2.1 - Basics of Functions

A relation is a set of ordered pairs. In this course the ordered pairs will usually consist of real numbers.

$\boxed{\text{Ex}}$ Relation R_1 is the set $R_1 = \{(-1,0), (4,8), (2,-3), (-1,7)\}$

$\boxed{\text{Ex}}$ Relation R_2 is the set $R_2 = \{(1,4), (4,4), (2,-3), (-1,7)\}$

$\boxed{\text{Ex}}$ Relation R_3 is the solution set of the inequality $y > x^2$.

Specifically this relation consists of all ordered pairs of the form (x,y) for which the y-coordinate is greater than the square of the x-coordinate. In set-builder notation:

$R_3 = \{(x,y) \mid y > x^2\}$ (It's understood that x & y must be real.)

Some ordered pairs belonging to the relation R_3:

- $(-1,4)$, since $y = 4 > 1 = (-1)^2 = x^2$ (i.e. $y > x^2$ when $x = -1$ and $y = 4$).
- $(\sqrt{2},3)$, since $y = 3 > 2 = (\sqrt{2})^2 = x^2$
- $(1/2, 1/2)$, since $y = \frac{1}{2} > \frac{1}{4} = \left(\frac{1}{2}\right)^2 = x^2$

$\boxed{\text{Ex}}$ Relation R_4 is the solution set of the equation $y = 2-3x$.

So relation R_4 consists of all ordered pairs (x,y) for which y=2-3x. In set-builder notation:

$R_4 = \{(x,y) \mid x \text{ is real } \& \ y = 2-3x\}$, or ...

$R_4 = \{(x, 2-3x) \mid x \text{ is real}\}$, or ...

$R_4 = \{(x, 2-3x) \mid x \in (-\infty,\infty)\}$

A function is a relation consisting of ordered pairs of the form (x,y) for which no two distinct pairs have the same x-coordinate.

More generally, without mentioning the form of the ordered pairs, we can say a function is a relation consisting of ordered pairs such that no two distinct pairs have the same first-coordinate value.

$\boxed{\text{Ex}}$ $R_1 = \{(-1,0), (4,8), (2,-3), (-1,7)\}$ is <u>not</u> a function since the pairs $(-1,0)$ & $(-1,7)$ have the same first-coordinate value. ■

$\boxed{\text{Ex}}$ $R_2 = \{(1,4), (4,4), (2,-3), (-1,7)\}$ is a function since the ordered pairs belonging to the relation all have different first-coordinate values (1, 4, 2, and -1). ■

$\boxed{\text{Ex}}$ $R_3 = \{(x,y) \mid y > x^2\}$ is not a function: $(-1,4)$ & $(-1,5)$ both belong to the relation, since when x=-1 and y=4 we have $y>x^2$ $(x^2 = (-1)^2 = 1 < 4 = y \rightarrow x^2 < y \rightarrow y > x^2)$, and also when x=-1 and y=5 we have $y>x^2$.

$\boxed{\text{Ex}}$ $R_4 = \{(x, 2-3x) \mid x \text{ is real}\}$ is a function, but how to show it?

To show that this relation is a function requires a general argument. Suppose that (a,b) and (a,c) are ordered pairs that belong to R_4.

$(a,b) \in R_4$ means $b = 2-3a$ (see the definition of R_4 above).

$(a,c) \in R_4$ means $c = 2-3a$.

Now $b = 2-3a = c$; that $b=c$, and so $(a,b) = (a,c)$.

So the pairs (a,b) and (a,c) are not distinct, they are identical. This shows that it is impossible for R_4 to have two distinct pairs with the same x-coordinate. Therefore R_4 is a function. ■

$\boxed{\text{Ex}}$ Let relation R_5 be the solution for $x = y^2$.

Specifically R_5 is the set of ordered pairs of the form (x,y) such that x is the square of y:

$$R_5 = \{(x,y) \mid y \text{ is real } \& \ x = y^2\} = \{(y^2, y) \mid y \text{ is real}\}.$$

We find that R_5 is NOT a function:

- If $y = 2$, then $y^2 = 4$ and we have $(y^2, y) = (4,2) \in R_5$.
- If $y = -2$, then $y^2 = 4$ and we have $(y^2, y) = (4,-2) \in R_5$.
- So R_5 contains two distinct ordered pairs having the same first-coordinate value (i.e. x-coordinate). Therefore R_5 is not a function. ∎

The domain of a relation is the set of first-coordinate values of its ordered pairs. The range of a relation is the set of second-coordinate values of its ordered pairs. If the ordered pairs have the form (x,y), then the domain is the set of x-coordinate values and the range is the set of y-coordinate values.

The domain of a relation f (whether a function or not) we denote by D_f, and the range of f we denote by R_f.

$$D_f = \{x \mid (x,y) \in f \text{ for some } y\}$$

$$R_f = \{y \mid (x,y) \in f \text{ for some } x\}$$

$\boxed{\text{Ex}}$ The relation $R_1 = \{(-1,0), (4,8), (2,-3), (-1,7)\}$ has domain

$D_{R_1} = \{-1, 2, 4\}$

(Note: sets don't recognize repeated elements, so that $\{-1,-1,2,4\} = \{-1,2,4\}$)

and range

$R_{R_1} = \{-3, 0, 7, 8\}$ ∎

$\boxed{\text{Ex}}$ We consider the relation that is the solution set to the equation $y=x^2$.

The relation happens to be a function, so we'll call it f. We want to find the domain & range of f.

$$D_f = \{x \mid \underbrace{(x,y)\in f}_{} \text{ for some } y\}$$

This means that x is real & $y=x^2$.

We have $f = \{(x,y) \mid x \text{ is real } \& \ y=x^2\}$,

or $f = \{(x,x^2) \mid x \text{ is real}\}$.

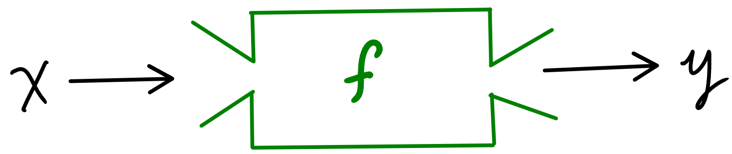
$$D_f = \{x \mid (x,x^2)\in f \text{ for } x \text{ real}\} = \{x \mid x \text{ is real}\} = \boxed{(-\infty,\infty)}$$

$$R_f = \{y \mid (x,y)\in f \text{ for some } x\}$$

$$= \{x^2 \mid (x,x^2)\in f \text{ for } x \text{ real}\}$$

$= \boxed{[0,\infty)}$, since $x^2 \geq 0$ for any real x. ■

It is convenient to think of a relation, and especially a function, as a machine that takes in an x value as input and returns a y value as output. So if the ordered pair (x,y) belong to the function f, we think of x as going "into" f, and y as coming "out of" f:



In function notation we write f(x)=y

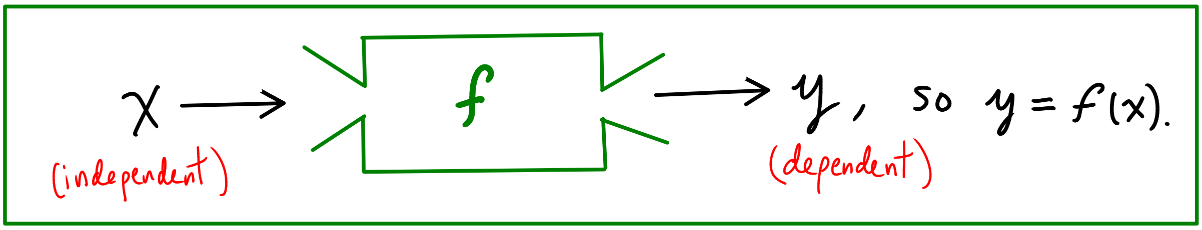
So to write $f(x)=y$ is equivalent to saying $(x,y)\in f$.

Since f, being a function, cannot have two ordered pairs (x,y) with the same x-coordinate value, putting an x value into the "machine" f results in NO MORE THAN ONE y value coming out of the "machine."

With this "machine interpretation" of a function, we call x the independent variable and y the dependent variable. This is because the output y is dependent on what the input x is.

If we write f(t)=u instead of f(x)=y, then t is the independent variable and u is the dependent variable.

The independent variable represents a function's input, and the dependent variable represents a function's output.



$f(x)$ is read as "f of x" or "f at x".

$\boxed{\text{Ex}}$ Consider the function $f(x) = x^2 + 3x - 5$. We evaluate $f(2)$, $f(-1)$, $f(x-2)$, $f(-x)$.

- $f(2) = 2^2 + 3(2) - 5 = 4 + 6 - 5 = \boxed{5}$
- $f(-1) = (-1)^2 + 3(-1) - 5 = 1 - 3 - 5 = \boxed{-7}$
- $f(x-2) = (x-2)^2 + 3(x-2) - 5 = x^2 - 4x + 4 + 3x - 6 - 5 = \boxed{x^2 - x - 7}$
- $f(-x) = (-x)^2 + 3(-x) - 5 = \boxed{x^2 - 3x - 5}$

The domain and range of a function f, using function notation, are:

$$
\begin{aligned}
D_f &= \{x \mid x \text{ is real } \& \ f(x) \text{ is real}\} \\
&= \{x \mid x \in \mathbb{R} \ \& \ f(x) \in \mathbb{R}\} \\
&= \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}
\end{aligned}
$$

$$R_f = \{f(x) \mid x \in D_f\}$$

$\boxed{\text{Ex}}$ Let $f(x) = |3-x| + 2$.

a) Find D_f & R_f.

b) Also find $f(-3)$, $f(10)$, $f(3)$, $f(2x-1)$, $f(-x)$

a) $D_f = \{x \mid x \text{ is real } \& \ f(x) \text{ is real}\}$

$= \{x \mid x \text{ is real } \& \ |3-x|+2 \text{ is real}\}$

$= \{x \mid x \text{ is real}\}$, since $|3-x|+2$ is real for <u>any</u> real x.

$= \boxed{(-\infty, \infty)}$

$R_f = \{f(x) \mid x \in D_f\}$

$= \{|3-x|+2 \mid x \text{ is real}\}$, since $x \in D_f$ means $x \in (-\infty,\infty)$ means "x is real"

Note: for any real x we have $|3-x| \geq 0$, so that $|3-x|+2 \geq 2$, and hence $f(x) \geq 2$. So...

