R.1/R.2: Sets, Numbers, Arithmetic & Absolute Value

A **set** is a collection of objects, the objects usually called **elements.** A set containing only the elements *a, b, c* can be written symbolically as $\{a,b,c\}$, which is called **roster form** (a roster being a list). Order does not matter, so for example $\{c,b,a\}$ represents the same set and we can write $\{a,b,c\} = \{c,b,a\}$. Repetition does not matter either, so for example $\{a,a,b,b,b,c,c,c,c\}$ is considered to be the same set as $\{a,b,c\}$ (both sets contain the letters *a, b, c* and nothing else). The set that contains no elements is called the **empty set** or **null set**, and symbolically is written as $\varnothing$ or $\{\ \}$. To write $\{\varnothing\}$ is *incorrect* since it represents the set that contains the empty set and therefore *is not empty.* (Consider that a bucket that contains an empty bucket is not itself generally considered to be empty.)

A set can safely contain anything except itself – which leads to a maddening scenario known as Russell's Paradox. For the purposes of this course we need only consider sets that contain numbers. First there is the set of **natural numbers** $\{1,2,3,\ldots\}$, also known as the **counting numbers** because they are the numbers that are most naturally used to count things. Notice how the set is being represented: the first few numbers are listed until a clear pattern is established, and then there are three dots to indicate that the pattern continues indefinitely. This is how roster form can be used to define sets that have an infinite number of elements. An even "nicer" way to denote the set of natural numbers is to give it the special symbol $\mathbb{N}$; that is, $\mathbb{N} = \{1,2,3,\ldots\}$.

Next there is the set of whole numbers $\{0,1,2,3,\ldots\}$, which can be seen to be $\mathbb{N}$ with zero thrown in.

More significantly there is the set of integers $\mathbb{Z} = \{\ldots-3,-2,-1,0,1,2,3,\ldots\}$, which is the set of all whole numbers and their opposites. The $\mathbb{Z}$ comes from *Zahl*, a German word for *number*. Only a few centuries ago many people, some mathematicians included, did not consider the negative integers to be actual numbers, since one could never have –3 stones in one's hand or count −12 ducks on a pond. Such objections have long vanished among math nerds everywhere, and it's a good bet that you yourself (nerd or not) have no qualms with the idea of –1 being a number. In this economy it makes perfect sense to talk of someone being worth, say, –2000 dollars. Thanks to the power of negative thinking the equation $x + 1 = 0$ has a solution.

The set of **rational numbers**, or **rationals**, consists of all quantities that can be expressed as a ratio of integers – that is, as a fraction with an integer in the numerator and a (nonzero) integer in the denominator. Some examples…

- 0 is rational, since $0 = 0/1$ (the ratio of the integer 0 to the integer 1)

- 1 is rational, since $1 = 1/1$ (the ratio of 1 to 1)
- –3 is rational, since $-3 = -3/1$ (the ratio of –3 to 1)
- $-\frac{2}{5}$ is rational, since $\frac{-2}{5}$ (the ratio of –2 to 5)
- 0.418 is rational: $0.418 = \frac{0.418}{1} = \frac{0.418}{1} \cdot \frac{1000}{1000} = \frac{418}{1000}$

In evidence above is that any integer or terminating decimal value is rational. What about decimals that repeat such as 2.1626262…? It turns out they're rational also, but there's a trick to getting them expressed as fractions of integers. Let N=2.1626262… Then 10N=21.626262… and 1000N=2162.6262… Check it out:

$$\begin{array}{r} 1000N \\ -\quad 10N \\ \hline 990N \end{array} \quad = \quad \begin{array}{r} 2162.6262... \\ -\quad 21.6262... \\ \hline 2141.0000... \end{array}$$

Thus we have $990N = 2141$, or $N = 2141/990$, which is a ratio of integers!

Keep in mind that something like 5/0 is not a rational number because division by 0 is undefined. That is, 5/0 is not a rational number because it's not a number to begin with. How many zero-pound objects *can* you stuff into a 5-pound bag, anyway? You can say an infinite number, but infinity ain't a number any more than ain't is a word.

Notice something about the rational numbers: they can't be listed in a neat row like the integers. After 0, what's the next rational number in line? Not 1, 'cause ½ comes before 1. Not ½, 'cause 1/3 comes before ½. And ¼ comes before 1/3, 1/5 before ¼, 1/6 before 1/5,… Long story less long: there is *no* "smallest" rational number that's greater than 0. We've bumped against an issue that's actually of great concern in calculus, so suffice it to say that while the rationals can indeed be listed (Cantor did it in the late 1800s), the list cannot be in ascending order. Nonetheless there is a "neat" way to precisely define the set of rational numbers using what's known as **set-builder notation**.

Here we go. Sets are often represented by capital letters like A or X, while the elements of sets are represented by lower-case letters like a or x. If a is an element of A, we write $a \in A$ (the symbol $\in$ is read as "is an element of"). If a is not an element of A, write $a \notin A$. Consider the set $S = \{1, 2, 3, ..., 99, 100\}$. So S is the set of integers from 1 to 100, inclusive. In set-builder notation we write $S = \{x \mid x \in \mathbb{Z} \ \ \& \ \ 1 \leq x \leq 100\}$, where the vertical bar | is read as "such that". The whole ensemble reads as follows: "S equals the set of all x such that x is an element of the set of integers, and x is greater than or equal to 1 and less than or equal to 100". A lot of words boil down to just a few symbols.

The set of positive even integers is $\{2,4,6,8,\ldots\}$, which in set-builder notation can be written as $\{x \mid x > 0 \ \ \& \ \ x \in \mathbb{Z} \ \ \& \ \ x \text{ is even}\}$ or $\{x \mid x \in \mathbb{N} \ \ \& \ \ x \text{ is even}\}$ or just $\{2x \mid x \in \mathbb{N}\}$ -- the last one being read as "the set of all $2x$ such that x is a natural number," which makes sense because $x \in \mathbb{N}$ means x can be 1, 2, 3, etc., and so $2x$ can be 2, 4, 6, etc.

