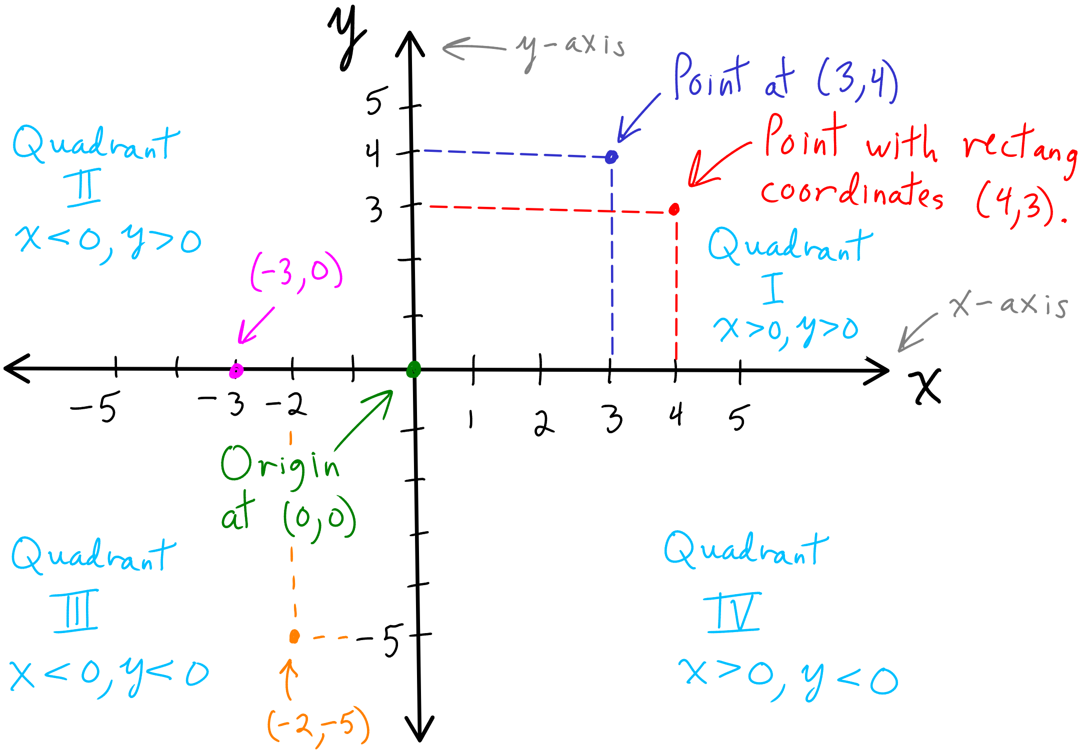
1.1 - Graphs

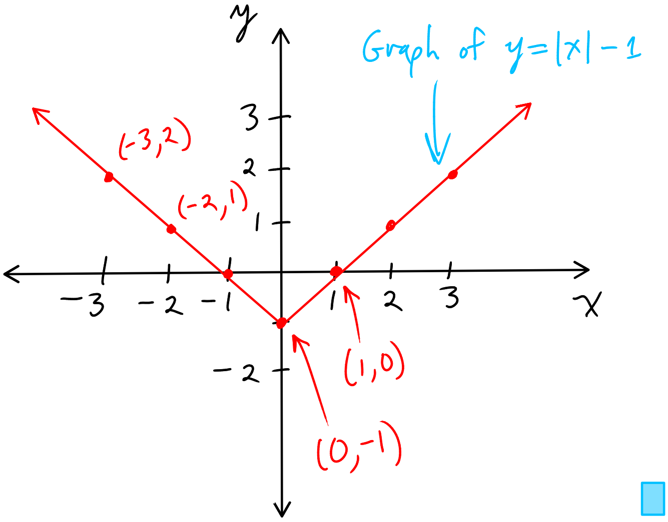
The rectangular coordinate system (or Cartesian coordinate system) is the only coordinate system we use in this course.



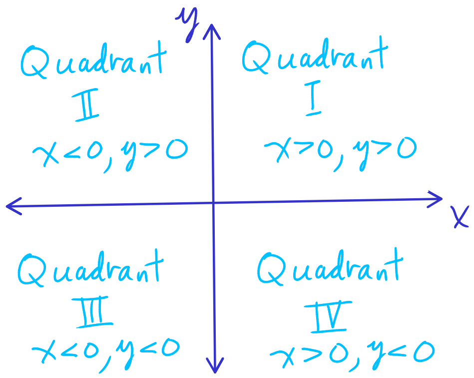
Rectangular coordinates are always in the format (x,y), which is to say an ordered pair with the x value 1st and the y value 2nd. So a point with coordinates (-2,-5) is located at x=-2 and y=-5.

Ex Graph $y=|x|-1$. Let $x=-3,-2,-1,0,1,2,3$.

x	Calculation	y	(x,y)
-3	$\lvert -3\rvert -1 = 3-1 = 2$	2	(-3,2)
-2	$\lvert -2\rvert -1 = 2-1 = 1$	1	(-2,1)
-1	$\lvert -1\rvert -1 = 1-1 = 0$	0	(-1,0)
0	$\lvert 0\rvert -1 = 0-1 = -1$	-1	(0,-1)
1	$\lvert 1\rvert -1 = 1-1 = 0$	0	(1,0)
2	$\lvert 2\rvert -1 = 2-1 = 1$	1	(2,1)
3	$\lvert 3\rvert -1 = 3-1 = 2$	2	(3,2)



$\boxed{\text{Ex}}$ List the quadrant(s) satisfying the condition : $\frac{x}{y} > 0$.



- To have $x/y > 0$ requires x & y to be either both positive $(x>0, y>0)$ or both negative $(x<0, y<0)$.
- $x>0, y>0$ implies (x,y) is in QI
- $x<0, y<0$ implies (x,y) is in QIII
- Answer: $\boxed{\text{Quadrants I \& III}}$ ■

1.2 - Linear & Rational Equations

A linear equation in one variable x has the standard form Ax+B=0, where A and B are constants, and A≠0.

$\boxed{\text{Ex}}$ Solve the linear equation $2(x-1)+3 = x-3(x+1)$.

1) Simplify each side: eliminate grouping symbols and combine like terms.

$$2x-2+3 = x-3x-3$$
$$2x+1 = -2x-3$$

2) Move all terms with x to the left side, and all terms without x to the right side.

$$\begin{array}{lcl} & 2x+1 & = -2x-3 \\ + & 2x-1 & \quad\;\; 2x-1 \\ \hline & 4x+0 & = 0-4 \end{array} \Rightarrow 4x=-4$$

3) Divide by the coefficient of x (i.e. the number that x is being multiplied by).

$$4x=-4 \xrightarrow{\text{Divide by 4}} \frac{4x}{4} = \frac{-4}{4} \rightarrow x=-1$$

4) State the solution set of the equation.

Solution set is $\boxed{\{-1\}}$

In a rational equation, both sides of the equation is a rational expression (remember that 0 is a rational expression). A rational expression is a ratio of polynomials (i.e. a fraction with a polynomial in the numerator and denominator).

$\boxed{\text{Ex}}$ $\frac{3}{x+3} = \frac{5}{2x+6} + \frac{1}{x-2}$ is a rational equation. Solve it.

1) Multiply both sides by the LCM of the denominators.

$\frac{3}{x+3} = \frac{5}{2(x+3)} + \frac{1}{x-2} \rightarrow$ LCM is $2(x+3)(x-2)$, so...

$$2\cancel{(x+3)}(x-2)\cdot\frac{3}{\cancel{x+3}} = \left(\frac{5}{2(x+3)} + \frac{1}{x-2}\right)\cdot 2(x+3)(x-2)$$

$$2(x-2)\cdot 3 = \frac{5}{\cancel{2}\cancel{(x+3)}}\cdot\cancel{2}\cancel{(x+3)}(x-2) + \frac{1}{\cancel{x-2}}\cdot 2(x+3)\cancel{(x-2)}$$

$$6(x-2) = 5(x-2) + 2(x+3)$$

2) Solve the resultant linear equation.

$$6x - 12 = 5x - 10 + 2x + 6$$

$$6x - 12 = 7x - 4$$

$$x = -8$$

3) Check for extraneous solutions, which are values of x that result in a division by 0 in the original equation.