$R_f = \boxed{[2,\infty)}$. (For $x \in D_f = (-\infty,\infty)$ we find that $f(x)$ assumes values from 2 on up)

b) $f(-3) = |3-(-3)| + 2 = |6| + 2 = 6+2 = \boxed{8}$

$f(10) = |3-10| + 2 = |-7| + 2 = 7+2 = \boxed{9}$

$f(3) = |3-3| + 2 = |0| + 2 = 0+2 = \boxed{2}$

$f(2x-1) = |3-(2x-1)| + 2 = \boxed{|4-2x| + 2}$

$f(-x) = |3-(-x)| + 2 = \boxed{|x+3| + 2}$ ∎

$\boxed{\text{Ex}}$ Find D_f & R_f for $f(x)=\sqrt{5-8x}$.

$$D_f = \{x\in\mathbb{R} \mid f(x)\in\mathbb{R}\} = \{x \mid \sqrt{5-8x}\in\mathbb{R}\}$$

$$= \{x \mid 5-8x\geq 0\} \qquad 5-8x\geq 0 \Rightarrow -8x\geq -5 \Rightarrow x\leq \frac{5}{8}$$

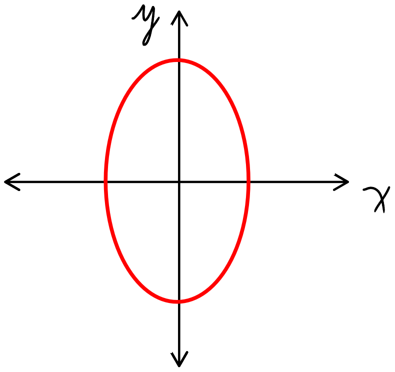
$$= \{x \mid x\leq 5/8\}$$

$$= \boxed{(-\infty, 5/8]}$$

$$R_f = \{f(x) \mid x\in D_f\} = \{\sqrt{5-8x} \mid x\leq 5/8\} = \boxed{[0,\infty)}$$

Note: $x\leq 5/8 \Rightarrow 5-8x\geq 0 \Rightarrow \sqrt{5-8x}\geq 0 \Rightarrow f(x)\geq 0$. ■

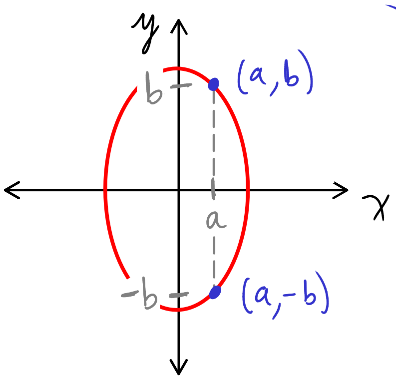
$\boxed{60}$ Determine whether the graph below, as a set of ordered pairs, defines y as a function of x.



We're supposed to determine whether the set of ordered pairs

$$\{(x,y) \mid \text{The point } (x,y) \text{ lies on the graph}\}, \qquad (1)$$

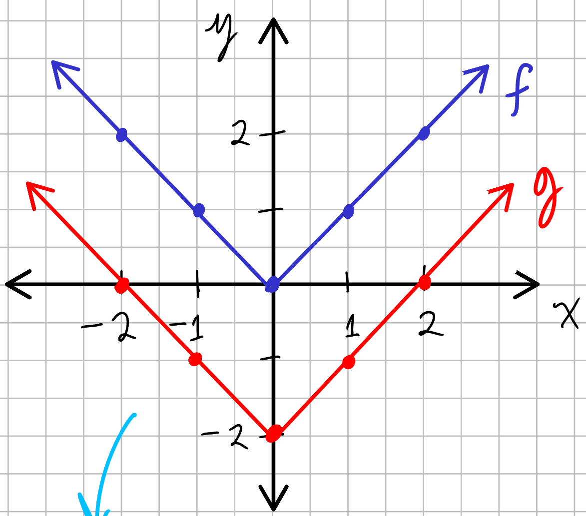
is a function or not.



We see that (a,b) & $(a,-b)$ both belong to the relation (1), which makes the relation not a function.

The graph of a function f is the set of points (x,f(x)) for $x \in \mathcal{D}_f$.

$\boxed{2.1.45}$ $f(x) = |x|$ & $g(x) = |x| - 2$



x	$f(x)$	$(x, f(x))$
-2	$\lvert -2 \rvert = 2$	$(-2, 2)$
-1	$\lvert -1 \rvert = 1$	$(-1, 1)$
0	$\lvert 0 \rvert = 0$	$(0, 0)$
1	$\lvert 1 \rvert = 1$	$(1, 1)$
2	$\lvert 2 \rvert = 2$	$(2, 2)$

The graph of g is the graph of f shifted down by 2 units.

x	$g(x)$	$(x, g(x))$
-2	$\lvert -2 \rvert - 2 = 0$	$(-2, 0)$
-1	$\lvert -1 \rvert - 2 = -1$	$(-1, -1)$
0	$\lvert 0 \rvert - 2 = -2$	$(0, -2)$
1	$\lvert 1 \rvert - 2 = -1$	$(1, -1)$
2	$\lvert 2 \rvert - 2 = 0$	$(2, 0)$

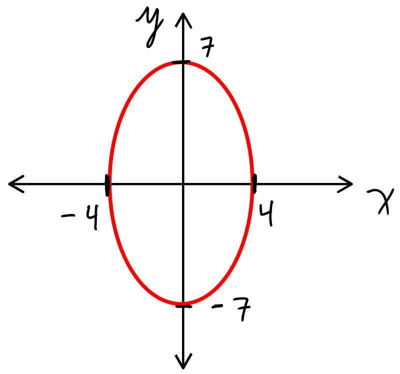
$\boxed{97}$ $f(x) = x^3 + x - 5.$

So: $f(-x) - f(x) = [(-x)^3 + (-x) - 5] - (x^3 + x - 5)$

$= -x^3 - x - 5 - x^3 - x + 5$

$= \boxed{-2x^3 - 2x}$

$\boxed{Ex}$ Find the domain and range of the relation S whose graph is below.



To find the domain of a relation using a graph, just find the x-coordinates of all the points on the graph.

To find the range of a relation using a graph, find the y-coordinates of all the points on the graph.

Domain of $S = D_S = \{x \mid (x,y) \text{ is a point on the graph for some } y \text{ value}\}$

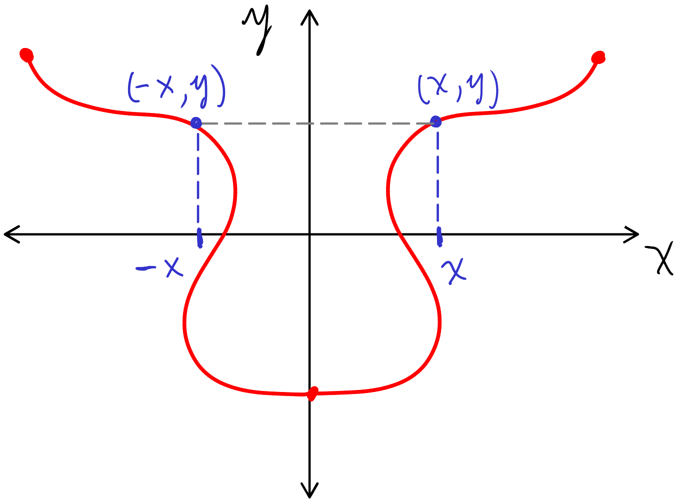
$= \boxed{[-4,4]}$.

Range of $S = R_S = \{y \mid (x,y) \text{ is a point on the graph for some } x \text{ value}\}$

$= \boxed{[-7,7]}$.

2.2 - More on Functions and Their Graphs

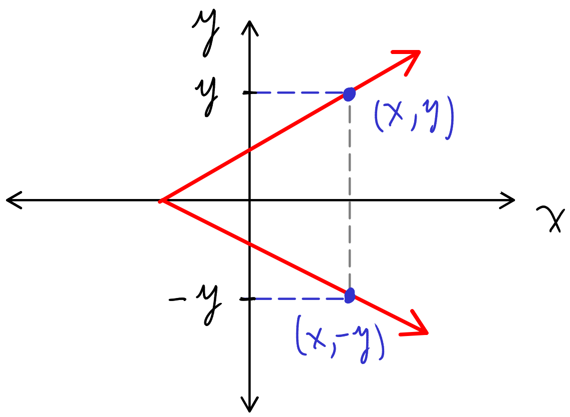
The graph of an equation is symmetric about the y-axis if (-x,y) is a point on the graph whenever (x,y) is.



Test for symmetry:

Substitute -x for x in the equation. If the equation is unchanged, then its graph will be symmetric about the y-axis.

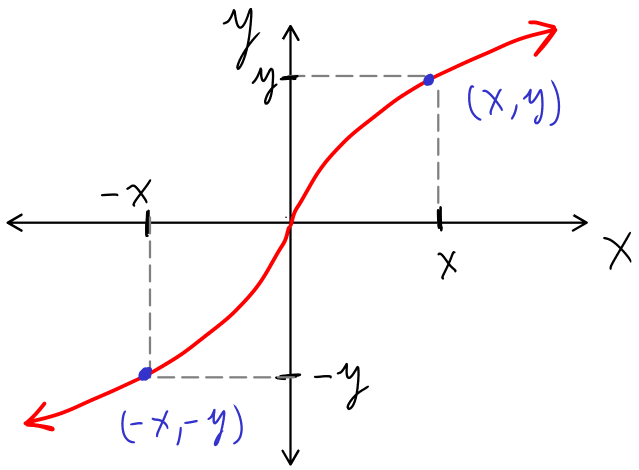
The graph of an equation is symmetric about the x-axis if (x,-y) is a point on the graph whenever (x,y) is.



Test for symmetry:

Substitute -y for y in the equation. If the equation is unchanged, then its graph will be symmetric about the x-axis.

The graph of an equation is symmetric about the origin if (-x,-y) is a point on the graph whenever (x,y) is.



Test for symmetry:

Substitute -x for x and -y for y in the equation. If the equation is unchanged, then its graph will be symmetric about the origin.