Now we're ready to define the set of rational numbers neatly. First, the set is customarily given the special symbol $\mathbb{Q}$ for "quotient" – since every ratio is one. So we have $\mathbb{Q} = \left\{\frac{p}{q} \mid \ p,q \in \mathbb{Z} \ \ \& \ \ q \neq 0\right\}$; that is, $\mathbb{Q}$ is the set of all quantities of the form $\frac{p}{q}$ such that p and q are integers, and q doesn't equal 0.

Many numbers are not rational, in which case they're called **irrational**. Expressed in decimal form, irrational numbers neither terminate nor repeat. Examples of irrational numbers are π, $\sqrt{2}$, $\sqrt[3]{5}$ and so on. Strangely, while there are an infinite number of rational numbers and an infinite number of irrational numbers, it can be proven that there are "more" irrationals than rationals! Sound intriguing? It all comes down to the difference between being "countably" infinite and "uncountably" infinite.

We come at last to the set of real numbers, often denoted by $\mathbb{R}$. Different texts define $\mathbb{R}$ in different ways, and for a "lower level" text the trick is always to come up with a definition that is "user-friendly" without being too sloppy. Your book's definition is too sloppy. After giving it some thought, I've come up with the following: $\mathbb{R} = \{d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots \mid n \in \mathbb{N} \ \ \& \ \ d_i \in \{0,1,\ldots,9\} \text{ for each } i\}$. That is, $\mathbb{R}$ is the set of all objects of the form $d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots$ such that n is a natural number and the digits d_1, d_2, d_3, etc. are each an integer from 0 to 9. So $70 \in \mathbb{R}$ since $70 = d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots$ if we simply let $n = 2$, $d_1 = 7$, $d_2 = 0$, $d_3 = 0$, $d_4 = 0$ and so on. And the famous number $\pi = 3.14159\ldots$ is also real since $\pi = d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots$ if we let n=1, $d_1 = 3$, $d_2 = 1$, $d_3 = 4$, $\mathrm{d}_4 = 1$ and so on. Notice that n is just the number of digits that come before the decimal point.

Basically, in our "base-10" numeration system, any number that can be expressed as some sequence of digits from 0 to 9, with the placement of a decimal point somewhere, is a real number. This is not a "rigorous" definition of the real numbers, but it'll carry you through calculus. A "rigorous" definition constructs the real numbers from scratch, as complete abstractions, by establishing around ten axioms. A few of these axioms we'll encounter soon, and you'll find them to be familiar. Later in the course (but not much later) we'll be looking at numbers that are *not* elements of $\mathbb{R}$ – that is, so-called *imaginary* and *complex* numbers.

We turn next to the universally agreed-upon **order of operations** that is observed when evaluating convoluted arithmetic expressions. You know the drill:

1) Grouping Symbols: Anything inside parentheses ( ), brackets [ ], or braces { } must be dealt with first.

2) Exponentiation: All exponents are evaluated once grouping symbols are eliminated. Recall that $a^n = a \cdot a \cdot a \cdot \cdots \cdot a$ (n a's in all). On calculators a^n is keyed in as a^n.

3) Multiplication/Division: Proceed from left to right. A common error is to assume that multiplication always comes before division!

4) Addition/Subtraction: Proceed from left to right.

Example 1 Evaluate $12 \div 4 \cdot 3$

Solution: Right way: divide *first*, since it lies to the *left* of multiplication here…

$12 \div 4 \cdot 3 = 3 \cdot 3 = 9$

Hugely *wrong* way: multiplying first, *then* dividing…

$12 \div 4 \cdot 3 = 12 \div 12 = 1$

Example 2 Evaluate $4^3 + 18 \div 3 - 2^4 - 3 \cdot (8-6)$

Solution:

$= 4^3 + 18 \div 3 - 2^4 - 3 \cdot 2$ (work inside parentheses first)
$= 64 + 18 \div 3 - 16 - 3 \cdot 2$ (evaluate exponents)
$= 64 + 6 - 16 - 3 \cdot 2$ (divide)
$= 64 + 6 - 16 - 6$ (multiply)
$= 70 - 16 - 6$ (add)
$= 54 - 6$ (subtract)
$= 48$

Example 3 Evaluate $\dfrac{15 \div 5 \cdot 4 \div 6 - 8}{-6-(-5)-8 \div 2}$

Solution: The fraction bar is a grouping symbol: it creates a top group and a bottom group. Groups are always evaluated first, so evaluate top & bottom separately.

Top: $15 \div 5 \cdot 4 \div 6 - 8 = 3 \cdot 4 \div 6 - 8 = 12 \div 6 - 8 = 2 - 8 = -6$

Bottom: $-6-(-5)-8 \div 2 = -6-(-5)-4 = -1-4 = -5$

Thus $\dfrac{15 \div 5 \cdot 4 \div 6 - 8}{-6-(-5)-8 \div 2} = \dfrac{-6}{-5} = \dfrac{6}{5}$.

Be mindful of negative signs! A negative sign is nothing more than shorthand for multiplication by –1, really. So when evaluating, say, the expression -5^2, you'll be better off in the long run seeing it as $-1 \cdot 5^2$. Now following the usual order of operations yields the correct outcome without fuss: $-1 \cdot 5^2 = -1 \cdot 25 = -25$. Note: $-5^2 \neq 25$! On the other hand, $(-5)^2 = (-5)(-5) = 25$. Be keenly aware of the difference between -5^2 and $(-5)^2$! It's the kind of thing I like to put on exams.