The only values for x that would yield a division by 0 are 2 and -3. So this means x=-8 is a valid solution.

4) State the solution set.

Solution set is $\boxed{\{-8\}}$

$\boxed{56}$ Solve $\dfrac{5}{x+2}+\dfrac{3}{x-2}=\dfrac{12}{(x+2)(x-2)}$.

Distributive Property

$a(b+c)=ab+ac$

1) LCM is $(x+2)(x-2)$, so...

$$(x+2)(x-2)\left(\frac{5}{x+2}+\frac{3}{x-2}\right)=\frac{12}{\cancel{(x+2)}\cancel{(x-2)}}\cdot\cancel{(x+2)}\cancel{(x-2)}$$

$$\cancel{(x+2)}(x-2)\cdot\frac{5}{\cancel{x+2}}+(x+2)\cancel{(x-2)}\cdot\frac{3}{\cancel{x-2}}=12$$

$$(x-2)\cdot 5+(x+2)\cdot 3=12$$

2) Solve linear equation...

$$5x-10+3x+6=12$$
$$8x-4=12$$
$$8x=16$$
$$x=2$$

3) Check: letting x=2 in the original equation results in a division by 0, so the "solution" x=2 is extraneous. This is to say 2 is not a solution at all. Since there is no other value of x that could be a solution, there is no solution.

4) Since there is no solution, the solution set is the empty set, denoted by $\boxed{\varnothing}$. □

An equation with variable x is conditional if the equation is a true statement for some values of x, and false for other values.

Equation $x+3=5$ is conditional, since it's true when $x=2$ $(2+3=5)$, and false when $x\neq 2$.

An equation is inconsistent if it is false for all values of x; that is, the solution set is the empty set.

#56 above features an inconsistent equation.

An equation is an identity if it is true for all values of x for which both sides of the equation are defined as real numbers.

$\frac{6x+4}{x} = \frac{12x+8}{2x}$ is an identity: the two sides are defined as real numbers for all $x \neq 0$, and when $x \neq 0$ we have:

$\frac{6x+4}{x} = \frac{\cancel{2}\cdot(6x+4)}{\cancel{2}\cdot x} \longrightarrow \frac{6x+4}{x} = \frac{6x+4}{x}$, which is satisfied for all $x \neq 0$. Solution set is $\{x \mid x \text{ is real } \& \ x \neq 0\}$.

(See p. 5 on textbook for a review of set-builder notation.)

$\boxed{84}$ Solve: $2(x+2)+2x = 4(x+1)$

- Simplify both sides: $2x+4+2x = 4x+4 \Rightarrow 4x+4 = 4x+4$.
- So now both sides of the equation look identical, which means the equation is an identity. Since both sides are defined for all real values of x, the solution set consists of all real numbers.
- Solution set in set-builder notation: $\boxed{\{x \mid x \text{ is real}\}}$

1.3 - Models and Applications

$\boxed{18}$ Including a 17.4% hotel tax, your room in Chicago cost $287.63 per night. Find the nightly cost before the tax was added.

The idea is to model the situation with an algebraic equation, using a variable like x, and then solving the equation to get the answer to the problem.

- Define a variable: Let x = nightly cost before the tax was added.
- Construct an equation involving x:

$$x + (17.4\% \text{ of } x) = \$287.63$$

$$x + 0.174x = 287.63$$

- Solve the equation:

$$1.174x = 287.63$$

$$x = \frac{287.63}{1.174} = 245$$

- Give the answer to the problem:

The nightly cost before tax is $\boxed{\$245.}$

$\boxed{24}$ Things did not quite go as planned. You invested $15,000, part of it in a stock that realized a 15% gain. However, the rest of the money suffered a 7% loss. If you had an overall gain of $1590, how much was invested at each rate?

- Let x = Amount invested in stock that gained 15%.
- Then 15,000 − x was invested at a 7% loss.
- Construct an equation...

- $$\underbrace{(15\% \text{ of } x)}_{\text{Amount gained from the stock investment}} + \underbrace{(-7\% \text{ of } 15{,}000-x)}_{\text{Amount "gained" from the other investment (note it is negative)}} = \$1590$$

$$0.15x - 0.07(15{,}000-x) = 1590$$

- Solve equation:

$$0.15x - 1050 + 0.07x = 1590$$
$$0.22x - 1050 = 1590$$
$$0.22x = 2640$$
$$x = \frac{2640}{0.22} = 12{,}000$$

- Answer: $12,000 was invested at 15% gain
  $3000 was invested at 7% loss.

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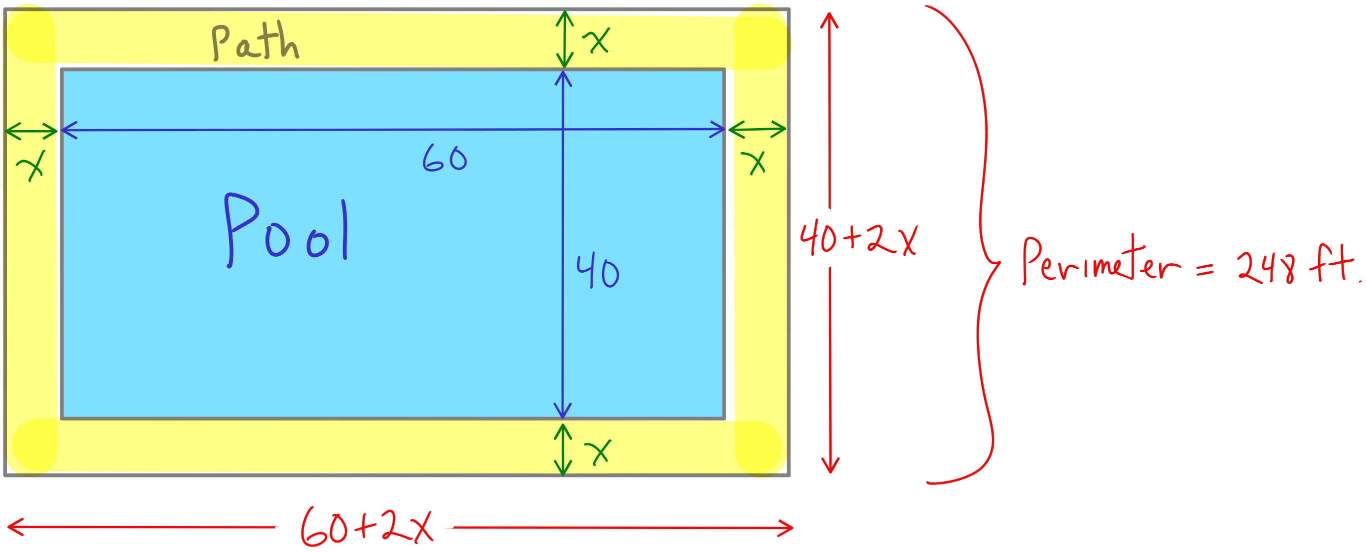
30 A rectangular swimming pool measures 40 feet by 60 feet, and is surrounded by a rectangular path of uniform width. The perimeter of the rectangle formed by the pool and the path is 248 feet. What is the width of the path?

Let x = Width of the path.



Perimeter = 2(length) + 2(width) = $2(60+2x)+2(40+2x)$

But also: Perimeter = 248

So: $2(60+2x)+2(40+2x)=248$

$120+4x+80+4x=248$

$8x+200=248$

$8x=48$

$x=6$

Answer: $\boxed{\text{Path is 6 ft. wide}}$

A formula (sometimes called a "literal equation") is an equation that has some particular application. It will consist of two or more variables, usually denoted by letters.

Ex Some formulas: $E=mc^2$, $A=\pi r^2$, $PV=nRT$, etc.

