Definition A function P is a <u>polynomial function</u> if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ for some positive integer n and constants $a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in \mathbb{R}$

Technically the definition above addresses only <u>single-variable</u> polynomial functions, meaning a polynomial function whose "output" depends on only one "input" — namely, the value of the variable x.

If we let $a_0 = c$, $a_1 = b$, $a_2 = a$, and let all other a_i's be 0 (that is, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, and so on unendingly), then $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ $= 0x^n + 0x^{n-1} + \cdots + 0x^3 + ax^2 + bx + c = ax^2 + bx + c$ — our old friend the quadratic function!

If we let $a_0 = b$, $a_1 = a$, and set all other a_i's equal to 0, then $P(x) = ax + b$ results — our long-time pal the linear function!

So, linear & quadratic functions are just special kinds of polynomial functions. Just in case it ever comes up in a quiz show, a function P given by $P(x) = ax^3 + bx^2 + cx + d$ is called a <u>cubic</u> function, while $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is a <u>quartic</u> function. Ten-dollar words for "higher-order" polynomial functions can be found in the literature should anyone wish to purchase them.

Much fuss & pencil-pushing has been generated by algebraists over the centuries regarding the zeros of polynomial functions. Remember what a "zero" is:

Definition Let f be a function. A number k is a <u>zero of f</u> if $f(k) = 0$.

Consider the quadratic function f given as usual by $f(x) = ax^2 + bx + c$. If k is a zero of f, then $f(k) = 0$. But $f(k) = ak^2 + bk + c$ also holds, and so the inevitable conclusion is that $ak^2 + bk + c = 0$. Hence, if we're interested in finding the zeros of f, then that means finding the values of the variable x for which $f(x) = 0$, and that in turn means we must find the solutions (or "roots") to the equation $ax^2 + bx + c = 0$. And we know how to do that, for the quadratic formula tells us that $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

What if we want to find the zeros of the cubic function f given by $f(x) = ax^3 + bx^2 + cx + d$? We would need to find the roots of the equation $ax^3 + bx^2 + cx + d = 0$. Is there a formula that will give us these roots? Yes, and it is a sight so nightmarish to behold that even professional mathematicians neither know it nor give it the time of day.

But there are ways to "crack open" certain kinds of "higher order" polynomial equations like $ax^3 + bx^2 + cx + d = 0$ and $ax^4 + bx^3 + cx^2 + dx + e = 0$ and so on. Some preliminaries follow...

Synthetic Division:

To divide a polynomial ax^3+bx^2+cx+d by $x-k$, we apply a streamlined procedure called synthetic division that goes much faster than long division...

$$
\begin{array}{c|cccc}
k & a & b & c & d \\
 & & ka & kb+k^2a & kc+k^2b+k^3a \\
\hline
 & a & b+ka & c+kb+k^2a & \underbrace{d+ck+k^2b+k^3a}_{\text{remainder } r}
\end{array}
$$

Procedure:

1) Drop a below bar
2) Multiply k by a
3) Place result of step 2 under b
4) Add result of Step 2 & b
5) Place result of step 4 under bar
6) Multiply k by result of Step 4
7) Place result of step 6 under c
8) Add result of step 6 & c
9) Place result of step 8 under bar
10) Multiply k by result of step 8
11) Place result of step 10 under d
12) Add result of step 10 & d

THE END.

$\Downarrow$ Outcome

$$\frac{ax^3+bx^2+cx+d}{x-k} = ax^2+(b+ka)x+(c+kb+k^2a)+\frac{r}{x-k}$$

$\Downarrow$ Disclaimer

"The procedure looks complicated but it really isn't." Some examples to allay all fears...

Example 1 Divide $x^3-9x^2+27x-27$ by $x-3$.

• Solution: Our $x-k$ here is $x-3$, so in our case k is 3. It goes in a box to the left while the coefficients of the powers of x in $x^3-9x^2+27x-27$ go in a row to the right of the box...

$$
\begin{array}{c|cccc}
3 & 1 & -9 & 27 & -27 \\
 & \downarrow & 3 & -18 & 27 \\
\hline
 & 1 & -6 & 9 & 0
\end{array}
$$

So, $\dfrac{x^3-9x^2+27x-27}{x-3} = 1x^2-6x+9+\dfrac{0}{x-3} = x^2-6x+9$ ✓

In Example 1, $x-3$ is the divisor, $x^3-9x^2+27x-27$ the dividend, x^2-6x+9 the quotient, & 0 the remainder.

Notice that $\dfrac{x^3-9x^2+27x-27}{x-3} = x^2-6x+9$ means $x^3-9x^2+27x-27 = (x-3)(x^2-6x+9)$.

We have managed to (partially) factor a cubic polynomial! This is important.

Example 2 Divide x^7-3x^2-1 by $x-3$

• Solution: We must look upon x^7-3x^2-1 as being $x^7+0x^6+0x^5+0x^4+0x^3-3x^2+0x-1$. As in the previous example, 3 goes in a box to the left & the coefficients of the powers of x (including zeros as place-holders for missing powers) go in a row to the right...

$$
\begin{array}{r|rrrrrrrr}
3 & 1 & 0 & 0 & 0 & 0 & -3 & 0 & -1 \\
 & \downarrow & 3 & 9 & 27 & 81 & 243 & 720 & 2160 \\
\hline
 & 1 & 3 & 9 & 27 & 81 & 240 & 720 & 2159
\end{array}
$$

So, $\dfrac{x^7-3x^2-1}{x-3} = x^6+3x^5+9x^4+27x^3+81x^2+240x+720+\dfrac{2159}{x-3}$ ✓