Now, on page 12 of the textbook some of those axioms concerning real numbers that I was on about are listed. The book calls the axioms "properties," but whatever you call them, they're unproved assumptions that we take as given throughout the course. Of particular importance is the Distributive Property: $a(b+c) = ab + ac$. For example, $5(2x+3) = 5 \cdot 2x + 5 \cdot 3 = 10x + 15$ and $x(2x-3) = x(2x+(-3)) = x \cdot 2x + x \cdot (-3) = 2x^2 + (-3x) = 2x^2 - 3x$, where we observe that a minus sign (subtraction) is merely shorthand for adding a negative: $a - b = a + (-b)$. (Pause and take note of the difference between a negative sign and a minus sign.)

Finally we come to the notion of **absolute value**. On page 14 of the textbook there is Figure 11, which depicts the real number line. The idea is that every real number corresponds to a point on the line, and the absolute value of a real number x, written $|x|$, is nothing more than the distance that x is from 0. So $|3| = 3$ since 3 is "3 steps" away from 0 on the number line; but also $|-3| = 3$ since –3 is also "3 steps" away from 0. Formally this is the definition of absolute value: $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Example 4 Evaluate $|-13+6|$, $\left|\sqrt{10}-3\right|$, $\left|\sqrt{2}-3\right|$

<u>Solution</u>:

$|-13+6|$. Absolute value bars are grouping symbols, so always evaluate what's *inside* the bars *first,* then go from there. So: $|-13+6| = |-7| = 7$.

$\left|\sqrt{10}-3\right|$. The absolute value of a *nonnegative* number is the number itself. Since $\sqrt{10} \approx 3.162$ is greater than 3, we find that $\sqrt{10}-3$ is *nonnegative.* Thus $\left|\sqrt{10}-3\right| = \sqrt{10}-3$ and we leave it at that. Unless instructed otherwise all answers must be exact – meaning no rounding occurs, so $\sqrt{10}$ must be left as $\sqrt{10}$, and π must remain π.

$\left|\sqrt{2}-3\right|$. This is a different story since $\sqrt{2} \approx 1.414$ and thus $\sqrt{2}-3$ is a *negative* quantity. The absolute value of a negative number is its opposite: $\left|\sqrt{2}-3\right| = -\left(\sqrt{2}-3\right) = 3-\sqrt{2}$.

Notice in Example 4 that $-\left(\sqrt{2}-3\right)=3-\sqrt{2}$, which comes about by applying the Distributive Property and the Commutative Property of Addition (along with the understanding that a negative sign represents multiplication by -1 and a minus sign represents addition of a negative quantity): $-\left(\sqrt{2}-3\right)=-1\cdot(\sqrt{2}+(-3))=(-1)\cdot\sqrt{2}+(-1)\cdot(-3)=-\sqrt{2}+3=3+(-\sqrt{2})$ $=3-\sqrt{2}$.

A set is a collection of objects, the objects usually called elements. A set consisting of the elements r, s, t can be represented symbolically by $\{r, s, t\}$, called roster form since the elements are listed explicitly between braces. The set that contains no elements is called the empty set and is represented by either $\{\ \}$ or $\emptyset$ (here $\emptyset$ is not the number zero, but is in fact a letter from the Norwegian alphabet). It is incorrect to write the empty set as $\{\emptyset\}$, for $\{\emptyset\}$ is not empty at all: it's the set that contains the empty set!

A set can contain anything, but for our purposes we need only consider sets which contain numbers. First, there is the set of natural numbers, which consists of all the numbers used in counting: $\{1, 2, 3, \ldots\}$. Note this set consists of an infinite number of elements, so we cannot list them all explicitly; nonetheless we can see from the pattern that the next element must be 4, and the one after that 5, and so on. For convenience the set of natural numbers is denoted by the symbol $\mathbb{N}$; that is, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The set of whole numbers is $\{0, 1, 2, 3, \ldots\}$; that is, the whole numbers are the natural numbers and 0.

The set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, often denoted by $\mathbb{Z}$ (for "Zahl", a German word for "number"); so $\mathbb{Z}$ consists of the whole numbers and their negatives. Only a few centuries ago many mathematicians did not consider negative numbers to be numbers at all, since, for example, you could never hold -3 stones in your hand or count -12 ducks on a pond. Such objections have long since vanished.

The set of rational numbers (or rationals) consists of all quantities that can be expressed as a ratio of integers — that is, a fraction with an integer in the numerator & a nonzero integer in the denominator.

- 0 is a rational number, since $0 = \frac{0}{1}$ (the ratio of 0 to 1)
- 1 is a rational since $1 = \frac{1}{1}$ (the ratio of 1 to 1)
- -3 is a rational since $-3 = \frac{-3}{1}$ (the ratio of -3 to 1)
- $-\frac{2}{5}$ is a rational since $-\frac{2}{5} = \frac{-2}{5}$ (the ratio of -2 to 5)

- Is 0.4186 a rational? Yes: $0.4186 = \frac{0.4186}{1} = \frac{0.4186}{1} \cdot \frac{10{,}000}{10{,}000} = \frac{4186}{10{,}000}$, the ratio of the integer 4186 to the integer 10,000.

- Is 2.1626262... a rational? Yes! Let $N = 2.1\overline{62}$ (the bar over 62 means the digits repeat indefinitely). Then $10N = 21.\overline{62}$ & $1000N = 2162.\overline{62}$, and we carry out the following clever subtraction...

$$\begin{array}{r} 1000N \\ -\ \ 10N \\ \hline 990N \end{array} = \begin{array}{r} 2162.626262\ldots \\ -\ \ 21.626262\ldots \\ \hline 2141.000000\ldots \end{array} \Rightarrow 990N = 2141 \Rightarrow N = \frac{2141}{990}$$

So $2.1\overline{62}$ equals the ratio of 2141 to 990. ✓

It turns out that all numbers expressible as repeating decimals are in fact rational numbers. One thing to keep in mind is that division by 0 is undefined, so creatures like $\frac{2}{0}$ & $-\frac{5}{0}$ are not rational numbers because they're not numbers at all!