To solve a formula for an indicated variable means to isolate that variable on one side of the equation.

50 Solve the formula $S=\frac{C}{1-r}$ for r.

- So we need to isolate r. Start by getting r out from under the fraction bar.

$$(1-r)\cdot S=\frac{C}{\cancel{1-r}}\cdot(\cancel{1-r})$$

$$(1-r)S=C$$

$$S-rS=C$$

$$S-C=rS$$

$$\frac{r\cancel{S}}{\cancel{S}}=\frac{S-C}{S}$$

$$\boxed{r=\frac{S-C}{S}}$$

Another Way: flip both sides...

$$S=\frac{C}{1-r} \longrightarrow \frac{S}{1}=\frac{C}{1-r} \longrightarrow \frac{1}{S}=\frac{1-r}{C}$$

$$C\cdot\frac{1}{S}=\frac{1-r}{\cancel{C}}\cdot\cancel{C} \longrightarrow \frac{C}{S}=1-r \longrightarrow \boxed{r=1-\frac{C}{S}}$$

$$\left(\text{Note: } r=\frac{S-C}{S} \rightarrow r=\frac{S}{S}-\frac{C}{S} \rightarrow r=1-\frac{C}{S}\right)$$

Homework Question:

$\boxed{1.2.33}$ Solve $\frac{3x}{5}-x=\frac{x}{10}-\frac{5}{2}$

$$10\cdot\left(\frac{3x}{5}-x\right)=\left(\frac{x}{10}-\frac{5}{2}\right)\cdot10$$

$$2\,\cancel{10}\cdot\frac{3x}{\cancel{5}}\,☆-10\cdot x=\frac{x}{\cancel{10}}\cdot\cancel{10}-\frac{5}{\cancel{2}}\cdot\cancel{10}\,5$$

$$2\cdot3x-10x=x-5\cdot5$$

$$6x-10x=x-25$$

$$-4x=x-25$$

$$-5x=-25$$

$$x=\frac{-25}{-5}=5$$

Solution set is $\boxed{\{5\}}$ ■

☆ $10\cdot\frac{3x}{5}=\frac{10}{1}\cdot\frac{3x}{5}$

$$=\frac{\cancel{5}\cdot2}{1}\cdot\frac{3x}{\cancel{5}}=2\cdot3x$$

1.4 - Complex Numbers

Definition We define the imaginary unit to be the number i for which $i^2 = -1$

Another symbol for i is $\sqrt{-1}$.

An imaginary number is a number of the form bi, where b is a real number.

Examples of imaginary numbers: i $(i = 1i)$, $-2i$, 0 $(0 = 0i)$

(So 0 is both real and imaginary)

Also: $-i$ $(-i = -1i)$, $\frac{3}{4}i$, $\sqrt{2}\,i$ (usually written $i\sqrt{2}$)

A complex number is a number of the form a+bi, where a and b are real numbers. We call a+bi the standard form of a complex number.

Examples: 2+3i, -2+i, 5i = 0+5i, 7 = 7+0i, etc.

2+3i	-2+i	5i = 0+5i	7 = 7+0i
$a=2$ $b=3$	$a=-2$ $b=1$	$a=0$ $b=5$	$a=7$ $b=0$

In general a+bi is imaginary if a=0, and real if b=0. As a result, the complex number system contains all real and imaginary numbers. In this sense the complex number system is an extension of the real number system.

Fact: Two complex numbers a+bi and c+di are equal if and only if a=c and b=d. (Note that a, b, c, and d are all intended to be real numbers.)

Much of this section is devoted to developing the rules of complex number arithmetic (addition, subtraction, multiplication, division, and exponentiation).

Let a, b, c, and d be real numbers.

Addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$.

$\boxed{\text{Ex}}$ Add $(2+3i) + (-1-7i) = (2-1) + (3i-7i)$

<u>Strict</u> standard form would be $-1+(-7)i$, but we can relax and just write $-1-7i$

$= 1+(-4i) = \boxed{1-4i}$ ∎

Subtraction: $(a+bi) - (c+di) = (a-c) + (b-d)i$

$\boxed{\text{Ex}}$ Subtract $(2+3i) - (-1-7i) = (2-(-1)) + (3i-(-7i))$

$= (2+1) + (3i+7i) = \boxed{3+10i}$ (We're done when we obtain a standard form) ∎

Multiplication: $(a+bi)(c+di) = (ac-bd) + (bc+ad)i$

One could memorize this formula, but all that's happening is a FOIL procedure...

$$(a+bi)(c+di) = ac + adi + bci + \boxed{(bi)(di)} \rightarrow$$

This is bdi^2
$= bd(-1) = -bd$

$$= ac + (ad+bc)i + (-bd)$$

$$= (ac-bd) + (bc+ad)i$$

$\boxed{\text{Ex}}$ Multiply: $(2-3i)(4+5i)$.

$$(2-3i)(4+5i) \underset{\text{FOIL}}{=} (2)(4)+(2)(5i)+(-3i)(4)+(-3i)(5i)$$

$$= 8+10i+(-12i)+\underline{(-15i^2)} \longrightarrow -15(-1)=15, \text{ since } i^2=-1$$

$$= 8+10i-12i+15$$

$$= \boxed{23-2i}$$

Standard form!

The conjugate of a complex number a+bi is a-bi, and the conjugate of a-bi is a+bi.

The product of a complex number and its conjugate is always a real number:

$$(a+bi)(a-bi) \underset{\text{FOIL}}{=} a^2 - \cancel{ab}i + \cancel{ab}i - b^2i^2$$

$$= a^2 - b^2i^2 \qquad i^2=-1$$

$$= a^2 - b^2(-1)$$

$$= a^2+b^2, \text{ which is real since } a \,\&\, b \text{ are real.}$$

It might be worth remembering, but FOIL always delivers this:

$$(a+bi)(a-bi) = a^2+b^2$$

Now we turn to complex number division. This is largely an exercise in rationalizing a denominator.

Ex Divide: $\frac{2+3i}{1-2i}$

Start by multiplying the numerator and denominator by the conjugate of the denominator. Conjugate of 1-2i is 1+2i, so...

$$\frac{2+3i}{1-2i}\cdot\frac{1+2i}{1+2i}=\frac{(2+3i)(1+2i)}{(1-2i)(1+2i)}=\frac{2+4i+3i+6i^2}{1+\cancel{2}i-\cancel{2}i\underbrace{-4i^2}_{-4(-1)=4}}$$

$$=\frac{2+7i-6}{1+4}=\frac{-4+7i}{5}=\boxed{-\frac{4}{5}+\frac{7}{5}i}$$ ■

Want standard form
$a+bi$

Definition If b>0 is a real number, then we define

$$\sqrt{-b}=i\sqrt{b}$$

Notice that $\sqrt{-1}=i\sqrt{1}=i\cdot 1=i$, so yes, $i=\sqrt{-1}$.

Ex $\sqrt{-9}=i\sqrt{9}=i\cdot 3=\boxed{3i}$ ■

Recall the following: If $\sqrt{a}$ & $\sqrt{b}$ are defined as real numbers, then...

P1) $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$

P2) $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$

Ex Simplify: $\sqrt{-2}\cdot\sqrt{-8}=\boxed{-4}$

wrong

Use (P1) to get
$\sqrt{(-2)(-8)}=\sqrt{16}=4$

right

$i\sqrt{2}\cdot i\sqrt{8}=i^2\cdot\sqrt{2}\cdot\sqrt{8}$
$=(-1)\cdot\sqrt{2\cdot 8}=(-1)\sqrt{16}=(-1)(4)=-4$ ■

$\boxed{42}$ Write in standard form a+bi: $(-5-\sqrt{-9})^2$.

