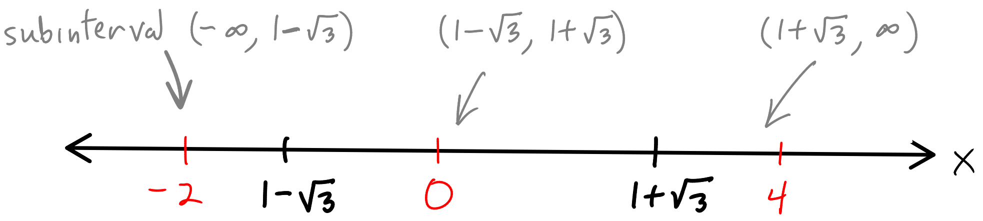
1) Get 0 on the right side, and find the zeros of f(x) at left:

- $x^2 - 2x - 2 \le 0$.
- Let $f(x) = x^2 - 2x - 2$, so inequality is $f(x) \le 0$.
- Find all real x s.t. $f(x) = 0$.

  This means solving $x^2 - 2x - 2 = 0$

  Quadratic formula: $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = 1 \pm \sqrt{3}$.

2) Use the zeros of f to partition the real line into subintervals, then evaluate f at a test value chosen from each subinterval.



$f(-2) = (-2)^2 - (-2) - 2 = 6$, so $f(-2) > 0$.

$f(0) = -2$, so $f(0) < 0$

$f(4) = 6$, so $f(4) > 0$.

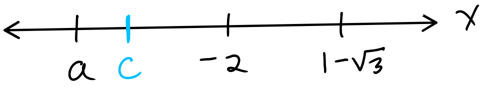
3) Use the IVT for polynomial functions to find all x for which f(x)>0 and f(x)<0.

- Since $f(-2)>0$, the IVT implies that $f(x)>0$ for all $x\in(-\infty,1-\sqrt{3})$. *
- Since $f(0)<0$, IVT implies $f(x)<0$ for all $x\in(1-\sqrt{3},1+\sqrt{3})$.
- Since $f(4)>0$, IVT implies $f(x)>0$ for all $x\in(1+\sqrt{3},\infty)$.

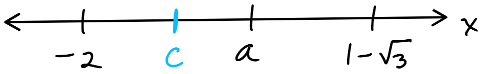
4) Find the solution set of the inequality in interval notation.

- Inequality is $f(x)\le 0$.
- Step 1 found $f(x)=0$ for $x=1\pm\sqrt{3}$.
- Step 3 found $f(x)<0$ for $x\in(1-\sqrt{3},1+\sqrt{3})$.
- Putting $1\pm\sqrt{3}$ together with $(1-\sqrt{3},1+\sqrt{3})$ yields the solution set $\boxed{[1-\sqrt{3},1+\sqrt{3}]}$

* Why this works: Suppose there is some a<1-$\sqrt{3}$ such that f(a)<0. Since f(-2)>0, the IVT implies there is some c between -2 and a such that f(c)=0. Since -2<1-$\sqrt{3}$, we must have c<1-$\sqrt{3}$:



a<-2 Case



a>-2 case

So, $c\in(-\infty,1-\sqrt{3})$ & $f(c)=0$. This is impossible: we have already found that f(x)=0 if and only if $x=1\pm\sqrt{3}$, yet here $f(c)=0$ for some $c\ne 1\pm\sqrt{3}$. Therefore we must have $f(x)>0$ for all $x\in(-\infty,1-\sqrt{3})$.

$\boxed{64}$ Find the domain of $f(x)=\sqrt{\dfrac{x}{2x-1}-1}$

By definition, $D_f=\{x\in\mathbb{R}:f(x)\in\mathbb{R}\}=\left\{x \;\middle|\; \dfrac{x}{2x-1}-1\ge 0\right\}$.

$\dfrac{x}{2x-1}-1\ge 0$ is a rational inequality.

Let $R(x) = \frac{x}{2x-1} - 1$, so the inequality is $R(x) \geq 0$. We solve it...

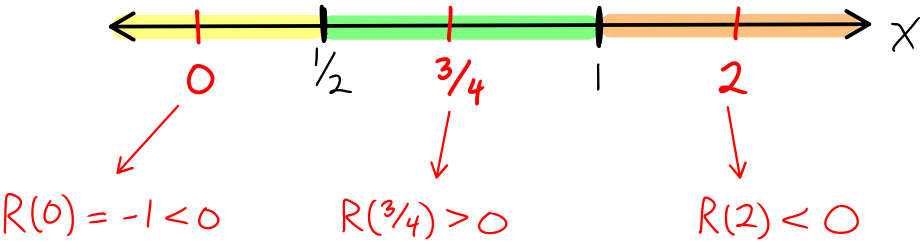
1) Get 0 on right side, find all x for which left side is 0 or undefined.