Can we express the set of rational numbers by listing them like we listed the integers? Alas, no! We all know the next integer after 0 is 1, but what's the next rational number after 0? Is it $1/2$? No, because $1/3$ fits between 0 & $1/2$. Is it $1/3$, then? No: $1/4$ fits between 0 & $1/3$. And $1/5$ fits between 0 & $1/4$, $1/6$ fits between 0 & $1/5$, and so on. There is no "next" rational that comes before or after a given rational, and so there is no obvious way to list to (not in a way that would be helpfw, anyway). We need a new way to define a set.

If A is a set & x is an element of A, we write $x \in A$ (read $\in$ as "is an element of"). Now we're ready to look at what's called set-builder notation. If a set contains the integers from 1 to 100, one way to represent the set is $\{1, 2, 3, \ldots, 100\}$. Here's another way:

$$\{x \mid x \in \mathbb{Z} \ \& \ 1 \le x \le 100\}$$

"The set... ...of all x... ...such that... ...x is an integer and $1 \le x \le 100$."

This is set builder notation, which typically has the structure $\{$Name of object $\mid$ Properties of Object$\}$, where the vertical bar, $\mid$, means "such that". Note that $x \in \mathbb{Z}$ literally translates into "x is an element of $\mathbb{Z}$", but since $\mathbb{Z}$ is the set of integers we can instead say "x is an element of the set of integers" or just simply "x is an integer".

The set of positive even integers is $\{2, 4, 6, 8, \ldots\}$, or in set-builder notation we can write as: $\{x \mid x > 0 \ \& \ x \in \mathbb{Z} \ \& \ x \text{ is even}\}$ or $\{x \mid x \in \mathbb{N} \ \& \ x \text{ is even}\}$ or $\{2x \mid x \in \mathbb{N}\}$. The last form is nicest, since $x \in \mathbb{N}$ means $x = 1, 2, 3, \ldots$ are all allowed, and thus $2x = 2, 4, 6, \ldots$ results; $\{2x \mid x \in \mathbb{N}\}$ could be read as "The set of all quantities of the form $2x$ such that x is a natural number," where a natural number is by definition the same thing as a positive integer.

Now we're ready to define the set of rational numbers (often denoted by $\mathbb{Q}$, for "quotient") in a precise manner. We can write: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z} \ \& \ q \neq 0\}$. That is, $\mathbb{Q}$ is the set of all quantities of the form $\frac{p}{q}$ such that p & q are integers, and $q \neq 0$.

If a number is not rational, it is called <u>irrational</u>.

Examples of irrational numbers are $\sqrt{2}$, π, $\sqrt[3]{7}$, $\frac{1}{\sqrt{3}}$. In fact it can be proven that there are more irrationals than rationals. Strange, but true.

The set of real numbers, denoted by $\mathbb{R}$ or $\mathbf{R}$, can be defined in many ways. One cheap way is to say the set of real numbers equals the following:

$$\{d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots \mid n \in \mathbb{N} \ \& \ d_i \in \{0,1,\ldots,9\} \text{ for each } i \in \mathbb{N}\}$$

That is, $\mathbb{R}$ is the set of all objects of the form $d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots$ such that n is a natural number and d_1, d_2, d_3, etc. are each a digit from 0 to 9. If you think about it, any number comprised of digits from 0 to 9 can be cast in the form $d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots$. For instance 48.6 is just $d_1 d_2 . d_3 d_4 d_5 \cdots$, so $n=2$ and we let $d_1 = 4$, $d_2 = 8$, $d_3 = 6$, $d_4 = 0$, $d_5 = 0$, and in general $d_i = 0$ for any $i \geq 4$. To get $d_1 d_2 d_3 \cdots d_n . d_{n+1} d_{n+2} \cdots$ to become π, set $n=1$ to obtain $d_1 . d_2 d_3 d_4 d_5 \cdots$, then let $d_1 = 3$, $d_2 = 1$, $d_3 = 4$, $d_4 = 1$, $d_5 = 5$, $d_6 = 9$, and so on.

Later in the course we'll encounter another set of numbers, the complex numbers, that contains all real numbers but also includes additional numbers that are not real.

Order of Operations

When evaluating a complex arithmetic expression there is a universally accepted (on Earth, anyway) order in which operations are performed. It is this:

1) Grouping Symbols. Anything inside parentheses, ( ), brackets, [ ], or braces, { }, must be dealt with first. Other grouping symbols: —— (fraction bar), | | (absolute value bars), $\sqrt[n]{\ }$ (root operation), and a few others.
2) Exponentiation. All exponents must be evaluated once grouping symbols have been eliminated.
3) Multiplication/Division. Proceed from left to right. A common error is to assume that multiplication always comes before division! Priority is given strictly on the basis of position.
4) Addition/Subtraction. Proceed from left to right.

Example 1 Evaluate $12 \div 4 \cdot 3$

- Right Way: division first, since it lies to the left of multiplication:

$$\underbrace{\underbrace{12 \div 4}_{3} \cdot 3}_{9} \checkmark$$

- Wrong Way: multiplication first:

$$\underbrace{12 \div \underbrace{4 \cdot 3}_{12}}_{1}$$

Example 2 Evaluate $4^3+18\div 3-2^4-3\cdot(8-6)$.