Use FOIL: $(-5-\sqrt{-9})^2 = (-5-i\sqrt{9})^2 = (-5-3i)^2$

$= (-5-3i)(-5-3i) = 25+15i+15i+9i^2$

$= 25+30i-9 = \boxed{16+30i}$

Definition: For any integer $n>0$, $i^{-n} = \frac{1}{i^n}$

Integer Powers of i:

$i^{-1} = -i$ $\left(i^{-1} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i\right)$

$i^0 = 1$ (by definition)

$i^1 = i$ (by definition)

$i^2 = -1$ (by definition)

$i^3 = -i$ $(i^3 = i^2 \cdot i^1 = (-1)i = -i)$ [Law of exponents: $a^m \cdot a^n = a^{m+n}$]

$i^4 = 1$ $(i^4 = i^2 \cdot i^2 = (-1)(-1) = 1)$

$i^5 = i$ $(i^5 = i^4 \cdot i^1 = (1)i = i)$

$i^6 = -1$ $(i^6 = i^4 \cdot i^2 = (1)(-1) = -1)$

$i^7 = -i$ $(i^7 = i^4 \cdot i^3 = (1)(-i) = -i)$

$\boxed{\text{Ex}}$ Write i^{319} in standard form a+bi.

As indicated in the powers of i listed above, the only possible answers are 1, i, -1, or -i.

Since $i^4 = 1$, it is convenient to break off i^4 factors from i^{319}.

We need to know how many 4's fit into 319, and what's left over. This calls for a simple long division...

$$\left.\begin{array}{r} 79 \\ 4\overline{)\,319} \\ \underline{28} \\ 39 \\ \underline{36} \\ 3 \end{array}\right\}\ \text{So } 319 = 4(79)+3$$

$a^{mn} = (a^m)^n$

$a^{m+n} = a^m \cdot a^n$

Now we have: $i^{319} = i^{4(79)+3} = i^{4(79)} \cdot i^3 = (i^4)^{79} \cdot i^3$

$= (1)^{79} \cdot i^3 = i^3 = \boxed{-i}$ (standard form a+bi with a=0 & b=-1) ■

1.5 - Quadratic Equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$ for constants a, b, c, where a ≠ 0.

We call $ax^2 + bx + c = 0$ the standard form of a quadratic equation.

We solve these equations one of four ways: factoring, the square root property, completing the square, and the quadratic formula.

To solve by factoring, the equation is put into standard form, then the polynomial is factored, and then the zero-product principle is used.

Zero-Product Principle (ZPP): If $AB = 0$, then $A = 0$ or $B = 0$.

Ex Solve $10x - 1 = (2x+1)^2$ by factoring.

1) 1st get the equation in standard form.

$$10x - 1 = 4x^2 + 4x + 1 \qquad \left[(2x+1)^2 = (2x+1)(2x+1) = 4x^2 + 2x + 2x + 1\right]$$

$$-4x^2 + 6x - 2 = 0$$

$$4x^2 - 6x + 2 = 0$$

Both are in standard form, so I tend to pick the form with a positive coefficient for x^2

2) Next, factor...

$$(2x-2)(2x-1) = 0$$

3) Use ZPP & solve for x.

Either $2x - 2 = 0$ or $2x - 1 = 0$

$2x = 2$ or $2x = 1$

$x = 1$ or $x = 1/2$

4) State the solution set: $\boxed{\{1/2, 1\}}$

To solve by the square root property requires having an equation of the form $(Ax+B)^2 = C$.

Square Root Property (SRP): If $u^2 = d$, then $u = \pm\sqrt{d}$.

Ex Solve using the SRP: $(x+2)^2 = -49$.

1) Apply the SRP with u=x+2 and d=-49.

$$(x+2)^2 = -49 \Rightarrow x+2 = \pm\sqrt{-49}$$

2) Solve the resultant equation.

$$x+2 = \pm\sqrt{-49} \Rightarrow x+2 = \pm i\sqrt{49} \Rightarrow x+2 = \pm 7i \Rightarrow$$

$$x = -2 \pm 7i$$

3) State the solution set.

$$\boxed{\{-2-7i, -2+7i\}} \quad \text{or} \quad \boxed{\{-2\pm 7i\}}$$

The most important way to solve a quadratic equation is by using the "complete the square" procedure.

Ex Solve by completing the square: $2x^2 - 4x - 1 = 0$

1) With the equation in standard form, first move the constant term to the other side of the equation.

$$2x^2 - 4x = 1$$

2) Divide both sides of the equation by the coefficient of x^2.

$$\frac{2x^2 - 4x = 1}{2} \Rightarrow x^2 - 2x = \frac{1}{2}$$

3) Add the square of half the coefficient of x to both sides.

$$x^2 - 2x + \underline{(-1)^2} = \frac{1}{2} + \underline{(-1)^2}$$

Half of -2
is -1.
Now square
this...

$$x^2 - 2x + 1 = \frac{3}{2}$$

4) Factor the perfect square trinomial that results from step 3, and solve the resultant equation using the SRP.

$$(x-1)^2 = \frac{3}{2} \Rightarrow x - 1 = \pm\sqrt{\frac{3}{2}} \Rightarrow x = 1 \pm \sqrt{\frac{3}{2}}$$

5) State the solution set.

$$\boxed{\left\{1 \pm \sqrt{\frac{3}{2}}\right\}} \quad \underline{\text{or}} \quad \boxed{\left\{1 + \frac{\sqrt{6}}{2}\right\}} \qquad \left(\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}\right)$$

A fuller description of the completing the square procedure for $ax^2 + bx + c = 0$:

1) Move c to RHS

$$ax^2 + bx = -c$$

2) Divide by a

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3) Add $\left(\frac{b}{2a}\right)^2$ to both sides

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

4) Factor the perfect-square trinomial

$$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a}+\frac{b^2}{4a^2}$$

5) Use SRP to solve for x

$$x+\frac{b}{2a} = \pm\sqrt{-\frac{c}{a}+\frac{b^2}{4a^2}}$$

$$x = -\frac{b}{2a}\pm\sqrt{-\frac{c}{a}+\frac{b^2}{4a^2}}$$

Optional but still important: Rewrite the RHS so that it is an expression consisting of a single fraction.

$$x = -\frac{b}{2a}\pm\sqrt{-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}} = -\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{\underbrace{\sqrt{4a^2}}} = -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2|a|}$$

$$\sqrt{4a^2} = \sqrt{4}\cdot\sqrt{a^2} = 2\cdot|a|$$

$$= \begin{cases} -\dfrac{b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2a}, & \text{if } a>0 \quad (\text{so } |a|=a) \\ -\dfrac{b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2(-a)}, & \text{if } a<0 \quad (\text{so } |a|=-a) \end{cases}$$

But $-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2(-a)} = -\frac{b}{2a}\mp\frac{\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$

So there is really on one outcome, regardless of whether a>0 or a<0:

$$x = -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}, \quad \text{or} \quad \boxed{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

This is the quadratic formula, which we will be looking at shortly.

$\boxed{\text{Ex}}$ Solve by completing the square: $x^2-4x+29=0.$

1) Move c to RHS (c = 29 here)

$$x^2-4x=-29$$

2) Divide by a (a = 1 here)

Since $a=1$, nothing changes: $x^2-4x=-29$

3) Add $\left(\frac{b}{2a}\right)^2$ to both sides $\left(\frac{b}{2a}\right)^2=\left(\frac{-4}{2\cdot1}\right)^2=(-2)^2=4$

$$x^2-4x+4=-29+4$$

4) Factor the perfect-square trinomial

$$(x-2)^2=-25$$

5) Use SRP to solve for x

$$x-2=\pm\sqrt{-25} \Rightarrow x-2=\pm5i \Rightarrow x=2\pm5i$$

6) State the solution set:

$\boxed{\{2\pm5i\}}$ ∎

$\boxed{1.5.51}$ Solve by completing the square: $x^2-6x-11=0.$

$a=1,\ b=-6,\ c=-11$

1) Move c to RHS

$$x^2-6x=11$$

2) Divide by a (a = 1, so we skip this)