$\boxed{30}$ Determine whether the graph of $x^2y^2+5xy=2$ is symmetric about the y-axis, the x-axis, the origin, more than one of these, or none of these.

- Test for symmetry about the y-axis: substitute -x for x in the given equation $x^2y^2+5xy=2$ and see if anything changes...

$$(-x)^2y^2+5(-x)y=2 \Rightarrow x^2y^2-5xy=2.$$

Equation has changed!

No symmetry about the y-axis.

If the equation hasn't changed, then subtracting the "new" equation from the original equation should result in 0=0, but it is not really necessary to do this. Still, here's a demo:

$$\begin{array}{r} x^2y^2+5xy=2 \\ -\quad x^2y^2-5xy=2 \\ \hline 10xy=0 \end{array}$$

Not 0=0, so equation changed.

- Test for symmetry about the x-axis: substitute -y for y in the given equation $x^2y^2+5xy=2$ and see if anything changes...

$$x^2(-y)^2+5x(-y)=2 \Rightarrow x^2y^2-5xy=2$$

Equation has changed! No symmetry about the x-axis.

- To test for symmetry about the origin we substitute -x for x and -y for y in the equation $x^2y^2+5xy=2$ and see if there's a change...

$$(-x)^2(-y)^2+5(-x)(-y)=2 \Rightarrow x^2y^2+5xy=2$$

Equation did not change, so its graph will be symmetric about the origin.

$\boxed{32}$ $y^5 = x^4 + 2$

• Test for symmetry about y-axis: substitute $-x$ for x...

$$y^5 = (-x)^4 + 2 \Rightarrow y^5 = (-1)^4 (x)^4 + 2 \Rightarrow y^5 = x^4 + 2$$

Equation is unchanged!

The equation's graph is symmetric about the y-axis.

• Test for symmetry about x-axis: substitute $-y$ for y...

$$(-y)^5 = x^4 + 2 \Rightarrow (-1)^5 y^5 = x^4 + 2 \Rightarrow -y^5 = x^4 + 2$$

Equation changed!

The equation's graph is not symmetric about the x-axis.

• Test for symmetry about the origin: substitute $-x$ for x and $-y$ for y...

$$(-y)^5 = (-x)^4 + 2 \Rightarrow -y^5 = x^4 + 2$$

Equation changed!

The equation's graph is not symmetric about the origin.

Definition

A function f is even if f(-x) = f(x) for all x in the domain of f.
A function f is odd if f(-x) = -f(x) for all x in the domain of f.

Fact: The graph of an even function is symmetric about the y-axis, and the graph of an odd function is symmetric about the origin.

If the graph of a relation is symmetric about the x-axis, then the relation cannot be a function!

$\boxed{38}$ Determine whether the function $f(x) = x^3 - x$ is even, odd, or neither.

For any x in the domain of f, we have:

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$$

So f is $\boxed{\text{odd.}}$ ■

$\boxed{40}$ Determine whether the function $g(x) = x^2 - x$ is even, odd, or neither.

$$g(-x) = (-x)^2 - (-x) = x^2 + x.$$

So $g(-x) \neq g(x) = x^2 - x$ & $g(-x) \neq -g(x) = -x^2 + x$

So g is $\boxed{\text{neither}}$ ■

A function f is piecewise defined if the expression f(x) changes depending on the value of x. For instance, the absolute value function is piecewise defined:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Two "pieces"

$\boxed{\text{Ex}}$ Consider the piecewise defined function f given by

$$f(x) = \begin{cases} -2x+1, & \text{if } x \leq 1 \\ 3x^2, & \text{if } 1 < x \leq 3 \\ \sqrt{x-2}, & \text{if } x > 3 \end{cases}$$

Find $f(-1)$, $f(1)$, $f(2)$, $f(3)$, $f(6)$.

- We have $f(x) = -2x+1$ for any $x \le 1$, so...

$f(-1) = -2(-1)+1 = \boxed{3}$

$f(1) = -2(1)+1 = \boxed{-1}$

- We have $f(x) = 3x^2$ for any $1 < x \le 3$, so...

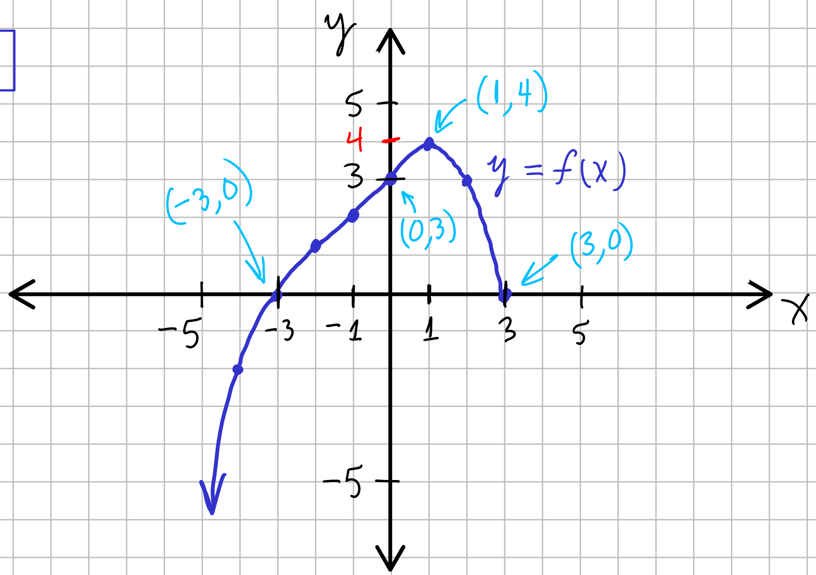
$f(2) = 3(2)^2 = \boxed{12}$

$f(3) = 3(3)^2 = 3^3 = \boxed{27}$

- We have $f(x) = \sqrt{x-2}$ for any $x > 3$, so...

$f(6) = \sqrt{6-2} = \sqrt{4} = \boxed{2}$ ■

$\boxed{51}$



a) $D_f = \{x \mid f(x) \text{ is real}\}$

$= \{x \mid (x,y) \text{ is a point on the graph for some } y \text{ value}\}$

$= \boxed{(-\infty, 3]}$

b) $R_f = \{f(x) \mid x \in D_f\} = \{f(x) \mid x \le 3\}$

$= \{y \mid (x,y) \text{ is a point on the graph for some } x \text{ value}\}$

$= \boxed{(-\infty, 4]}$

c) A value c is a zero for f if $f(c) = 0$. An x-intercept for f is a real value c for which $f(c) = 0$. That is, the real zeros of a function are its x-intercepts.

Zeros of f here: $\boxed{-3 \ \& \ 3}$

d) $\boxed{f(0) = 3}$

e) f is increasing on $\boxed{(-\infty, 1)}$

f) f is decreasing on $\boxed{(1, 3)}$

g) Find the values of x for which f(x) ≤ 0. $\boxed{(-\infty, -3] \cup \{3\}}$

(This is where the graph is <u>on</u> or <u>below</u> the x-axis.)

h) Function f has one relative maximum at $\boxed{x=1,}$ and the value of the relative maximum is $\boxed{4.}$

i) What is the value of x for which f(x) = 4?

That is, what is the x-coordinate of the point on the graph that has y-coordinate 4? $\boxed{1}$

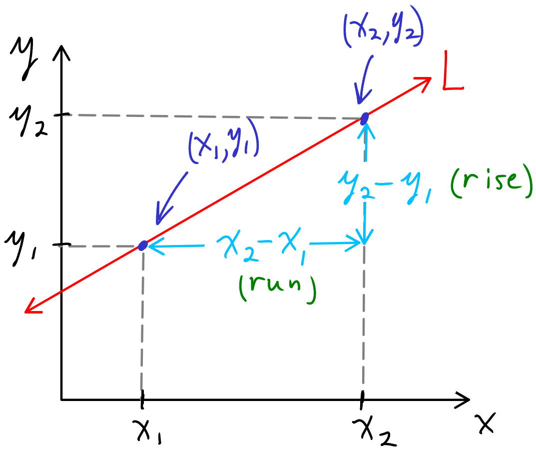
j) $f(-1) = 2$ (or something close to 2), so $f(-1)$ is $\boxed{\text{positive}}$

2.3 - Linear Functions and Slope

- A function f given by $f(x) = mx+b$ for all real x, where m & b are constants, is called a linear function.

- The graph of a linear function is a (non-vertical) line.

- The slope of a line containing points (x_1,y_1) & (x_2,y_2) is defined to be $\dfrac{y_2-y_1}{x_2-x_1}$



- For $f(x) = mx+b$ let $y_1 = f(x_1) = mx_1+b$ and let $y_2 = f(x_2) = mx_2+b$.

Then: $$\frac{y_2-y_1}{x_2-x_1} = \frac{(mx_2+b)-(mx_1+b)}{x_2-x_1} = \frac{mx_2-mx_1}{x_2-x_1} = \frac{m\cancel{(x_2-x_1)}}{\cancel{x_2-x_1}} = m$$

So the slope of a line given by a linear functions $f(x) = mx+b$ is m

$$\boxed{\text{Slope Formula: } m = \frac{y_2-y_1}{x_2-x_1}}$$

Ex Find the slope of the line containing the points (-3,-2) and (-1,5).

Let $(x_1,y_1) = (-3,-2)$ & $(x_2,y_2) = (-1,5)$.

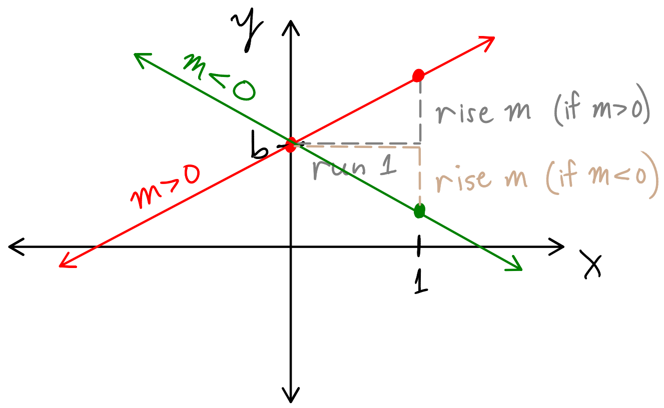
Slope m of the line is $m = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{5-(-2)}{-1-(-3)} = \dfrac{5+2}{-1+3} = \boxed{\dfrac{7}{2}}$.