- Solution:

$$\begin{aligned}
&= 4^3+18\div 3-2^4-3\cdot 2 && \text{(work inside parentheses)}\\
&= 64+18\div 3-16-3\cdot 2 && \text{(evaluate exponents)}\\
&= 64+6-16-3\cdot 2 && \text{(divide)}\\
&= 64+6-16-6 && \text{(multiply)}\\
&= 70-16-6 && \text{(add)}\\
&= 54-6 && \text{(subtract)}\\
&= \boxed{48} && \text{(subtract)}
\end{aligned}$$

Example 3 Evaluate $\dfrac{15\div 5\cdot 4\div 6-8}{-6-(-5)-8\div 2}$

- Solution: The fraction bar creates two groups: a top group (the numerator) & a bottom group (the denominator). Groups are always evaluated first, so evaluate top & bottom separately:

$$\begin{aligned}
&15\div 5\cdot 4\div 6-8 && \\
&= 3\cdot 4\div 6-8 && \text{(divide)}\\
&= 12\div 6-8 && \text{(multiply)}\\
&= 2-8 && \text{(divide)}\\
&= -6 &&
\end{aligned}$$

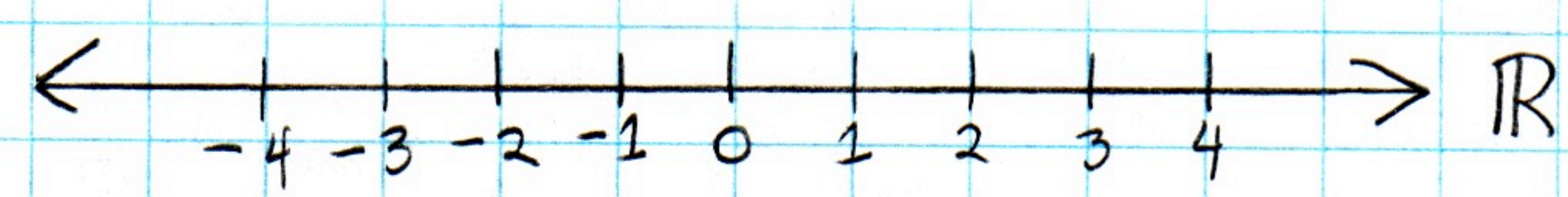
$$\begin{aligned}
&-6-(-5)-8\div 2 && \\
&= -6-(-5)-4 && \text{(divide)}\\
&= -1-4 && \text{(subtract)}\\
&= -5 && \text{(subtract)}
\end{aligned}$$

Thus the fraction becomes $\dfrac{-6}{-5}$, which simplifies to $\boxed{\dfrac{6}{5}}$

Be mindful of negative signs! A negative sign is nothing more than shorthand for multiplication by -1. A common error is to write $-3^2=9$, but this is wrong! To write -3^2 means to write $-1\cdot 3^2$, and by the Order of Operations $-1\cdot 3^2=-1\cdot 9=-9$ (exponentiation 1st, multiplication 2nd). That is, $-3^2=-9$.

On the other hand, $(-3)^2=(-3)(-3)=9$, which is a very different result.

R.2 — ABSOLUTE VALUE

The real number line is depicted as: 

Each point on the line corresponds to a particular real number. All the real numbers are there. You'll find $\sqrt{2}$ somewhere between 1 & 2, and π just to the right of 3. Smaller reals keep to the left whilst larger reals are found to the right.

The absolute value of a real number x, written $|x|$, is simply the distance x is from 0 on the number line. Hence $|2| = 2$ & $|-2| = 2$ since both 2 & −2 are a distance of 2 steps away from 0. Also $|-2/3| = 2/3$, $|\sqrt[3]{5}| = \sqrt[3]{5}$, and $|0| = 0$.

No matter what real number x is, it's always true that $|x| \geq 0$. More formally the absolute value operation is defined thus:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \quad , \quad (*)$$

Example 1 Write each expression without absolute value bars.

a) $|-13+6|$

• Solution: absolute value bars are grouping symbols, so evaluate $-13+6$ first, then get the absolute value...

$|-13+6| = |-7| = 7$ ✓

b) $|\sqrt{10} - 3|$

• Solution: a calculator will tell you that $\sqrt{10} - 3 \geq 0$; so, using (*) about, we find that $|\sqrt{10} - 3| = \sqrt{10} - 3$ ✓

c) $|\sqrt{3} - 3|$

• Solution: here $\sqrt{3} - 3 < 0$, so by (*) we get $|\sqrt{3} - 3| = -(\sqrt{3} - 3) = -\sqrt{3} + 3$
$= 3 + (-\sqrt{3}) = 3 - \sqrt{3}$ ✓

Definition 1

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n\ a\text{'s}} \quad \text{for any } n \in \mathbb{N}\text{ ; } a \text{ is called the \underline{base}, } n \text{ the \underline{exponent}}$$

From this definition (henceforth called D1) many laws arise:

Laws of Exponents For any $m, n \in \mathbb{N}$ with $m > n$ we have...

$\boxed{L1}$ $a^m \cdot a^n = a^{m+n}$

$\boxed{L2}$ $\dfrac{a^m}{a^n} = a^{m-n}$ for $a \neq 0$ (we need $m > n$ so that $m - n > 0$)

$\boxed{L3}$ $(a^m)^n = a^{mn}$

$\boxed{L4}$ $(a \cdot b)^m = a^m \cdot b^m$

$\boxed{L5}$ $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ for $b \neq 0$

Note Exponents are merely short-hand for repeated multiplication just as multiplication is shorthand for repeated addition:

$3+3+3+3 \longrightarrow 3 \cdot 4$

$3 \cdot 3 \cdot 3 \cdot 3 \longrightarrow 3 \wedge 4 \longrightarrow 3^4$

The laws make sense even if we don't bother with their formal proofs. According to Law 1 (L1), $a^2 \cdot a^5 = a^{2+5} = a^7$; but that makes sense since $a^2 = a \cdot a$ & $a^5 = a \cdot a \cdot a \cdot a \cdot a$, and thus $a^2 \cdot a^5 = (a \cdot a) \cdot (a \cdot a \cdot a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \overset{D1}{=} a^7$ as the law says. (The D1 over the equal sign means Definition 1 is being applied).

L2 says $\dfrac{a^5}{a^2} = a^{5-2} = a^3$, but that jibes with reality since $\dfrac{a^5}{a^2} = \dfrac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = \dfrac{a}{a} \cdot \dfrac{a}{a} \cdot a \cdot a \cdot a =$ $1 \cdot 1 \cdot a \cdot a \cdot a = a \cdot a \cdot a \overset{D1}{=} a^3$ (note $\frac{a}{a} = 1$ for any $a \neq 0$).