- $R(x) = 0 \Rightarrow \frac{x}{2x-1} - 1 = 0 \Rightarrow (2x-1)\left(\frac{x}{2x-1} - 1\right) = 0 \Rightarrow$

  $x - (2x-1) = 0 \Rightarrow \boxed{x = 1}$

- $R(x)$ undefined $\Rightarrow 2x-1 = 0$ (so a division by 0 occurs) $\Rightarrow$ $\boxed{x = 1/2}$

2) Use the values of x for which R(x) is 0 or undefined to partition the real line into subintervals. At a test value t chosen from each subinterval determine whether R(t)>0 or R(t)<0.



3) Use the IVT to determine whether R>0 or R<0 on each of the subintervals.

- $R(0) < 0$, so IVT implies $R(x) < 0$ for all $x \in (-\infty, 1/2)$.
- $R(3/4) > 0$, so IVT implies $R(x) > 0$ for all $x \in (1/2, 1)$.
- $R(2) < 0$, so IVT implies $R(x) < 0$ for all $x \in (1, \infty)$.

4) Find the solution set of the inequality in interval notation.

- Inequality is $R(x) \geq 0$.
- Step 1 found $R(x) = 0$ when $x = 1$.
- Step 3 found $R(x) > 0$ when $x \in (1/2, 1)$.
- Combining all values, solution set is $\boxed{(1/2, 1]}$

Since f(x) is real-valued if and only if R(x) ≥ 0, it follows that the domain of f is $\boxed{D_f = (\tfrac{1}{2}, 1]}$ □

$\boxed{\text{Ex}}$ Solve $\frac{x}{x+2} \le \frac{1}{x}$.

Do NOT multiply by $x(x+2)$, as when solving $\frac{x}{x+2} = \frac{1}{x}$. The problem is that we can't treat $x(x+2)$ as being positive or negative, and so we can't know what to do with the inequality sign $\le$ if we were to multiply by $x(x+2)$.

1) Get 0 on right side, find all x for which left side is 0 or undefined.

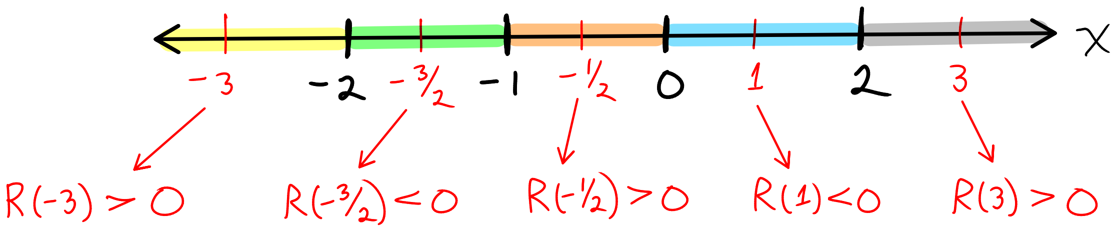
$$\frac{x}{x+2} \le \frac{1}{x} \Rightarrow \frac{x}{x+2} - \frac{1}{x} \le 0 \Rightarrow \frac{x^2-(x+2)}{x(x+2)} \le 0 \Rightarrow$$

$$\frac{x^2-x-2}{x(x+2)} \le 0. \quad \text{Let } R(x) = \frac{x^2-x-2}{x(x+2)} = \frac{(x-2)(x+1)}{x(x+2)}.$$

So inequality is $R(x) \le 0$.

- $R(x) = 0$ implies $\frac{(x-2)(x+1)}{x(x+2)} = 0$, which implies $\boxed{x=-1}$ or $\boxed{x=2}$.
- $R(x) =$ undefined implies $\boxed{x=0}$ or $\boxed{x=-2}$

2) Use the values of x for which R(x) is 0 or undefined to partition the real line into subintervals. At a test value t chosen from each subinterval determine whether R(t)>0 or R(t)<0.



3) Use the IVT to determine whether R>0 or R<0 on each of the subintervals.

$R(x) > 0$ on $(-\infty, -2)$, $R(x) < 0$ on $(-2, -1)$, $R(x) > 0$ on $(-1, 0)$,
$R(x) < 0$ on $(0, 2)$, $R(x) > 0$ on $(2, \infty)$.

4) Find the solution set of the inequality in interval notation.

- Inequality is $R(x) \leq 0$.
- Step 1 found $R(x) = 0$ when $x = -1$ or $x = 2$.
- Step 3 found $R(x) < 0$ when $x \in (-2,-1)$ or $x \in (0,2)$.
- Put together, solution set is $\boxed{(-2,-1] \cup (0,2]}$ ■