So we have $\dfrac{\text{rise}}{\text{run}} = \dfrac{7}{2}$; that is, the line rises 7 units for every 2 units it runs to the right. ■

- For $f(x) = mx+b$ we have $f(0) = m \cdot 0 + b = b$, and so $(0,b)$ is a point on the line corresponding to f. That is,

$$\boxed{b \text{ is the y-intercept of the line}}$$

- Slope m gives the direction of the line, and the y-intercept b gives a particular point on the line. The two together can quickly give us a graph of the line generated by f(x)=mx+b.



Slope $m = \frac{\text{rise}}{\text{run}} = \frac{m}{1}$, so a rise of m units is followed by a run of 1 unit to get from point $(0,b)$ on the line to another point on the line.

Letting $y = f(x)$, then the linear function f(x)=mx+b can be written as the equation y=mx+b.

$$\boxed{\text{Slope-intercept form: } y = mx+b}$$

Suppose a line has slope m and point (x_1, y_1). We want to find an equation of the line. Let (x,y) be any other point on the line. This variable point (x,y) stands in for our second point (x_2, y_2) on the line. We use the slope formula with (x_1,y_1) and $(x_2,y_2)=(x,y)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1} \Rightarrow \cancel{(x-x_1)}\frac{y-y_1}{\cancel{x-x_1}} = m(x-x_1) \Rightarrow$$

$$y - y_1 = m(x-x_1)$$

$$\boxed{\text{Point-slope formula: } y - y_1 = m(x-x_1)}$$

$\boxed{16}$ Find the equation of the line with slope -5 and point (-4,-2).

Since we are given a point and a slope, we use the point-slope formula to get the line's equation...

We have $m=-5$ & $(x_1,y_1)=(-4,-2)$, so equation is:

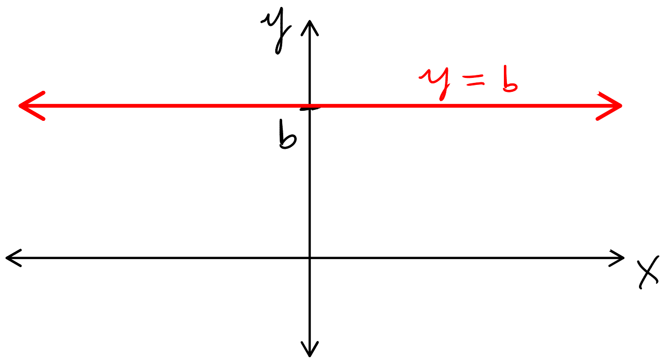
$$\boxed{y-(-2) = -5(x-(-4))} \leftarrow \text{"point-slope form"}$$

$$y+2 = -5(x+4)$$

$$\boxed{y = -5x-22} \leftarrow \text{slope-intercept form}$$

Note that the y-intercept for the line is -22! ■

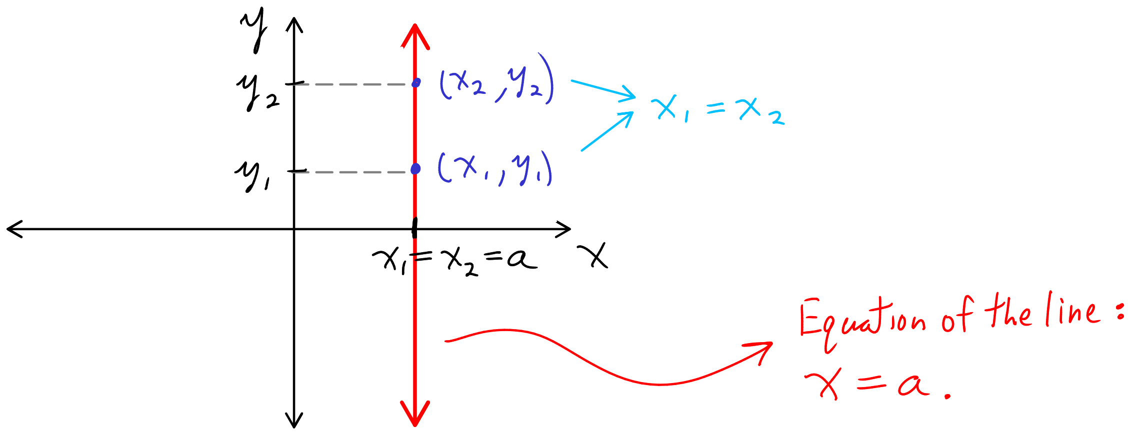
Suppose a line has slope m=0. Then f(x)=mx+b becomes f(x)=b, which is a constant function (a function with only one value in its range, or in other words can return only one value as output). If we let y=f(x), then we get the equation y=b. The graph is a horizontal line with y-intercept b.



So: a line with slope 0 is horizontal.

The slope of a line is undefined if and only if the line is vertical.

Given that $m = \dfrac{y_2-y_1}{x_2-x_1}$, we see that m is undefined if and only if $x_1=x_2$ (division by 0 occurs). To have $x_1=x_2$ means to have a line with two points having the same x-coordinate.



The general form of the equation of a line is $Ax+By+C=0$ for some real constants A, B, and C, where A and B are not both 0.

$\boxed{23}$ Find the equation of the line with slope -2/3 and point (6,-2).

Since we are given a point and a slope, we use the point-slope formula to get the line's equation. We use

$y - y_1 = m(x - x_1)$ with $m = -\frac{2}{3}$, $x_1 = 6$, $y_1 = -2$ ...

$y - (-2) = -\frac{2}{3}(x-6) \Rightarrow \boxed{y + 2 = -\frac{2}{3}(x-6)}$ (Point-slope form)

Now we get the slope-intercept form that the instructions call for...

$\underset{-2}{y+2} = -\frac{2}{3}x + \underset{-2}{4} \Rightarrow \boxed{y = -\frac{2}{3}x + 2}$ (Slope-intercept form) ■

$\boxed{35}$ Find the equation of the line with point (2,4) and x-intercept -2.

- x-intercept $= -2$ means point $(-2,0)$ lies on the line.
- With two points $(\underset{x_2}{2}, \underset{y_2}{4})$ & $(\underset{x_1}{-2}, \underset{y_1}{0})$ we get slope $m = \frac{4-0}{2-(-2)} = \frac{4}{4} = 1$
- Point-slope formula gives $y - 0 = 1\cdot(x-(-2)) \Rightarrow \boxed{y = x+2}$,

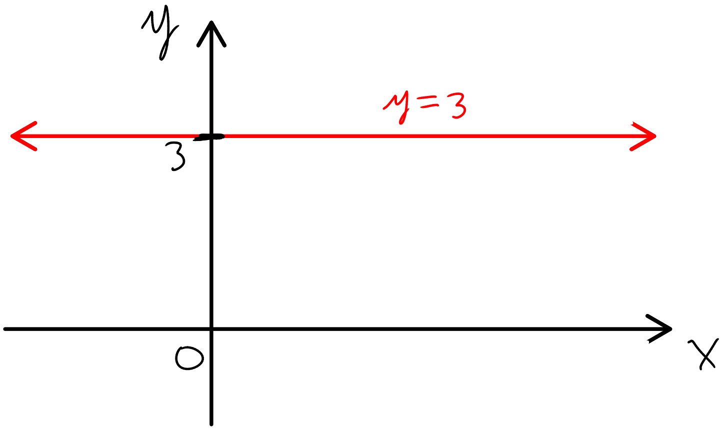
which is both the point-slope form and slope-intercept form. ■

$\boxed{65}$ $3y - 9 = 0$.

a) Slope-intercept form: $3y = 9 \Rightarrow y = 3$

b) Compare $y=3$ with $y = mx+b$ to see that slope is $m=0$ & y-intercept is $b=3$

c)



2.4 - More on Slope

- Let line L_1 have slope m_1 & line L_2 have slope m_2.
- We say L_1 & L_2 are parallel (written $L_1 \parallel L_2$) if $m_1 = m_2$.
- We say L_1 & L_2 are perpendicular (written $L_1 \perp L_2$) if $m_1 m_2 = -1$, or if one of the slopes is 0 & the other is undefined.

8 Find the equation of the line L_1 that passes through (-4,2) and is perpendicular to the line L_2 whose equation is $y = \frac{1}{3}x + 7$.

- The line L_2 has mx+b = $\frac{1}{3}x + 7$, and so the slope of this line is $\frac{1}{3}$. That is, $m_2 = 1/3$.
- Since $L_1 \perp L_2$, the slope m_1 of L_1 must be such that $m_1 m_2 = -1$. Thus $\frac{1}{3}m_1 = -1 \Rightarrow m_1 = -3$.
- So L_1 has point $(-4,2)$ & slope -3. With the point-slope formula we get

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = -3(x - (-4)) \Rightarrow$$

$\boxed{y - 2 = -3(x+4)}$ (point-slope form) $\Rightarrow y - 2 = -3x - 12 \Rightarrow$
$ +2 \qquad +2$

$\boxed{y = -3x - 10}$ (slope-intercept form) ■

---

10 Find the equation of the line L_1 that passes through (-1,3) and is parallel to the line L_2 whose equation is 3x-2y-5=0.

- Get the slope m_2 of L_2 by putting its equation into slope-intercept form:
$3x - 2y = 5 \Rightarrow -2y = -3x + 5 \Rightarrow y = \frac{-3x+5}{-2} \Rightarrow y = \frac{3}{2}x - \frac{5}{2}$
Comparing $\frac{3}{2}x - \frac{5}{2}$ with $mx+b$, we see that $m_2 = \frac{3}{2}$.
- Since $L_1 \parallel L_2$, to slope of L_1 is $m_1 = m_2 = \frac{3}{2}$.

- So L_1 has point $(-1,3)$ & slope $3/2$. Equation is: $y-3=\frac{3}{2}(x-(-1))$
- Point-slope form: $\boxed{y-3=\frac{3}{2}(x+1)}$ (point-slope form)
- General form: $2(y-3)=2\cdot\frac{3}{2}(x+1) \Rightarrow 2y-6=3(x+1) \Rightarrow$

$2y-6=3x+3 \Rightarrow \boxed{-3x+2y-9=0}$ (general form)

If we multiply both sides of the general form equation by -1, we get...