L3 says $(a^5)^2 = a^{5 \cdot 2} = a^{10}$, which we can see is reasonable if we take another (longer) approach making liberal use of D1: $(a^5)^2 = a^5 \cdot a^5 = (a \cdot a \cdot a \cdot a \cdot a) \cdot (a \cdot a \cdot a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$ $= a^{10}$.

Diddle around with L4 & L5 to see that they mesh with expectations as well. Note: $a^1 = a$ by default.

Definition 2

$$a^0 = 1 \quad \text{for any } a \neq 0$$

The reason for the invention of D2 stems from the observation that, for $a \neq 0$, $\dfrac{a^m}{a^m} = 1$; but if we allowed L2 to be employed in this case, it would say $\dfrac{a^m}{a^m} = a^{m-m} = a^0$. So, letting $a^0 = 1$ fits the general scheme of things.

Some matters to be mindful of: a negative sign indicates multiplication by -1 (and -1 in turn is defined to be the "additive opposite" of 1, which means 1 & -1 sum to zero). Thus when encountering an expression like -3^2, it must be read as $-1\cdot 3^2$; then, by Order of Operations, the exponent is dispatched first so that 3^2 becomes 9, after which multiplication is performed so that $-1\cdot 9$ becomes -9. To summarize: $-3^2 = -1\cdot 3^2 = -1\cdot 9 = -9$. This was mentioned in Section R.1, but it's worth repeating since writing $-3^2 = 9$ is such a common error.

Similarly, while $1^0 = 1$, $5^0 = 1$, $17.32^0 = 1$, and $(-83)^0 = 1$, it must be seen that $-5^0 = -1$ since $-5^0 = -1\cdot 5^0 = -1\cdot 1 = -1$.

When variables are introduced the logic is the same: if $x \neq 0$, then $-x^0 = -1\cdot x^0 = -1\cdot 1 = -1$ while $(-x)^0 = 1$. Another demo: $-6x^0 = -6\cdot x^0 = -6\cdot 1 = -6$, $(-6x)^0 = 1$, and finally $-(6x)^0 = -1\cdot(6x)^0 = -1\cdot 1 = -1$. The problems algebra students encounter here, I believe, stem from the frequent "invisibility" of multiplication in the conventional notation! That is, $2\cdot x$ will usually be written as $2x$, and $-1\cdot 2$ will be written as -2.

Some terminology now follows. A <u>term</u> is generally a quantity in a sum, so $2x+5y$ is an expression consisting of the terms $2x$ & $5y$, and $8x^2-4x+5$ consists of the terms $8x^2$, $-4x$, and 5 (write the expression as $8x^2+(-4x)+5$ to see that $-4x$, not $4x$, is actually the term in the middle). <u>Like terms</u> are terms that consist of the same variables with the same exponents. Here's an illustration, with like terms connected by like-colored underscoring:

$$2x^2 - 5x + 3xy + 9x^2y - \tfrac{1}{5}xy^2 - 0.21x^2y + \sqrt{2}\,xy - 87xyz - 7xy^2 + x - \tfrac{3}{4}y - 15.8x^2 + 6$$

Note $-87xyz$ is a term not like to any other since no other term has the variable z. Also the last term, 6, has no buddy since it has no variable at all. Finally, notice that terms such as $3xy$ & $9x^2y$ are not like because, though they have the same variables, the exponents don't entirely match up ($3xy$ has x^1 while $9x^2y$ has x^2).

Now, a <u>polynomial</u> is nothing more than an expression consisting of one or more terms, each term consisting of constants (i.e. numbers) and/or variables, each variable having a nice whole-number variable (i.e. $0, 1, 2, 3, \ldots$). The giant 13-term expression above is a polynomial. Other examples of polynomials: 3, $3x$, $3xy$, $-3x^2y^{95}$, $7xy^2zu^{14} - \frac{1}{2}x^5y^3zv^{26} + a^2b^3c^{129} - 14.2$. Indeed polynomials are to algebra what numbers are to arithmetic. They can be added, subtracted, multiplied, divided, and even exponentiated.

To add or subtract two polynomials, simply combine their like terms...

Example 1 Add the polynomials $3x^2-4x-5$ and $-2x^2-12x+3$

• <u>Solution:</u> $(3x^2-4x-5)+(-2x^2-12x+3) = 3x^2-4x-5-2x^2-12x+3$

$= \underbrace{3x^2-2x^2}_{x^2}\underbrace{-4x-12x}_{-16x}\underbrace{-5+3}_{-2}$ (terms can be rearranged so that birds of a feather fly together)

$= \underline{\underline{x^2-16x-2}}$ (note that $1x^2 = 1\cdot x^2 = x^2$, another "invisible" multiplication)

Example 2 Subtract $3x^2-4x-5$ from $-2x^2-12x+3$

• <u>Solution:</u> $(-2x^2-12x+3)-(3x^2-4x-5)$ (note "subtract A from B" means $B-A$)

$= -2x^2-12x+3-3x^2+4x+5$ (Distributive Property: $-(3x^2-4x-5) = -1\cdot(3x^2-4x-5)$
$= -3x^2+4x+5$)

$= -2x^2-3x^2-12x+4x+3+5$

$= \underline{\underline{-5x^2-8x+8}}$

Recall the general statement of the Distributive Property: $a(b+c) = ab+ac$. This easily extends to $a(b+c+d) = ab+ac+ad$* & so on. It's actually a handy tool for doing mental multiplications: $7(13) = 7(10+3) = 7\cdot 10+7\cdot 3 = 70+21 = 91$.

It should be pointed out that while a negative sign is short-hand for multiplication by -1, a minus sign is short-hand for addition of a negative quantity. So -8 means $-1\cdot 8$, and $x-8$ means $x+(-8)$. So $-(3x^2-4x-5)$ in Example 2 becomes $-1\cdot[3x^2+(-4x)+(-5)]$. Now apply * to get: $-1\cdot[3x^2+(-4x)+(-5)] = (-1)(3x^2)+(-1)(-4x)+(-1)(-5) = -3x^2+4x+5$, as desired.