↓

3) Add $\left(\frac{b}{2a}\right)^2$ to both sides $\left(\left(\frac{-6}{2\cdot 1}\right)^2 = (-3)^2 = 9\right)$

$$x^2-6x+9 = 11+9$$

4) Factor the perfect-square trinomial

$$(x-3)^2 = 20$$

5) Use SRP to solve for x

$$x-3 = \pm\sqrt{20} \Rightarrow x-3 = \pm 2\sqrt{5} \Rightarrow x = 3 \pm 2\sqrt{5}$$

6) State the solution set:

$$\boxed{\{3 \pm 2\sqrt{5}\}} \text{ or } \boxed{\{3 \pm \sqrt{20}\}}$$

$\boxed{1.5.55}$ Solve $x^2-5x+6=0$ $(a=1, b=-5, c=6)$

1) Move c to RHS

$$x^2-5x=-6$$

2) Divide by a

3) Add $\left(\frac{b}{2a}\right)^2$ to both sides $\left(\left(\frac{b}{2a}\right)^2 = \left(\frac{-5}{2\cdot 1}\right)^2 = \left(-\frac{5}{2}\right)^2 = \frac{25}{4}\right)$

$$x^2-5x+\frac{25}{4} = -6+\frac{25}{4}$$

4) Factor the perfect-square trinomial

$$\left(x-\frac{5}{2}\right)^2 = \frac{1}{4}$$

5) Use SRP to solve for x

$$x-\frac{5}{2}=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2} \Rightarrow x=\frac{5}{2}\pm\frac{1}{2}=2,3$$

6) State the solution set:

$$\boxed{\{2,3\}}$$

$\boxed{106}$ Solve $\frac{1}{x}+\frac{1}{x+3}=\frac{1}{4}$

This is a rational equation, so to solve it we multiply by the LCM of the denominators as in section 1.2. LCM = 4x(x+3).

$$4x(x+3)\left[\frac{1}{x}+\frac{1}{x+3}=\frac{1}{4}\right]$$

$$4\cancel{x}(x+3)\cdot\frac{1}{\cancel{x}}+4x\cancel{(x+3)}\cdot\frac{1}{\cancel{x+3}}=\cancel{4}x(x+3)\cdot\frac{1}{\cancel{4}}$$

$$4(x+3)+4x=x(x+3)$$

$$4x+12+4x=x^2+3x \longleftarrow$$

Quadratic equation results, unlike in section 1.2. Get in standard form.

$$x^2+3x=8x+12$$

$$x^2-5x-12=0$$

We'll solve this using the quadratic formula: $\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

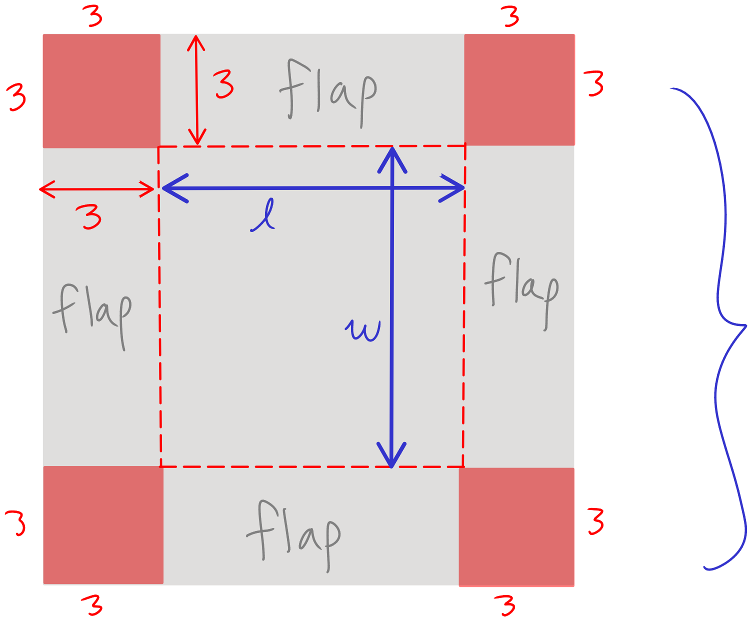
Here: $a=1$, $b=-5$, $c=-12$

Then $x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-12)}}{2(1)}$ (Note: $-5^2=-25$, $(-5)^2=+25$)

$$x=\frac{5\pm\sqrt{25+48}}{2}=\frac{5\pm\sqrt{73}}{2}$$

Neither of these values of x result in division by 0 in the original rational equation, so they are valid solutions. Solution set: $\boxed{\left\{\frac{5\pm\sqrt{73}}{2}\right\}}$

1.5.152 A machine produces open-topped boxes using square sheets of metal. The machine cuts equal-sized squares measuring 3 inches on a side from the corners and then shapes the metal into an open-topped box by turning up the sides. If each box must have a volume of 75 cubic inches, find the length and width of the box.



Need to find length ℓ & width w. Note $\ell = w$.

- Folding up the flaps creates an open-topped box with dimensions ℓ by w by 3.
- Volume $= 3\ell w$.
- But $\ell = w$ here, so Volume $= 3\ell^2$
- Box must have volume 75 in.^{3}, so $3\ell^2 = 75 \Rightarrow \ell^2 = 25 \Rightarrow \ell = 5$ in.
- So the length and width of the box will be $\boxed{\text{5 inches.}}$

1.6 - Other Types of Equations

A polynomial equation is an equation having a polynomial on each side (recall that 0 is considered a polynomial). Linear and quadratic equations are polynomial equations.

We solve polynomial equations by factoring and using the zero-product principle.

$\boxed{1.6.6}$ Solve the polynomial equation $x+1=9x^3+9x^2$.

1) Move everything to one side and get 0 on the other side.

$$0=9x^3+9x^2-x-1$$

$$9x^3+9x^2-x-1=0$$

2) Fully factor the polynomial. This requires factoring by grouping.

$$(9x^3+9x^2)+(-x-1)=0$$

$$9x^2(x+1)-(x+1)=0$$

$$(x+1)(9x^2-1)=0$$

$$(x+1)[(3x)^2-1^2]=0$$

$$A^2-B^2=(A+B)(A-B)$$

$$(x+1)(3x-1)(3x+1)=0$$

3) Use the zero-product principle...

$x+1=0$ or $3x-1=0$ or $3x+1=0$

4) Solve the linear equations and get the solution set.

$x=-1$ or $x=\frac{1}{3}$ or $x=-\frac{1}{3}$

Solution set: $\boxed{\{-1, 1/3, -1/3\}}$

A radical equation has at least one radical in it. Here we consider radical equations that have one, two, or even three square root radicals.

To solve a square root radical equation:

1) If there is one radical in the equation, isolate it on one side; and if there's two, get them on opposite sides.

2) Square both sides of the equation.

3) If any radical remains, isolate it and square again.

4) Solve the resultant polynomial equation.

5) Check for extraneous solutions, and state the solution set.

$\boxed{20}$ Solve $\underbrace{\sqrt{2x+15}}_{\text{A radical term, so we have a radical equation here.}} - 6 = x$

1) Isolate radical:

$$\sqrt{2x+15} = x+6$$

2) Square both sides.

$$\left(\sqrt{2x+15}\right)^2 = \underbrace{(x+6)^2}_{\text{FOIL } (x+6)(x+6)} \Rightarrow 2x+15 = x^2+12x+36$$

3) There are no radicals left, so we skip this step.

4) Solve the quadratic equation.

$$x^2+10x+21=0 \Rightarrow (x+7)(x+3)=0 \Rightarrow$$

$$x+7=0 \ \underline{\text{or}} \ x+3=0 \Rightarrow x=-7, -3$$

5) Check for extraneous solutions using the ORIGINAL EQUATION and state solution set.

Check $x=-7$: $\sqrt{2x+15}-6=x$

$$\sqrt{2(-7)+15}-6=-7$$
$$\sqrt{1}-6=-7$$
$$1-6=-7$$
$$-5=-7 \longrightarrow$$

False! So -7 is not a solution.

Check $x=-3$: $\sqrt{2x+15}-6=x$

$$\sqrt{2(-3)+15}-6=-3$$
$$\sqrt{9}-6=-3$$
$$3-6=-3$$
$$-3=-3 \longrightarrow$$

True! So -3 is a solution.