$\boxed{3x-2y+9=0}$ (also general form) ■

Ex A line L has y-intercept -3 and is perpendicular to the line $y-2x+5=0$. Find the equation for L in slope-intercept form.

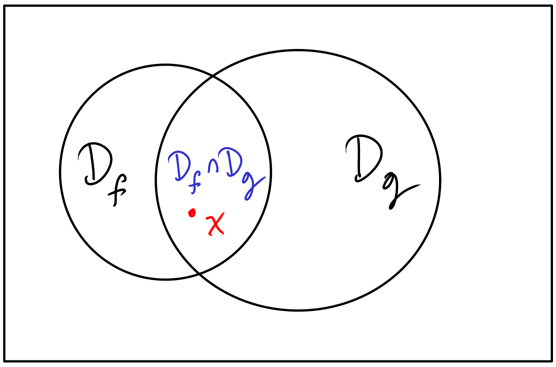
- Find slope of the line $y-2x+5=0$: $y=2x-5$, so slope is 2.
- Since L is $\perp$ to $y-2x+5=0$, the slope of L must be $-\frac{1}{2}$. (Note: $-\frac{1}{2}\cdot 2=-1$, as required.)
- So L has slope $-\frac{1}{2}$ & y-intercept $b=-3$. From $y=mx+b$ we get $\boxed{y=-\frac{1}{2}x-3}$ ■

We skip section 2.5!

2.6 - Function Combinations & Compositions

Let f and g be functions with domain D_f and D_g. We define a new function, denoted by f+g, as follows:

$$(f+g)(x) = f(x) + g(x) \quad \text{with domain } D_{f+g} = D_f \cap D_g.$$



Similarly we define f-g, fg, and f/g as follows:

$$(f-g)(x) = f(x) - g(x) \quad \text{with domain } D_{f-g} = D_f \cap D_g.$$

$$(fg)(x) = f(x) \cdot g(x) \quad \text{with domain } D_{fg} = D_f \cap D_g.$$

$$(f/g)(x) = \frac{f(x)}{g(x)} \quad \text{with domain } D_{f/g} = \{x \mid x \in D_f \cap D_g \ \& \ g(x) \neq 0\}.$$

We call f+g, f-g, fg, and f/g function combinations.

$\boxed{2.6.48}$ Find f+g, f-g, fg, f/g, and also the domain for each, where

$$f(x) = \sqrt{x+6} \quad \& \quad g(x) = \sqrt{x-3}.$$

- To "find f+g" means to obtain an expression in terms of x for (f+g)(x). By definition,

$$(f+g)(x) = f(x) + g(x) = \boxed{\sqrt{x+6} + \sqrt{x-3}.} \quad (f+g \text{ now found!})$$

Also...

$$(f-g)(x) = f(x) - g(x) = \boxed{\sqrt{x+6} - \sqrt{x-3}.}$$

$$(fg)(x) = f(x)\,g(x) = \boxed{\sqrt{x+6}\cdot\sqrt{x-3}} = \boxed{\sqrt{(x+6)(x-3)}}$$

$$= \boxed{\sqrt{x^2+3x-18}}$$

Can stop here.

$$(f/g)(x) = \frac{f(x)}{g(x)} = \boxed{\frac{\sqrt{x+6}}{\sqrt{x-3}}} = \boxed{\sqrt{\frac{x+6}{x-3}}}$$

- Now we find the domain for each function combination. This means finding D_f and D_g first.

$$D_f = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}$$

Real input... ...resulting in real output.

$$= \{x \mid f(x) \text{ is real}\} = \{x \mid \sqrt{x+6} \text{ is real}\}$$

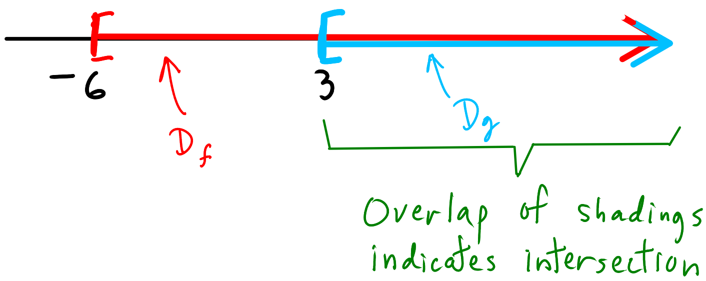
It's understood that x must be real here.

$$= \{x \mid x+6 \geq 0\} = \{x \mid x \geq -6\} = \boxed{[-6,\infty)}$$

$$D_g = \{x \mid g(x) \in \mathbb{R}\} = \{x \mid \sqrt{x-3} \text{ is real}\} = \{x \mid x \geq 3\}$$

$$= \boxed{[3,\infty)}.$$

Now we get: $D_f \cap D_g = [-6,\infty) \cap [3,\infty) = [3,\infty)$



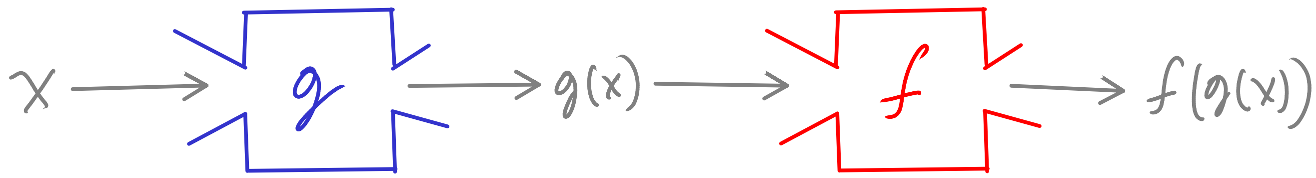
$$D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g = \boxed{[3,\infty)}$$

$$D_{f/g} = \{x \mid x \in D_f \cap D_g \ \& \ g(x) \neq 0\} = \{x \mid x \in [3,\infty) \ \& \ \sqrt{x-3} \neq 0\}$$

$$= \{x \mid x \geq 3 \ \& \ x \neq 3\} = \{x \mid x > 3\} = \boxed{(3,\infty)}$$

The composition of f with g, denoted by f ∘ g (read as "f composed with g" or "f circle g"), is defined as follows:

$$\boxed{(f \circ g)(x) = f(g(x))}$$



The domain of $f \circ g$ is...

$$\boxed{D_{f\circ g} = \{x \mid x \in D_g \ \& \ g(x) \in D_f\}}$$

$\boxed{\text{Ex}}$ Let $f(x) = \sqrt{x+6}$ & $g(x) = \sqrt{x-3}$.

- Find $f \circ g$ & its domain.

$$(f\circ g)(x) = f(g(x)) = f\left(\sqrt{x-3}\right) = \boxed{\sqrt{\sqrt{x-3}+6}} \quad (f\circ g \text{ is found})$$

Work from the inside out.

In the previous example we found $D_f = [-6,\infty)$ & $D_g = [3,\infty)$, so:

$$D_{f\circ g} = \{x \mid x \in D_g \ \& \ g(x) \in D_f\}$$

$$= \{x \mid x \in [3,\infty) \ \& \ \sqrt{x-3} \in [-6,\infty)\}$$

$x \geq 3$ $\qquad$ $\sqrt{x-3} \geq -6$

Since a square root is never negative, $\sqrt{x-3} \geq -6$ implies we must have $\sqrt{x-3} \geq 0 \Rightarrow x-3 \geq 0 \Rightarrow x \geq 3$.

$$D_{f\circ g} = \{x \mid x \geq 3 \ \& \ x \geq 3\} = \{x \mid x \geq 3\} = \boxed{[3,\infty)}.$$

• Now find $g\circ f$ & its domain.

$$(g\circ f)(x) = g(f(x)) = g(\sqrt{x+6}) = \boxed{\sqrt{\sqrt{x+6}-3}}.$$

(Recall: $(f\circ g)(x) = \sqrt{\sqrt{x-3}+6} \neq \sqrt{\sqrt{x+6}-3} = (g\circ f)(x)$
Thus $f\circ g \neq g\circ f$, which is usually the case.)

$$D_{g\circ f} = \{x \mid x\in D_f \ \& \ f(x)\in D_g\}$$

$$= \{x \mid x\in[-6,\infty) \ \& \ \sqrt{x+6}\in[3,\infty)\}$$

$x \geq -6$ $\qquad$ $\sqrt{x+6}\geq 3 \Rightarrow x+6\geq 9 \Rightarrow x\geq 3$

$$= \{x \mid x\geq -6 \ \& \ x\geq 3\}$$

$$= \{x \mid x\geq 3\} = \boxed{[3,\infty)}$$

In this case we have $D_{f\circ g} = D_{g\circ f}$, but in general this is not the case.

• Evaluate $(f\circ g)(4)$ & $(g\circ f)(4)$.