L1 is the first result that indicates how to mutiply polynomials, the Distributive Property the second.

Example 3 Multiply $5x^2y$ & $-6x^8y^3$

• <u>Solution:</u> $(5x^2y)(-6x^8y^3)$

$= 5\cdot x^2\cdot y\cdot(-6)\cdot x^8\cdot y^3$ (bring all the "invisible" multiplications out into the light)

$= 5\cdot(-6)\cdot x^2\cdot x^8\cdot y\cdot y^3$ (rearrange so that factors that can be meaningfully multiplied are together)

$= -30\cdot x^{2+8}\cdot y^{1+3}$ (by L1 $y\cdot y^3 = y^1\cdot y^3 = y^{1+3}$ & $x^2\cdot x^8 = x^{2+8}$)

$= \underline{\underline{-30x^{10}y^4}}$

$5x^2y$ & $-6x^8y^3$ are one-term polynomials, also known as <u>monomials</u>. Two-term polynomials are <u>binomials</u>, while three-termers are <u>trinomials</u>. There's probably a \$10 word for a four-termer, but I don't think I've ever heard it.

Example 4 Multiply $-5x$ & $3x^2-2$

• Solution: $-5x(3x^2-2) = \underbrace{-5x\left[3x^2+(-2)\right]}_{\text{Turn subtraction into addition of a negative } \underline{\text{if}} \text{ it helps you}} = (-5x)(3x^2)+(-5x)(-2) = \underline{\underline{-15x^3+10x}}$

To multiply a binomial with a binomial requires three applications of the Distributive Property (DP):

$$(a+b)(c+d) \overset{DP}{=} (a+b)c+(a+b)d = c(a+b)+d(a+b) \overset{DP}{=} ac+bc+ad+bd, \quad (1)$$

But there's a short-cut method known as FOIL:

$$(a+b)(c+d) = \underbrace{ac}_{F}+\underbrace{ad}_{O}+\underbrace{bc}_{I}+\underbrace{bd}_{L}$$

Here F = First Terms, O = Outer Terms, I = Inner Terms, L = Last Terms. We know that FOIL works the result features the same four terms ac, ad, bc, and bd as (1). The order the terms are written in does not matter, of course.

Example 5 Multiply $(3x^2-4)(9x^2-5)$

• Solution: Apply FOIL, starting with $[(3x^2)+(-4)][(9x^2)+(-5)]$ if it helps (it's not required).

Then: $[(3x^2)+(-4)][(9x^2)+(-5)] = \underbrace{(3x^2)(9x^2)}_{F}+\underbrace{(3x^2)(-5)}_{O}+\underbrace{(-4)(9x^2)}_{I}+\underbrace{(-4)(-5)}_{L}$

$= 27x^4+(-15x^2)+(-36x^2)+20 = 27x^4\underbrace{-15x^2-36x^2}_{\text{Always combine like terms!}}+20 = \underline{\underline{27x^4-51x^2+20}}$

The FOIL procedure establishes a pattern that is easily extrapolated to situations involving polynomials having more than two terms:

$$(a+b)(c+d+e) = ac+ad+ae+bc+bd+be$$

Another method, which is also a viable alternative to FOIL itself, multiplies big polynomials in the same way that big numbers are multiplied:

Example 6 Multiply $(2x-1)(3x^2-8x-4)$

• Solution: Set up in vertical format

$$\begin{array}{r} 3x^2-8x-4 \\ \times \quad 2x-1 \\ \hline -3x^2+8x+4 \\ +\ 6x^3-16x^2-8x \quad \\ \hline 6x^3-19x^2+0x+4 \end{array}$$

Keep like terms aligned

$\longrightarrow \underline{\underline{6x^3-19x^2+4}}$

Example 7 Multiply $(x^2+3x-6)(3x^3+4x^2-x+5)$

• Solution:

$$
\begin{array}{rrrrrrl}
 & & 3x^3 & +4x^2 & -x & +5 & \\
\times & & & x^2 & +3x & -6 & \\
\hline
 & & -18x^3 & -24x^2 & +6x & -30 & \leftarrow \text{Multiply top polynomial by } -6 \\
 & 9x^4 & +12x^3 & -3x^2 & +15x & & \leftarrow \text{Mult. top poly. by } 3x \text{ (indent to align like terms!)} \\
+\ 3x^5 & +8x^4 & -x^3 & +5x^2 & & & \leftarrow \text{Mult. top poly. by } x^2 \text{ (indent again to keep like terms aligned!!)} \\
\hline
3x^5 & +17x^4 & -7x^3 & -22x^2 & +21x & -30 & \leftarrow \text{Simple addition combines the like terms \& yields answer}
\end{array}
$$

Example 8 Exponentiate $(3x+5)^2$

• Solution: Wrong: $(3x+5)^2 = (3x)^2 + 5^2 = 9x^2 + 25$

Right: use FOIL... $(3x+5)^2 = (3x+5)(3x+5) = (3x)(3x) + (3x)(5) + (5)(3x) + (5)(5)$

$= 9x^2 + 15x + 15x + 25 = \underline{\underline{9x^2 + 30x + 25}}$

R.7 — RATIONAL EXPONENTS

Definition 4 For any $n \in \mathbb{N}$, $a^{1/n} = b \iff b^n = a$, where $b \geq 0$ is required if n is even.

That is, $a^{1/n}$ equals that number b for which $b^n = a$. In the case where n is an even number there are usually two choices for b, but the definition resolves the matter by mandating the <u>nonnegative</u> choice.