Solution set: $\boxed{\{-3\}}$

$\boxed{24}$ Solve $\sqrt{x+5}-\sqrt{x-3}=2$.

1) There are two radicals, so get them on opposite sides.

$$\sqrt{x+5}=2+\sqrt{x-3}$$

2) Square both sides.

$$\left(\sqrt{x+5}\right)^2=\left(2+\sqrt{x-3}\right)^2$$

$$x+5=\left(2+\sqrt{x-3}\right)\left(2+\sqrt{x-3}\right)$$

$$x+5=4+4\sqrt{x-3}+(x-3)$$

$$x+5=4\sqrt{x-3}+x+1$$

3) One radical remains, so we isolate it and square again.

$4 = 4\sqrt{x-3} \Rightarrow \sqrt{x-3} = 1 \Rightarrow x-3 = 1$

4) Solve the polynomial equation for x.

$x = 4$.

5) Check to see if the "solution" is valid, and state solution set.

Check $x = 4$ using original equation...

$\sqrt{4+5} - \sqrt{4-3} = 2 \Rightarrow \sqrt{9} - \sqrt{1} = 2 \Rightarrow 3 - 1 = 2$, which is true!

Solution set: $\boxed{\{4\}}$

Recall that $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

To solve an equation with a rational exponent, we use a procedure that is very similar to the steps in solving a radical equation. Only difference: if an equation with a rational exponent is equivalent to an nth-root radical equation, then instead of squaring in the procedure (steps 2 & 3), we raise to the nth power.

$\boxed{38}$ Solve $(x+5)^{2/3} = 4$.

This is the same as $\sqrt[3]{(x+5)^2} = 4$, so solve like a cube root radical equation — cubing instead of squaring.

1) Isolate the term with the rational exponent.

Already done

2) Cube both sides.

$$\left[(x+5)^{2/3}\right]^3 = (4)^3 \Rightarrow (x+5)^2 = 64$$

$$(a^m)^n = a^{mn}$$

3) No term with a rational exponent remains to isolate, so we skip this step.

4) Solve the polynomial equation.

$(x+5)^2 = 64 \overset{\text{SRP}}{\Longrightarrow} x+5 = \pm\sqrt{64} = \pm 8 \Rightarrow x = -5 \pm 8 \Rightarrow$
$x = -13, 3$

5) Check for extraneous solutions, state solution set.

<u>Check -13:</u> $(x+5)^{2/3} = 4 \Rightarrow (-13+5)^{2/3} = 4 \Rightarrow (-8)^{2/3} = 4 \Rightarrow$
$[(-8)^{1/3}]^2 = 4 \Rightarrow [-2]^2 = 4 \Rightarrow 4 = 4$ ✓

<u>Check 3:</u> $(x+5)^{2/3} = 4 \Rightarrow (3+5)^{2/3} = 4 \Rightarrow (+8)^{2/3} = 4 \Rightarrow$
$[(+8)^{1/3}]^2 = 4 \Rightarrow [2]^2 = 4 \Rightarrow 4 = 4$ ✓

Solution set: $\boxed{\{-13, 3\}}$ ■

Suppose an equation has variable x in it. Let u=f(x). If we substitute u for f(x) in the equation, and the resultant equation (with variable u) is quadratic, then we say the original equation (with variable x) has a quadratic form.

$\boxed{\text{Ex}}$ $x^4 - 13x^2 + 36 = 0$ has a quadratic form: letting $u=x^2$, then the equation becomes $u^2 - 13u + 36 = 0$, which is a quadratic equation. (Note: $x^4 = (x^2)^2 = u^2$).

- We solve $x^4 - 13x^2 + 36 = 0$ by 1st solving $u^2 - 13u + 36 = 0$ for u:
$u^2 - 13u + 36 = 0 \Rightarrow (u-4)(u-9) = 0 \Rightarrow u-4=0$ or $u-9=0$
$\Rightarrow u=4$ or $u=9$

- Now solve for x, where $x^2 = u$...
We have $x^2 = 4$ <u>or</u> $x^2 = 9$, so $x = \pm 2$ or $x = \pm 3$

- Solution set: $\boxed{\{-2, 2, -3, 3\}}$ ■

An absolute value equation is an equation that has at least one absolute value containing the variable.

$$\text{Recall: } |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$(\text{So } |-2| = -(-2) = 2)$

Note that $|z| = c$ for some $c \geq 0$ implies that $z = \pm c$.

If there is one absolute value term in an equation, isolate it first, then use the fact that $|z|=c$ implies $z=\pm c$.

$\boxed{\text{Ex}}$ Solve $|2x-7| = 13$.

We have $2x-7=13$ or $2x-7=-13$

$2x = 20$ or $2x = -6$

$x = 10$ or $x = -3$

Solution set: $\boxed{\{-3, 10\}}$ ■

If there are two absolute value terms in an equation, get them on opposite sides. Fact: $|a|=|b|$ means that a and b are the same distance from 0 on the real number line, and so either $a=b$ or $a=-b$.

$\boxed{\text{Ex}}$ Solve $|1-2x| - |3x+4| = 0$.

1) Get absolute values on opposite sides.

$$|1-2x| = |3x+4|$$

2) Use that fact that $|a|=|b|$ implies $a=b$ or $a=-b$.

$1-2x = 3x+4$ or $1-2x = -(3x+4)$

3) Solve the polynomial equations.

$1-2x = 3x+4$ or $1-2x = -(3x+4)$

$-5x = 3$ or $1-2x = -3x-4$

$x = -3/5$ or $x = -5$

4) State solution set.

$\boxed{\left\{-3/5, -5\right\}}$ ■

Note: $|x| = -4$ has no solution. Absolute values are never negative!

1.7 - Linear and Absolute Value Inequalities

Linear equation in standard form: $Ax+B=0$.

Linear inequality in standard form: $Ax+B \square 0$, where $\square$ can be replaced by $>$, $<$, $\geq$, $\leq$, $\neq$ (i.e. some kind of inequality symbol)

The solution set of a linear inequality usually consists of one or more intervals of real numbers. These can be nicely expressed using what's called interval notation.

The set of real numbers x for which x>a and x<b, usually written a<x<b, we express in set-builder notation as $\{x \mid a < x < b\}$. In interval notation we write (a,b). That is,

$$(a,b) = \{x \mid a < x < b\}$$

Brackets are used to denote inclusion:

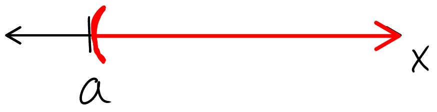
$$(a,b] = \{x \mid a < x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

$$[a,b] = \{x \mid a \leq x \leq b\}$$

Infinity and negative infinity denote an unbounded interval:

$$(a,\infty) = \{x \mid x > a\}$$



$$(-\infty,b) = \{x \mid x < b\}$$

$$[a,\infty) = \{x \mid x \geq a\}$$

$$(-\infty,b] = \{x \mid x \leq b\}$$

$$(-\infty,\infty) = \{x \mid x \text{ is real}\} = \text{Set of all real numbers}$$

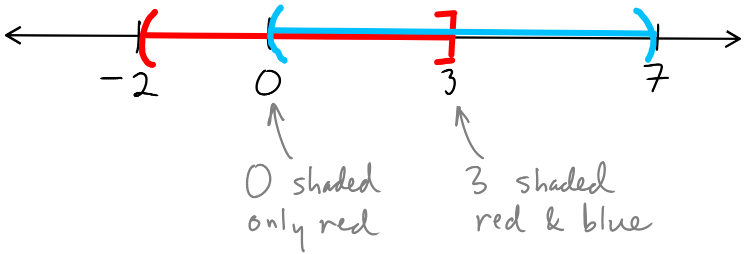
The union of two intervals (a,b) and (c,d), written $(a,b) \cup (c,d)$, is the set of real numbers that lie in one interval or the other...

$$(a,b)\cup(c,d) = \{x \mid \underset{\substack{\uparrow \\ x \text{ is in } (a,b)}}{x\in(a,b)} \text{ or } \underset{\substack{\uparrow \\ x \text{ is in } (c,d)}}{x\in(c,d)}\}$$

The intersection of two intervals (a,b) and (c,d), written $(a,b)\cap(c,d)$, is the set of real numbers that lie in both intervals...

$$(a,b)\cap(c,d) = \{x \mid x\in(a,b) \text{ and } x\in(c,d)\}$$

$\boxed{\text{Ex}}$ Find $(-2,3]\cup(0,7)$ & $(-2,3]\cap(0,7)$



The union of the two intervals consists of all numbers that are shaded either red or blue (or both), so that

$$(-2,3]\cup(0,7) = \boxed{(-2,7)}.$$

The intersection of the two intervals consists of all numbers that are shaded both red and blue, so that

$$(-2,3]\cap(0,7) = \boxed{(0,3]}.$$ ■

Note that the union $(-2,3]\cup(4,19]$ cannot be expressed as a single interval: they are "disjoint" intervals, meaning they have no real number in common.