$(f\circ g)(x) = \sqrt{\sqrt{x-3}+6}$, so:

$$(f\circ g)(4) = \sqrt{\sqrt{4-3}+6} = \sqrt{\sqrt{1}+6} = \sqrt{1+6} = \boxed{\sqrt{7}}.$$

$(g\circ f)(x) = \sqrt{\sqrt{x+6}-3}$, so:

$$(g\circ f)(4) = \sqrt{\sqrt{4+6}-3} = \boxed{\sqrt{\sqrt{10}-3}}$$

Note that $(f\circ g)(4) \neq (g\circ f)(4)$. ∎

$\boxed{30}$ Find the domain of $f(x) = \dfrac{7x+2}{x^3-2x^2-9x+18}$

$$D_f = \{x \mid f(x) \text{ is real}\} = \left\{x \;\middle|\; \frac{7x+2}{x^3-2x^2-9x+18} \text{ is real}\right\}$$

$$= \{x \mid x^3-2x^2-9x+18 \neq 0\}$$

$$(x^3-2x^2)+(-9x+18) \neq 0$$

$$x^2(x-2)+(-9)(x-2) \neq 0$$

$$x^2(x-2)-9(x-2) \neq 0$$

$$(x-2)(x^2-9) \neq 0$$

$$(x-2)(x-3)(x+3) \neq 0$$

$$x \neq 2, 3, -3$$

$$= \{x \mid x \neq -3, 2, 3\} = \boxed{(-\infty,-3)\cup(-3,2)\cup(2,3)\cup(3,\infty)}$$

$\boxed{74}$ Let $f(x) = x^2+1$ & $g(x) = \sqrt{2-x}$.

a) Find $f\circ g$

$$(f\circ g)(x) = f(g(x)) = f\left(\sqrt{2-x}\right) = \left(\sqrt{2-x}\right)^2+1 = (2-x)+1$$

$$= \boxed{3-x}$$

b) Find $D_{f\circ g}$

$$D_f = (-\infty,\infty) \quad \& \quad D_g = \{x \mid 2-x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2].$$

$$D_{f\circ g} = \{x \mid x \in D_g \;\&\; g(x) \in D_f\}$$

$$= \{x \mid x \in (-\infty, 2] \;\&\; \sqrt{2-x} \in (-\infty,\infty)\}$$

So $\sqrt{2-x}$ must be real

$$= \{x \mid x \le 2 \ \& \ 2 - x \ge 0\}$$

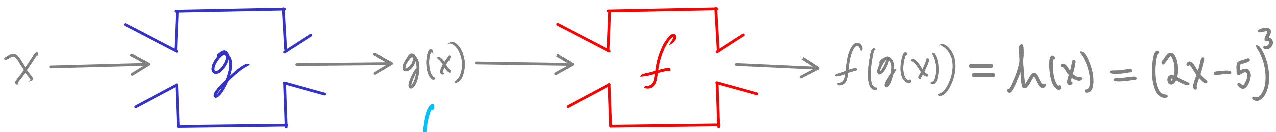
$$= \{x \mid x \le 2 \ \& \ x \le 2\}$$

$$= \{x \mid x \le 2\} = \boxed{(-\infty, 2]}$$

The identity function is the function that takes x as input and returns x as output; that is, the output is identical to the input. If I denotes the identity function, then I(x)=x for all real x.

$\boxed{76}$ Let $h(x) = (2x-5)^3$. Find functions f & g such that $f \circ g = h$. Neither f nor g should be the identity function!

So we need to have $(f \circ g)(x) = h(x)$ for all $x \in D_h$:



If this is $2x-5$, then we want $f(2x-5) = (2x-5)^3$. This will work if we define $f(x) = x^3$
Meanwhile we also want $g(x) = 2x-5$

We want $\boxed{f(x) = x^3 \ \& \ g(x) = 2x-5.}$

Check it: $(f \circ g)(x) = f(g(x)) = f(2x-5) = (2x-5)^3 = h(x)$

More than two functions can be put together through the composition operation:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Q&A session...

$\boxed{2.4.11}$ Find the equation of the line L_1 passing through (4,-7) and perpendicular to the line L_2 with equation x-2y-3=0,

- We need the slope of line L_1. We start by getting the slope of L_2.
- $x-2y-3=0 \Rightarrow x-3=2y \Rightarrow y=\frac{1}{2}x-\frac{3}{2}$.

  Compare with slope-intercept form, which is $y=mx+b$.

  We see that the slope of L_2 is $m_2=\frac{1}{2}$

- Since $L_1 \perp L_2$, the slope m_1 of line L_1 is such that

$$m_1 m_2 = -1.$$

We get $m_1 = -\frac{1}{m_2} = -\frac{1}{1/2} = -2$

- Using the point-slope formula $y-y_1=m(x-x_1)$ with $m=m_1=-2$ & $x_1=4$ & $y=-7$, we get

  $y-(-7)=-2(x-4) \Rightarrow \boxed{y+7=-2(x-4)}$ (point-slope form)

- The general form is Ax+By+C=0, so...

  $y+7=-2x+8 \Rightarrow \boxed{2x+y-1=0}$ (general form)

$\boxed{2.6.13}$ Find the domain of $h(x)=\dfrac{4}{\frac{3}{x}-1}$.

$D_h=\{x \mid h(x) \text{ is real}\} = \left\{x \;\middle|\; \dfrac{4}{\frac{3}{x}-1} \text{ is real}\right\}$

$$= \{x \mid x \neq 0 \ \& \ \tfrac{3}{x} - 1 \neq 0\}$$

Otherwise $\frac{3}{x}$ is undefined! Otherwise $\frac{4}{0}$ results — undefined!

We have: $\frac{3}{x} - 1 \neq 0 \Rightarrow \frac{3}{x} \neq 1 \Rightarrow x \neq 3$, and so...

$$D_h = \{x \mid x \neq 0 \ \& \ x \neq 3\} = \boxed{(-\infty,0)\cup(0,3)\cup(3,\infty)}$$

In the example above one might be tempted to simplify the complex fraction in

$$h(x) = \frac{4}{\frac{3}{x} - 1} \cdot \frac{x}{x} = \frac{4x}{3-x}$$

Looks like 0 is in the domain of h now, but we cannot get this fraction without assuming $x \neq 0$.

We must assume $x \neq 0$ to do this simplification!

$\boxed{\text{Ex}}$ Find the domain of $f(x) = \dfrac{\sqrt{x-3} + \sqrt{10-x}}{x-5}$

$$D_f = \{x \mid f(x) \text{ is real}\}$$

$$= \{x \mid x - 3 \geq 0 \ \& \ 10 - x \geq 0 \ \& \ x - 5 \neq 0\}$$

$$= \{x \mid x \geq 3 \ \& \ x \leq 10 \ \& \ x \neq 5\}$$

$$= \{x \mid 3 \leq x \leq 10 \ \& \ x \neq 5\}$$

$$= \boxed{[3,5)\cup(5,10]}$$

2.7 - Inverse Functions

A function f is one-to-one (I might write 1-1) if f(a)=f(b) implies a=b for every a and b in the domain of f.

Equivalently, a function f is one-to-one if f(a) $\neq$ f(b) whenever a $\neq$ b. So different inputs always yield different outputs!

$\boxed{\text{Ex}}$ $f(x) = x^2 - 2x - 15$ is <u>not</u> 1-1. We have $f(x) = (x-5)(x+3)$, so $f(5) = 0$ & $f(-3) = 0$. That is, $f(5) = f(-3)$, and yet $5 \neq -3$. ∎

$\boxed{\text{Ex}}$ Show that $f(x) = \dfrac{x+3}{2x-1}$ is 1-1.

Let a and b be two values in the domain of f, and suppose that f(a)=f(b).

Then $\dfrac{a+3}{2a-1} = \dfrac{b+3}{2b-1} \Rightarrow$

$(a+3)(2b-1) = (b+3)(2a-1) \Rightarrow$

$\cancel{2ab} - a + 6b \cancel{-3} = \cancel{2ab} - b + 6a \cancel{-3} \Rightarrow$

$-a + 6b = -b + 6a \Rightarrow$

$-7a = -7b \Rightarrow$

$a = b$ ✓

So supposing that $f(a) = f(b)$ necessarily implies that $a = b$. Therefore f is 1-1. ∎

A 1-1 function f has what is called an inverse function, denoted by the symbol f^{-1} (read as "f inverse" or "the inverse of f").

Note: the -1 in the symbol f^{-1} is NOT an exponent. That is,

$$f^{-1} \neq \frac{1}{f}.$$

The relationship between f & f^{-1} is as follows:

$$\boxed{\begin{array}{l} f(x) = y \quad \Leftrightarrow \quad f^{-1}(y) = x \\ \text{for all } x \in D_f \ \& \ y \in R_f \end{array}} \qquad (1)$$

(Recall: R_f means "range of f")

Since inputs of f are outputs of f^{-1} and inputs of f^{-1} are outputs of f, we find the following to be true:

$$\boxed{R_f = D_{f^{-1}} \ \& \ D_f = R_{f^{-1}}} \qquad (2)$$

From definition (1) we find that:

$$\boxed{\begin{array}{l} f(f^{-1}(y)) = y \quad \& \quad f^{-1}(f(x)) = x \\ \text{for all } x \in D_f \ \& \ y \in R_f \end{array}} \qquad (3)$$

Here (3) is a theorem that is derived from (1). We could write (3) as ...

$$\boxed{\begin{array}{l} (f \circ f^{-1})(y) = y \quad \& \quad (f^{-1} \circ f)(x) = x \\ \text{for all } x \in D_f \ \& \ y \in R_f \end{array}} \qquad (3')$$

$\boxed{10}$ Determine whether $f(x)=\sqrt[3]{x-4}$ is the inverse of $g(x)=x^3+4$.

Using (3) or (3′), we see if we have $f(g(y))=y$ & $g(f(x))=x$ for all $x\in D_f$ & $y\in R_f$.

Let $x\in D_f$ & $y\in R_f$. Then...

$f(g(y)) = f(y^3+4) = \sqrt[3]{(y^3+4)-4} = \sqrt[3]{y^3} = y.$

$g(f(x)) = g(\sqrt[3]{x-4}) = (\sqrt[3]{x-4})^3+4 = (x-4)+4 = x.$

So $f(g(y))=y$ & $g(f(x))=x$, and we conclude that g is the inverse of f; that is, $g=f^{-1}$. ∎

$\boxed{\text{Ex}}$ The function $f(x)=\dfrac{3-4x}{2x+5}$ is one-to-one, and so has an inverse. Find the inverse.

To do this, we use (1): $f(x)=y \;\Leftrightarrow\; f^{-1}(y)=x$

Procedure is as follows:

1) Let $y=f(x)$, so that $y=\dfrac{3-4x}{2x+5}$.

2) Solve for x...

$$y=\frac{3-4x}{2x+5} \Rightarrow y(2x+5)=3-4x \Rightarrow 2xy+5y=3-4x \Rightarrow$$

$$2xy+4x=3-5y \Rightarrow x(2y+4)=3-5y \Rightarrow$$

$$x = \frac{3-5y}{2y+4}.$$

3) Since $f(x) = y \Leftrightarrow f^{-1}(y) = x$, we have:

$$\boxed{f^{-1}(y) = \frac{3-5y}{2y+4}}.$$

We have found f^{-1}!

4) Optional: restore the symbol x in its traditional role as the independent variable...

$$\boxed{f^{-1}(x) = \frac{3-5x}{2x+4}}$$

Note: the x that appears in Step 4 is not the same x as in the earlier steps! The x is just a symbol to denote the input for a function.