Example 1 Evaluate each.

a) $81^{1/2}$

• <u>Solution:</u> By definition $81^{1/2} = b$, where $b^2 = 81$. Note that $9^2 = 81$ & $(-9)^2 = 81$, so it appears we could let $b = 9$ or -9. However here $n = 2$, which is even, so $b = 9$ is the answer. That is, $81^{1/2} = 9$. ✓

b) $(-27)^{1/3}$

• <u>Solution:</u> Let $(-27)^{1/3} = b$, where $b^3 = -27$ must hold by definition. The only solution is $b = -3$, so $(-27)^{1/3} = -3$. ✓

c) $(-16)^{1/4}$

• <u>Solution:</u> Let $(-16)^{1/4} = b$, where $b^4 = -16$ must hold. But no real number will work here! We have $2^4 = 16$ & $(-2)^4 = 16$, but nothing that will yield -16. There is no solution to $(-16)^{1/4} = b$, so we say $(-16)^{1/4}$ is undefined. ✓

Definition 4 (D4) is motivated by Law 3 of exponents: $(a^m)^n = a^{mn}$. For if $a^{1/n} = b$, it seems reasonable that $(a^{1/n})^n = b^n$, where $(a^{1/n})^n = a^{1/n \cdot n} = a^1 = a$ by Law 3; then we have $b^n = a$ just as the definition says. But we need the definition, since without it we can't actually apply Law 3 to simplify $(a^{1/n})^n$ because $1/n$ isn't an integer. Now, with the definition, we can logically "extend" Law 3 to cases like $(a^{1/n})^n$ to get a.

In general we have, by definition, $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$, for any $m, n \in \mathbb{Z}$, $n \neq 0$.

Example 2 Evaluate each.

a) $\left(16/81\right)^{-3/4}$

• <u>Solution:</u> By L5 we get $\dfrac{16^{-3/4}}{81^{-3/4}} \overset{L3}{=} \dfrac{(16^{1/4})^{-3}}{(81^{1/4})^{-3}}$, where $16^{1/4} = b \Rightarrow b^4 = 16 \Rightarrow b = 2$, so $16^{1/4} = 2$; and $81^{1/4} = b \Rightarrow b^4 = 81 \Rightarrow b = 3$, so $81^{1/4} = 3$. Then...

$$\left(\frac{16}{81}\right)^{-3/4} = \frac{(16^{1/4})^{-3}}{(81^{1/4})^{-3}} = \frac{2^{-3}}{3^{-3}} = \frac{3^3}{2^3} = \frac{27}{8} \checkmark$$

b) $\left(-\frac{125}{216}\right)^{2/3}$

• Solution: Have $\left(-\frac{125}{216}\right)^{2/3} \overset{L5}{=} \frac{(-125)^{2/3}}{216^{2/3}} \overset{L3}{=} \frac{[(-125)^{1/3}]^2}{(216^{1/3})^2}$; now, $(-125)^{1/3} = b \Rightarrow b^3 = -125 \Rightarrow b = -5 \Rightarrow (-125)^{1/3} = -5$, and $216^{1/3} = b \Rightarrow b^3 = 216 \Rightarrow b = 6 \Rightarrow 216^{1/3} = 6$;

Then $\left(-\frac{125}{216}\right)^{2/3} = \frac{(-5)^2}{6^2} = \frac{25}{36}$ ✓

c) $(x^4)^{1/4}$

• Solution: Let $(x^4)^{1/4} = b$. By definition $b^4 = x^4$. So do we have $b = x$? Maybe not! The value of x is unknown. If $x < 0$, then letting $b = x$ is wrong since $b \geq 0$ is required for n even (here $n = 4$). If $x \geq 0$, then letting $b = x$ is right. Not knowing what x is, we're forced to have $b = |x|$; then, $x \geq 0 \Rightarrow |x| = x$, & $x < 0 \Rightarrow |x| = -x$. Hence, $(x^4)^{1/4} = |x|$. ✓

Since this is an algebra course we're mostly interested in simplifying expressions containing at least one variable, so...

Example 3 Simplify each expression, writing each answer without negative exponents.

a) $\frac{x^7 \cdot (x^{-2})^5}{x^{-6}}$

• Solution: $\overset{L3}{=} \frac{x^7 \cdot x^{-10}}{x^{-6}} \overset{L1}{=} \frac{x^{7+(-10)}}{x^{-6}} = \frac{x^{-3}}{x^{-6}} = \frac{x^6 x^{-3}}{1} \overset{L1}{=} x^{6+(-3)} = x^3$,

where $\frac{1}{x^{-6}} = \frac{x^6}{1}$ by the definition of a negative exponent. ✓

b) $\frac{(k^{-2/3})^2 (k^{1/2})^3}{6k^{-1/2}}$

• Solution: $\overset{L3}{=} \frac{k^{-4/3} \cdot k^{3/2}}{6k^{-1/2}} \overset{L1}{=} \frac{k^{-4/3+3/2}}{6k^{-1/2}} = \frac{k^{1/6}}{6k^{-1/2}} = \frac{k^{1/6} \cdot k^{1/2}}{6} \overset{L1}{=} \frac{k^{1/6+1/2}}{6} = \frac{k^{2/3}}{6}$ ✓

c) $\frac{(x^{-3}y^2)^{-6} x^{-10} y^{-3}}{8x^{-5}y^{-9}}$

• Solution: $\overset{L4}{=} \frac{(x^{-3})^{-6}(y^2)^{-6} x^{-10} y^{-3}}{8x^{-5}y^{-5}} \overset{L3}{=} \frac{x^{18}y^{-12}x^{-10}y^{-3}}{8x^{-5}y^{-5}} \overset{L1}{=} \frac{x^{18-10}y^{-12-3}}{8x^{-5}y^{-5}}$

$= \frac{x^8 y^{-15}}{8x^{-5}y^{-5}} \underset{\uparrow}{=} \frac{x^8 x^5}{8y^{-5}y^{15}} \overset{L1}{=} \frac{x^{13}}{8y^{10}}$ ✓

Def. of Negative Exponent