$\boxed{40}$ Solve $5(3-x)\le 3x-1$, and state the solution set in interval notation.

Solve like the linear equation 5(3-x) = 3x-1, only recall that if the two sides of the inequality are multiplied or divided by a negative number, then the inequality sign must reverse direction:

$1<2 \longrightarrow -1>-2$

$$5(3-x) \le 3x-1 \Rightarrow \underset{-15}{15}-5x \le 3x\underset{-15}{-1} \Rightarrow \underset{-3x}{-5x} \le \underset{-3x}{3x}-16 \Rightarrow$$

$$-8x \le -16 \Rightarrow \frac{-8x}{-8} \ge \frac{-16}{-8} \Rightarrow x \ge 2$$

So the solution set is {x | x ≥ 2}, which in interval notation is:

$$\boxed{[2,\infty)}$$

$\boxed{50}$ Solve $6(x-1)-(4-x) \ge 7x-8$.

$$6x-6-4+x \ge 7x-8$$

$$7x-10 \ge 7x-8$$

$$\underline{-7x \qquad\qquad -7x}$$

$$-10 \ge -8 \longrightarrow \text{False!}$$

The inequality has no solution. Solution set is $\boxed{\emptyset}$

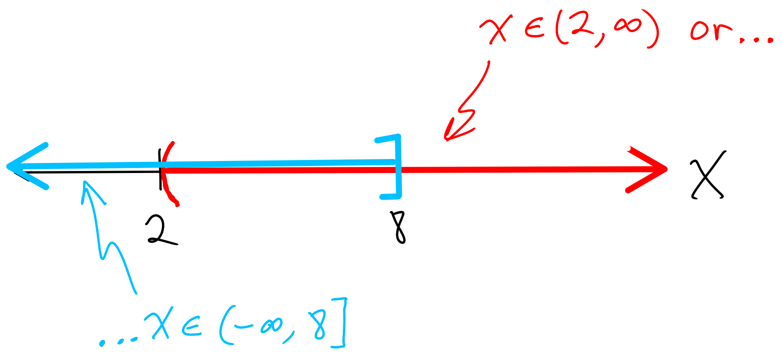
Yes, inequalities can have no solution. This has no solution: x+1<x.

A compound inequality is a statement consisting of two or more inequalities joined one to another by "or" or "and."

$\boxed{\text{Ex}}$ A compound inequality: $2x-1>3$ $\underline{\text{or}}$ $3-x \ge -5$

To find the solution to the compound inequality, we first find the solution to each of the inequalities that comprise it.

- $2x-1>3 \Rightarrow 2x>4 \Rightarrow x>2$, so $x \in (2,\infty)$
- $3-x \ge -5 \Rightarrow -x \ge -8 \Rightarrow (-1)(-x) \le -8(-1) \Rightarrow x \le 8$, so $x \in (-\infty,8]$.
- So: $x \in (2,\infty)$ or $x \in (-\infty,8]$



By the meaning of "or", x satisfies the compound inequality if it satisfies either one of the inequalities in the compound inequality. The first inequality is satisfied by any x in interval $(2,\infty)$, and and the second inequality is satisfied by any x in the interval $(-\infty, 8]$.

So the solution set consists of both intervals. The solution set is:

$$(-\infty,8] \cup (2,\infty) = \boxed{(-\infty,\infty)}$$ ■

a < x < b is another compound inequality. It translates as follows:

$$x > a \ \underline{\text{and}} \ x < b$$

$\boxed{56}$ Solve $\underset{\substack{\uparrow \\ \text{left}}}{3} \le \underset{\substack{\uparrow \\ \text{middle}}}{4x-3} < \underset{\substack{\uparrow \\ \text{right}}}{19}$.

For this kind of compound inequality the strategy is to isolate x in the middle part.

$$\underset{+3}{3} \le \underset{+3}{4x-3} < \underset{+3}{19} \Rightarrow 6 \le 4x < 22 \Rightarrow \frac{6}{4} \le \frac{4x}{4} < \frac{22}{4} \Rightarrow$$

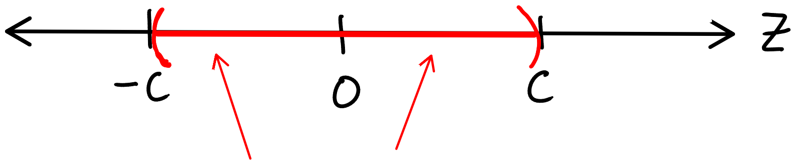
$$\frac{3}{2} \le x < \frac{11}{2}.$$

Solution set is $\left\{x \,\middle|\, \frac{3}{2} \le x < \frac{11}{2}\right\} = \boxed{\left[\frac{3}{2}, \frac{11}{2}\right)}$ ■

An absolute value inequality is an inequality in which the variable is between absolute value bars somewhere.

For any real number z, |z| denotes the distance between z and 0 on the real number line.

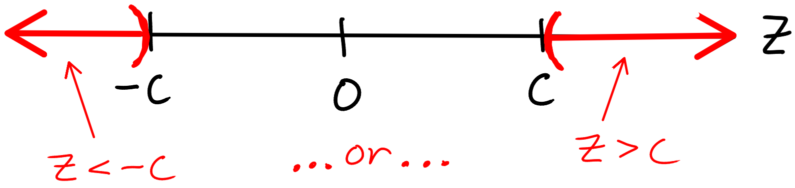
So for any c > 0, to write |z|<c means to say "The distance between z and 0 on the number line is less than c units."



$|z|<c$ means z can be anywhere here

That is, we have $-c<z<c$

To write |z|>c means to say "The distance between z and 0 on the number line is greater than c units."



$|z|>c$ means z could be less than $-c$ or greater than c

That is, $z<-c$ or $z>c$

Theorem For any $c>0$,

① $|z|\leq c$ means $-c\leq z\leq c$.

② $|z|\geq c$ means $z\leq -c$ or $z\geq c$

$\boxed{66}$ Solve $|3(x-1)+2|\leq 20$.

- First simplify what's inside the absolute value bars.