For the next example we recall (2) from a couple pages ago:

$$\boxed{R_f = D_{f^{-1}} \ \& \ D_f = R_{f^{-1}}}$$

$\boxed{\text{Ex}}$ Find the domain and range of both f and f^{-1} for

$$f(x) = \frac{3-4x}{2x+5}.$$

In the previous example we found that $f^{-1}(y) = \dfrac{3-5y}{2y+4}$

$$R_{f^{-1}} = D_f = \{x \mid f(x) \text{ is real}\} = \{x \mid 2x+5 \neq 0\}$$

$$= \{x \mid x \neq -5/2\} = \boxed{(-\infty, -5/2) \cup (-5/2, \infty)}$$

$$R_f = D_{f^{-1}} = \{x \mid 2x+4 \neq 0\} = \{x \mid x \neq -2\}$$
$$= \boxed{(-\infty,-2)\cup(-2,\infty)}$$

So there is no $x \in D_f$ such that $f(x) = -2$.

For nonbelievers...

$$f(x) = -2 \Rightarrow \frac{3-4x}{2x+5} = -2 \Rightarrow 3-4x = -2(2x+5) \Rightarrow$$

$$3-4x = -4x-10 \Rightarrow 3 = -10$$
$$+4x \qquad +4x$$

$\boxed{\text{Ex}}$ Let $f(x) = (x-1)^2$, $x \leq 1$.

The condition x ≤ 1 is part of the definition of f. Specifically it is declaring the domain of f to be $(-\infty, 1]$ instead of $(-\infty, \infty)$, which is the "native domain." This results in f being one-to-one.

a) Find f^{-1}.

1) Let $y = f(x)$, so $y = (x-1)^2$.

2) Solve for x: $\sqrt{(x-1)^2} = \sqrt{y} \Rightarrow |x-1| = \sqrt{y}$,

but $x \leq 1 \Rightarrow x-1 \leq 0 \Rightarrow |x-1| = -(x-1) = 1-x$

and so $1-x = \sqrt{y} \Rightarrow x = 1-\sqrt{y}$

3) Since $f(x) = y \Leftrightarrow f^{-1}(y) = x$, we have

$f^{-1}(y) = 1-\sqrt{y}$.

4) So $f^{-1}(x) = 1-\sqrt{x}$

b) Find D_f, R_f, $D_{f^{-1}}$, $R_{f^{-1}}$

$R_{f^{-1}} = D_f = (-\infty, 1]$ (which was given)

$R_f = D_{f^{-1}} = [0, \infty)$ (need $x \geq 0$ so that $\sqrt{x}$ is real) ■

2.7.17 Let $f(x) = x^3 + 2$.

a) Find f^{-1}

- Let $y = f(x)$, so $y = x^3 + 2$.
- Since $x = f^{-1}(y)$, we solve $y = x^3 + 2$ for x:

  $x^3 = y - 2 \Rightarrow x = \sqrt[3]{y-2}$.
- We now have $f^{-1}(y) = \sqrt[3]{y-2}$.
- Could write $f^{-1}(x) = \sqrt[3]{x-2}$.

b) Verify we have the inverse of f.

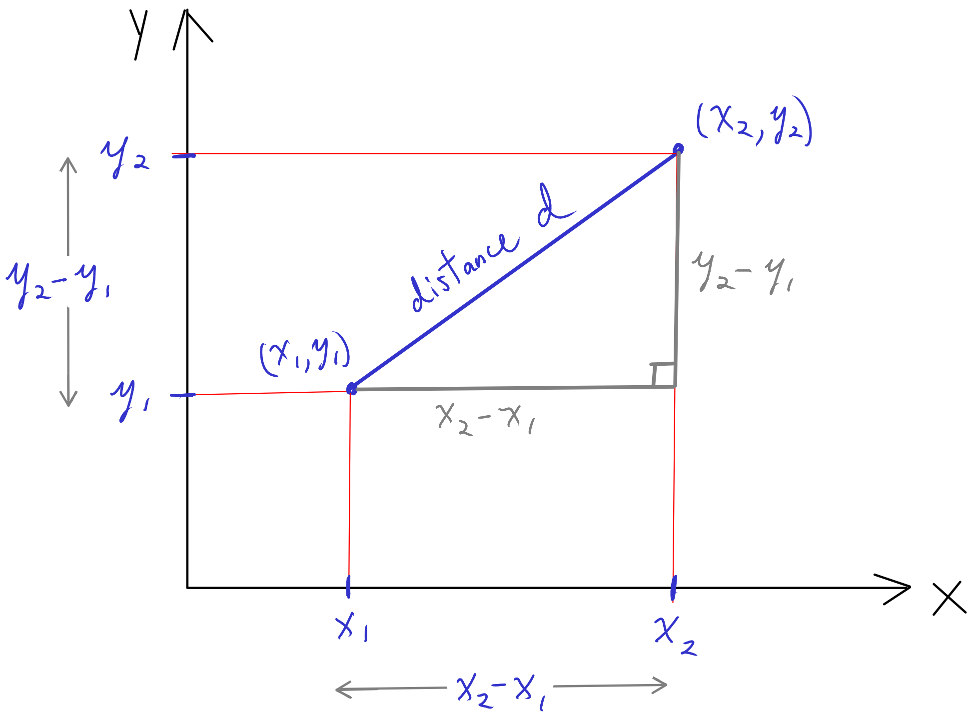
We check that $f(f^{-1}(y)) = y$ & $f^{-1}(f(x)) = x$

$f(f^{-1}(y)) = f(\sqrt[3]{y-2}) = (\sqrt[3]{y-2})^3 + 2 = (y-2) + 2 = y$

$f^{-1}(f(x)) = f^{-1}(x^3+2) = \sqrt[3]{(x^3+2) - 2} = \sqrt[3]{x^3} = x$ ■

2.8 - Distances, Midpoints, and Circles

We start by finding the distance between two points in the rectangular coordinate system (i.e. the xy-plane). The definition for distance we'll arrive at will derive from the Pythagorean theorem.



$$\downarrow$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\downarrow *$$

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

(Distance Formula)

* $\sqrt{d^2} = |d| = d$, since d is a length (and so not negative)

Ex Find the exact distance between points (-2,7) and (-5,-8).

We can let $(x_1, y_1) = (-2, 7)$ & $(x_2, y_2) = (-5, -8)$.
By distance formula we get:

$$d = \sqrt{(-5-(-2))^2 + (-8-7)^2} = \sqrt{(-3)^2 + (-15)^2} = \boxed{\sqrt{234}}$$

exact answer required!

$\boxed{\text{Ex}}$ Find the distance between (-2,7) & (-5,-8) rounded to two decimal places and also to one decimal place.

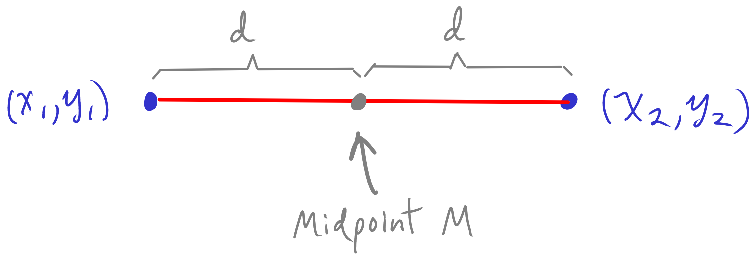
Rounding to "two decimal places" means rounding to the nearest hundredth (which keeps two digits to the right of the decimal point).

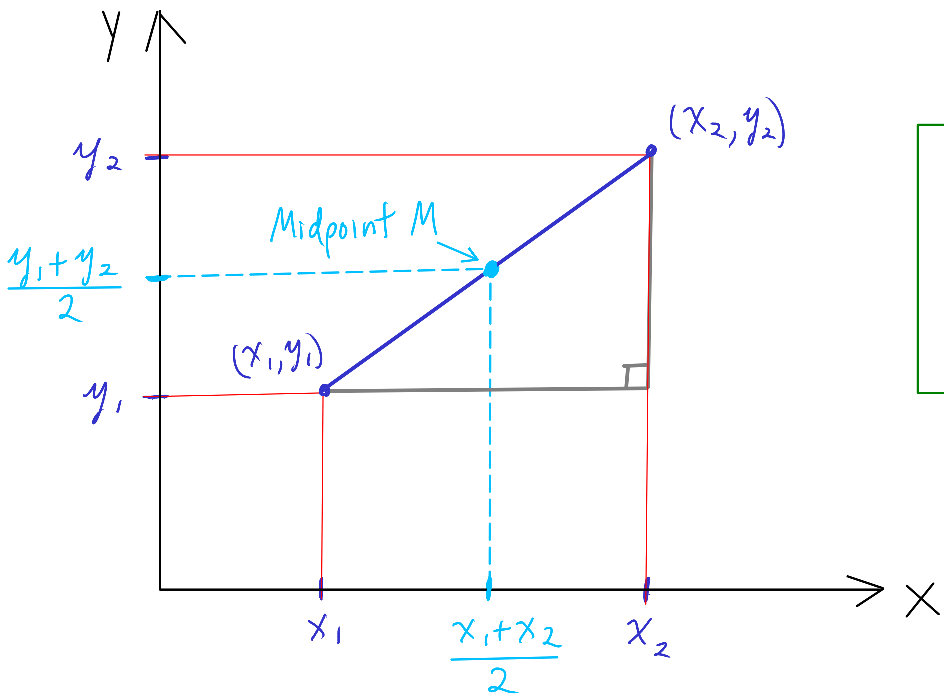
So: $d = \sqrt{234} = 15.29705\ldots \approx \boxed{15.30}$

Note: writing 15.3 is not quite correct, since it loses information that is contained in the representation 15.30.

Rounded to ONE decimal place, the distance d is $\boxed{15.3}$

Now we derive a midpoint formula. Given two points (x_1,y_1) and (x_2,y_2), the midpoint between these points is the point on the line segment connecting the points that is equidistant from (x_1,y_1) and (x_2,y_2).





Midpoint M is at

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$\boxed{\text{Ex}}$ Find the midpoint between the points (-2,7) & (-5,-8).

We can let $(x_1,y_1) = (-2,7)$ & $(x_2,y_2) = (-5,-8)$.

With the midpoint formula we then get the coordinates of the midpoint:

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-2+(-5)}{2}, \frac{7+(-8)}{2}\right) = \boxed{\left(-\frac{7}{2}, -\frac{1}{2}\right)}$$

$\boxed{\text{Ex}}$ Find the distance between (-7/2,-1/2) and (-2,7).

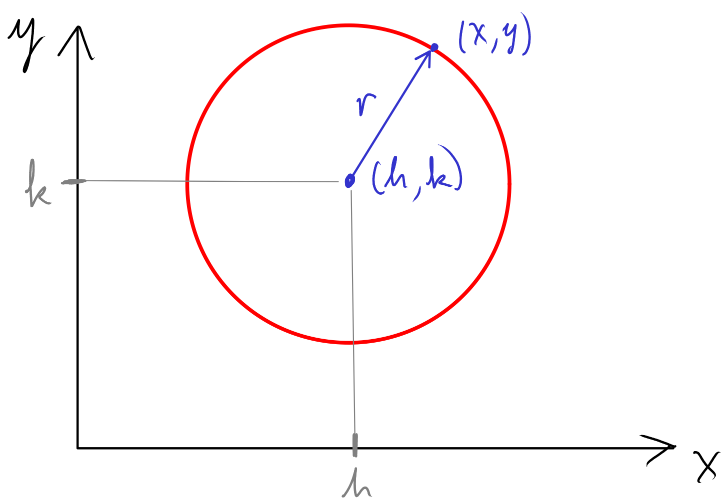
Let $(x_1,y_1) = \left(-\frac{7}{2}, -\frac{1}{2}\right)$ & $(x_2,y_2) = (-2,7)$.

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{\left(-2-\left(-\frac{7}{2}\right)\right)^2+\left(7-\left(-\frac{1}{2}\right)\right)^2}$$

$$= \sqrt{\left(-2+\frac{7}{2}\right)^2+\left(7+\frac{1}{2}\right)^2} = \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{15}{2}\right)^2} = \sqrt{\frac{3^2+15^2}{2^2}}$$

$$= \sqrt{\frac{234}{4}} = \boxed{\frac{\sqrt{234}}{2}} = \boxed{\frac{1}{2}\sqrt{234}.}$$

A circle with center located at point (h,k) and radius r>0 is defined to be the set of points (x,y) in the xy-plane that are a distance r from (h,k).



So (x,y) lies on the circle if & only if (x,y) is a distance r from (h,k). With the distance formula we have, letting $(x_1,y_1)=(h,k)$ & $(x_2,y_2)=(x,y)$,

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

The center-radius form for the equation of our circle (called the "standard form" in the textbook), is derived by squaring the equation above, to eliminate the radical:

$$\boxed{(x-h)^2+(y-k)^2=r^2} \quad \text{(center-radius form)}$$

$\boxed{38}$ Write the equation of the circle having center (-5,-3) and radius $\sqrt{5}$.

We have center (h,k) with $h=-5$ & $k=-3$, and radius $r=\sqrt{5}$. Equation of circle is:

$$\left(x-(-5)\right)^2+\left(y-(-3)\right)^2=(\sqrt{5})^2$$

$$\boxed{(x+5)^2+(y+3)^2=5}$$

$\boxed{58}$ Find the center and radius of the circle having equation

$$x^2+y^2+12x-6y-4=0^{*}$$

We will want to get the equation into center-radius form, which will show us h, k, and r at a glance. To do this we need to complete the square twice: once for each variable.

* This is the form that is usually called the "standard form" for the equation of a circle: $Ax^2+By^2+Cx+Dy+E=0$

$$(x^2+12x)+(y^2-6y)=4$$

$$(x^2+12x+6^2)+(y^2-6y+(-3)^2)=\underbrace{4+6^2+(-3)^2}_{4+36+9=49}$$

halve $\quad 6 \quad$ square

-3

$$(x+6)^2+(y-3)^2=49$$

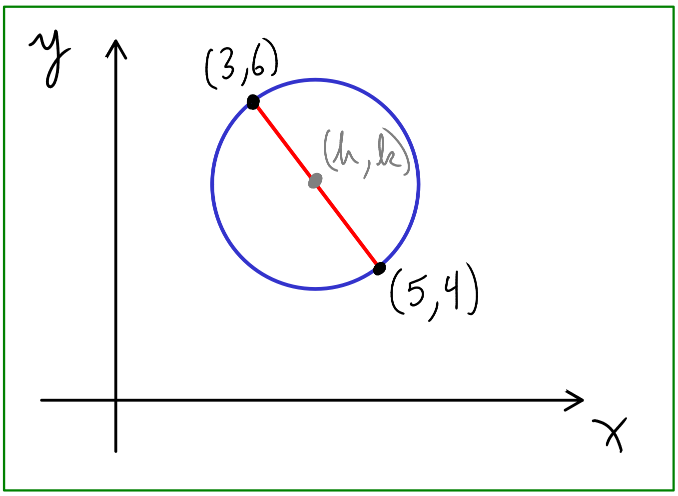
$$(x-h)^2+(y-k)^2=r^2$$

Compare: $h=-6$, $k=3$, and $r=7$ $(49=7^2)$

$\boxed{\text{Circle has center (-6,3) and radius 7.}}$

$\boxed{66}$ A line segment through the center of a circle intersects the circle at points (3,6) and (5,4).

a) Find the coordinates of the circle's center.
b) Find the radius of the circle.
c) Write the circle's equation.
d) Being a set of points (and so a set of ordered pairs), the circle is a relation. Find the domain and range of the relation.



a) Center of circle is midway between (3,6) & (5,4). With the midpoint formula we find the center to be at:

$$\left(\frac{3+5}{2}, \frac{6+4}{2}\right)=\boxed{(4,5)}$$

b) Radius of circle will be the distance between the center (4,5) and, say, the point (3,6).

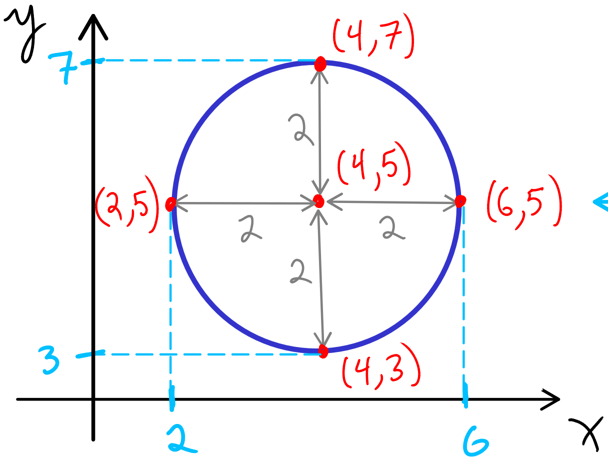
$$r = \sqrt{(4-3)^2 + (5-6)^2} = \sqrt{1^2 + (-1)^2} = \boxed{\sqrt{2}}$$

c) We have $(h,k) = (4,5)$ & $r = \sqrt{2}$, so...

$$(\sqrt{2})^2 = 2$$

$$(x-h)^2 + (y-k)^2 = r^2 \Rightarrow \boxed{(x-4)^2 + (y-5)^2 = 2}$$

d) Whenever given a graph of a relation, the domain of the relation will be the set of x-coordinates of the points lying on the graph, and the range will be the set of y-coordinates.



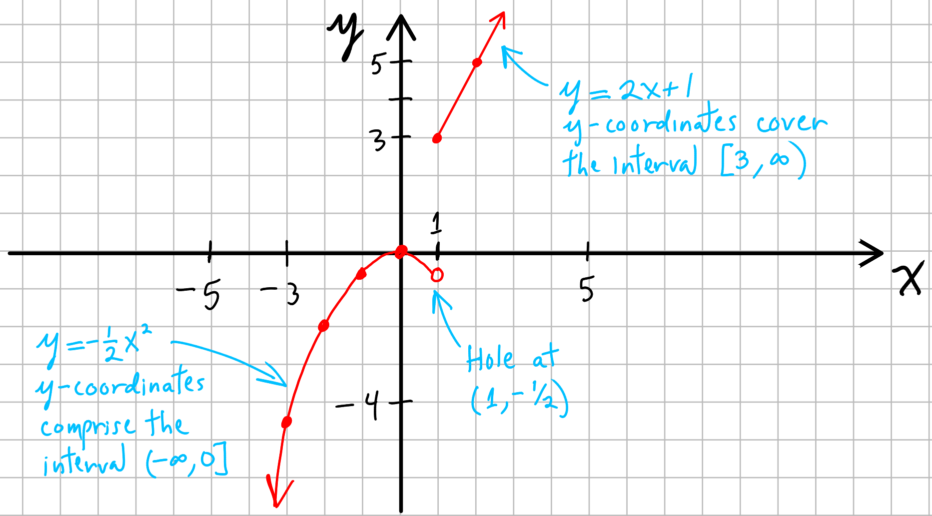
Domain: [2,6]

Range: [3,7]

Q & A

$\boxed{2.2.68}$ Let $f(x) = \begin{cases} -\frac{1}{2}x^2, & \text{if } x<1 \\ 2x+1, & \text{if } x \geq 1 \end{cases}$

Graph f, and find the range of f.



$f(-3) = -\frac{1}{2}(-3)^2 = -\frac{9}{2} \rightarrow (-3,-\frac{9}{2})$ is a point on the graph.

$f(-2) = -\frac{1}{2}(-2)^2 = -2 \rightarrow (-2,-2)$ is on graph

$f(-1) = -\frac{1}{2}(-1)^2 = -\frac{1}{2} \rightarrow (-1,-\frac{1}{2})$

$f(0) = -\frac{1}{2}(0)^2 = 0 \rightarrow (0,0)$

f has a hole at coordinates $(1, -\frac{1}{2}(1)^2) = (1,-\frac{1}{2})$

$f(1) = 2(1)+1 = 3 \rightarrow (1,3)$

$f(2) = 2(2)+1 = 5 \rightarrow (2,5)$

The range of f will be the set of y-coordinates of the points lying on the graph of f…

$\boxed{(-\infty,0] \cup [3,\infty).}$ Note: $D_f = (-\infty,\infty)$ ■