  $|3x-1|\leq 20$

- Next, use (1) in the theorem above, with z = 3x-1 and c = 20:

$$\underset{+1}{-20}\leq \underset{+1}{3x-1}\leq \underset{+1}{20} \Rightarrow \frac{-19}{3}\leq\frac{3x}{3}\leq\frac{21}{3} \Rightarrow -\frac{19}{3}\leq x\leq 7.$$

- Solution set: $\boxed{\left[-\frac{19}{3},7\right]}$ ■

$\boxed{80}$ Solve $5|2x+1|-3 \ge 9$

- There's nothing to simplify between the absolute value bars, but the absolute value term is not isolated. So first we isolate it...

$$5|2x+1| \ge 12 \Rightarrow |2x+1| \ge \frac{12}{5}.$$

- Next, use (2) in the theorem above with z = 2x+1 and c = 12/5.

$$2x+1 \le -\frac{12}{5} \quad \underline{\text{or}} \quad 2x+1 \ge \frac{12}{5}$$

We solve each inequality...

$$2x \le -\frac{17}{5} \quad \underline{\text{or}} \quad 2x \ge \frac{7}{5}$$

Multiply both sides by 1/2

$$x \le -\frac{17}{10} \quad \underline{\text{or}} \quad x \ge \frac{7}{10}$$

$$x \in \left(-\infty, -\frac{17}{10}\right] \quad \text{or} \quad x \in \left[\frac{7}{10}, \infty\right)$$

- Solution set: $\boxed{\left(-\infty, -\frac{17}{10}\right] \cup \left[\frac{7}{10}, \infty\right)}$

$\boxed{\text{Ex}}$ Solve $|5-2x|+6 < 2.$

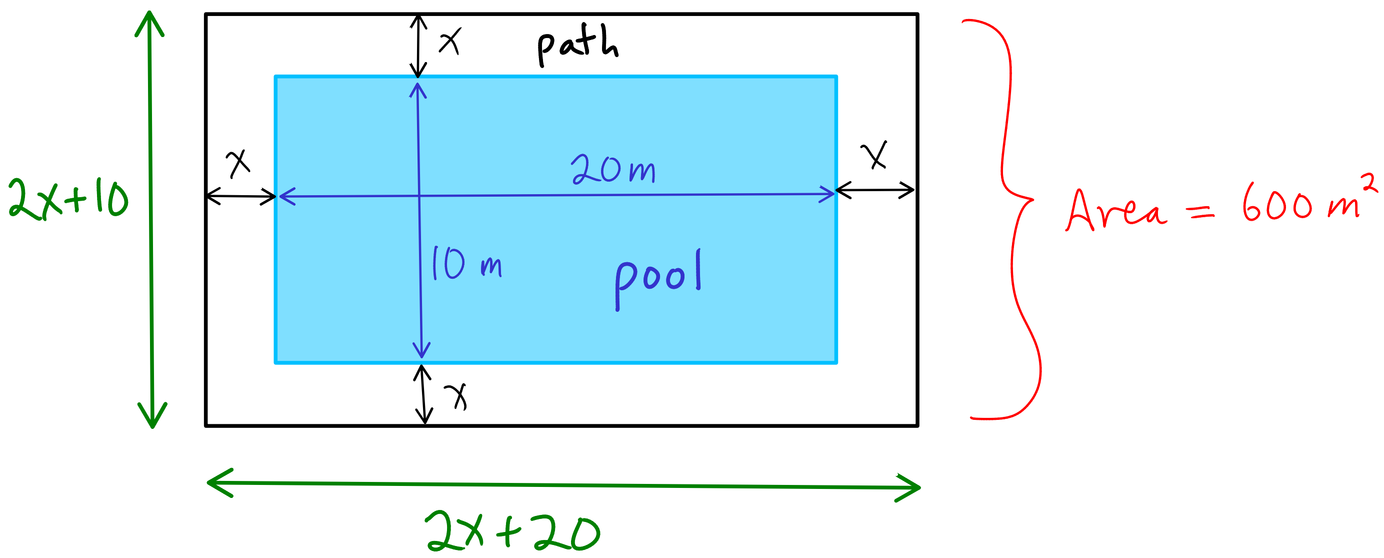
$-6 \quad -6$

We get $|5-2x| < -4.$

We cannot use (1) in the theorem because here c=-4, so c<0, and the theorem requires c>0.

BUT we should remember that an absolute value can never be less than 0, and so having |5-2x| be less than -4 is impossible, no matter what x is. Solution set is the empty set: $\boxed{\emptyset}$

$\boxed{1.5.149}$ A pool measuring 10 meters by 20 meters is surrounded by a path of uniform width. If the area of the pool and the path combined is 600 square meters, what is the width of the path?



We have (length)(width) = 600 $\Rightarrow$ $(2x+20)(2x+10) = 600$.

$$4x^2 + 20x + 40x + 200 = 600$$

$$4x^2 + 60x - 400 = 0$$

$$x^2 + 15x - 100 = 0$$

$$(x+20)(x-5) = 0$$

$x + 20 = 0$ or $x - 5 = 0$

$x = -20$ or $x = 5$

$\downarrow$ Can't have path with negative width.

$\downarrow$ Physically possible $\downarrow$

Width of the path is $\boxed{5 \text{ meters}}$ ▢

- Should know: Area of rectangle formula, area of circle formula, Pythagorean theorem, quadratic formula, how to find the perimeter of a polygon (such as a rectangle or triangle), volume of a box, circumference of a circle formula, area of a triangle. See the table on page 133 of the textbook.

- Don't need to know the volume of a sphere or cone or cylinder formulas, or the area of a trapezoid.

- Know the interval notation: you'll be asked to give solution sets for inequalities in interval notation.

1.3.9 You are choosing between two gyms. One gym offers membership for a fee of \$40 plus a monthly fee of \$25. The other gym offers membership for a fee of \$15 plus a monthly fee of \$30. After how many months will the total cost at each gym be the same? What will be the total cost for each gym at that point?

Let x = # of months until the cost of each gym is the same.

Cost of 1st gym after x months: $40+25x$ dollars

Cost of 2nd gym after x months: $15+30x$ dollars.

We want x such that: $40+25x = 15+30x$.

$$25 = 5x$$

$$\boxed{x = 5 \text{ months.}}$$

Total cost after 5 months is: $40+25(5) = \boxed{\$165}$ ■

1.6.29   Solve $\sqrt{3\sqrt{x+1}} = \sqrt{3x-5}$

1) There are two radical terms, and they're already on opposite sides.

2) Square both sides...

$$\left(\sqrt{3\sqrt{x+1}}\right)^2 = \left(\sqrt{3x-5}\right)^2 \Rightarrow 3\sqrt{x+1} = 3x-5$$

3) There is a radical remaining, so isolate it and square again.

$$\sqrt{x+1} = \frac{3x-5}{3} \Rightarrow \left(\sqrt{x+1}\right)^2 = \left(\frac{3x-5}{3}\right)^2 \Rightarrow$$

$$x+1 = \left(x - \frac{5}{3}\right)^2 \Rightarrow x+1 = x^2 - \frac{10}{3}x + \frac{25}{9}$$

4) Solve the quadratic equation.

$$9\left(x+1 = x^2 - \frac{10}{3}x + \frac{25}{9}\right) \Rightarrow 9x+9 = 9x^2 - 30x + 25 \Rightarrow$$

$$9x^2 - 39x + 16 = 0.$$

Let's solve this using the quadratic formula...

$$x = \frac{-(-39) \pm \sqrt{(-39)^2 - 4(9)(16)}}{2(9)} = \frac{39 \pm \sqrt{945}}{18} = \frac{39 \pm 3\sqrt{105}}{18}$$

$$x = \frac{13 \pm \sqrt{105}}{6}$$

5) Check for extraneous solutions.

- Check $\frac{13-\sqrt{105}}{6}$: $\sqrt{3\sqrt{\frac{13-\sqrt{105}}{6}+1}} = \sqrt{3\left(\frac{13-\sqrt{105}}{6}\right)-5}$

$\sqrt{a} = \sqrt{b}$ if & only if $a=b$, so we can write ...

$$3\sqrt{\frac{13-\sqrt{105}}{6}+1} = 3\left(\frac{13-\sqrt{105}}{6}\right)-5$$

But $3\left(\frac{13-\sqrt{105}}{6}\right)-5$ is negative, and a square root can never be negative. So: $\frac{13-\sqrt{105}}{6}$ is extraneous.

- Check $\frac{13+\sqrt{105}}{6}$: See if the equation

$$3\sqrt{\frac{13+\sqrt{105}}{6}+1} = 3\left(\frac{13+\sqrt{105}}{6}\right)-5$$ is true.

Using a calculator, we find that it is true, and so the solution is valid.

- Solution set is $\boxed{\left\{\frac{13+\sqrt{105}}{6}\right\}}$