Example 3 Divide $3x^4-2x^2+15x$ by $x+2$

• Solution: Our $x-k$ here is $x-(-2)$, so $k=-2$ is the value that goes in the box. (Note how k must always be a number after a minus sign.) Meanwhile the dividend of $3x^4-2x^2+15x$ will be a couple of zero terms as place-holders, since it is equal to $3x^4+0x^3-2x^2+15x+0$. The set-up and execution follows...

$$
\begin{array}{r|rrrrr}
-2 & 3 & 0 & -2 & 15 & 0 \\
 & & -6 & 12 & -20 & 10 \\
\hline
 & 3 & -6 & 10 & -5 & 10
\end{array}
$$

So, $\dfrac{3x^4-2x^2+15x}{x+2} = 3x^3-6x^2+10x-5+\dfrac{10}{x+2}$ ✓

Time to move on to bigger and better things. To recap, a function P is a <u>polynomial</u> function if $P(x)$ is a <u>polynomial</u> expression. (Always be mindful of the distinction between the symbol f and the symbol $f(x)$. For us f always represents a function, which mathematically is a set of ordered pairs, but which can also be fairly regarded as a "machine" that receives inputs & returns outputs. In contrast $f(x)$ just represents the output f returns when it receives the value x as input.)

Now, if we divide a polynomial $P(x)$ by a divisor of the form $x-k$ (k a constant), we usually would like to get a remainder of 0 if for no other reason than it appeals to our sense of aesthetics. But even if we don't get a remainder of 0, we still have <u>something</u>...

Remainder Theorem

If the polynomial $P(x)$ is divided by $x-k$, then the remainder is equal to $P(k)$.

How bizarre is that? Let's verify the theorem's conclusion in a few instances...

- In Example 1 our polynomial $P(x)$ was $x^3-9x^2+27x-27$ (that is, $P(x)=x^3-9x^2+27x-27$). Dividing this $P(x)$ by $x-3$, a remainder of 0 resulted. According to the Remainder Theorem, then, $P(3)=0$. And sure enough, $P(3)=3^3-9(3)^2+27(3)-27$ $= 27-9(9)+81-27 = 27-81+81-27 = -54+54 = 0$, as predicted!

- In Example 2, $P(x)=x^7-3x^2-1$. Dividing this $P(x)$ by $x-3$ yielded a remainder of 2159. According to the Remainder Theorem $P(3)=2159$. The direct verification is straightforward: $P(3)=3^7-3(3)^2-1 = 2187-3(9)-1 = 2187-27-1 = 2159$. Hallelujah.

- In Example 3, $P(x)=3x^4-2x^2+15x$. Dividing $P(x)$ by $x+2$ gave us a remainder of 10. Now note that $P(-2)=3(-2)^4-2(-2)^2+15(-2) = 48-8-30 = 10$.

The above lends credence to the Remainder Theorem, but it doesn't prove it. That's not our problem. But here's another theorem...

FACTOR THEOREM

$P(k)=0$ if and only if $x-k$ is a factor of $P(x)$.

The phrase "if and only if" is quite important in mathematics. If I say "Math is fun if and only if you have the right attitude", then I mean "If math is fun then you have the right attitude, and if you have the right attitude then math is fun."

More prosaically "A if and only if B" means "If A then B, and if B then A."

So the Factor Theorem is saying: "If $P(k)=0$, then we know $x-k$ is a factor of $P(x)$; and also, if $x-k$ is a factor of $P(x)$, then we know $P(k)=0$." It's a two-way street.

The Remainder Theorem is worded differently, as a (more strict) "if-then" statement. It's a one-way street. If we happen to know that the remainder in a synthetic division process is equal to the number $P(4)$, we can <u>not</u> conclude that $P(x)$ was divided by $x-4$! Can you see why? (Consider the polynomial $P(x)=x^2-x-7$, and notice that $P(4)=P(-3)$.)

What's the use of the Remainder and Factor theorems? Taken together they give us a means to factor certain higher-degree polynomials such as cubic (ax^3+bx^2+cx+d) & quartic $(ax^4+bx^3+cx^2+dx+e)$ polynomials — or even higher! Call it an algorithm, or even a "cookbook recipe":

1) Divide a polynomial by divisors of the form $x-k$ (use synthetic division)
2) If a remainder of 0 results, then by Remainder Theorem $P(k)=0$
3) Thus by the Factor Theorem $P(x)$ has $x-k$ as a factor
4) Therefore $P(x)=(x-k)\,Q(x)$, where $Q(x)$ is the quotient.

Truth be told, there is nothing "miraculous" about the foregoing results. If a polynomial $P(x)$ is divided by $x-k$, and what results is a quotient $Q(x)$ and a remainder r equal to 0, then...

$$\frac{P(x)}{x-k} = Q(x) + \frac{r}{x-k} \quad \text{becomes} \quad \frac{P(x)}{x-k} = Q(x) + \frac{0}{x-k} \quad \text{so that} \quad \frac{P(x)}{x-k} = Q(x),$$

and finally $\boxed{P(x) = (x-k)\,Q(x)}^{*}$ results.

Let's look specifically at cubic polynomials, which have the "standard" form ax^3+bx^2+cx+d. To factor such a beast, we could define a function P such that $P(x) = ax^3+bx^2+cx+d$, then find the <u>zeros of P</u> in order to bag the <u>factors of $P(x)$</u> (using the Factor Theorem, or just running through the manipulations above to get $*$). An example...

Example 4 Completely factor the polynomial x^3+x-2.

• <u>Solution:</u> Define function P by $P(x) = x^3+x-2$.

It might be noticed that $P(1) = 1^3+1-2 = 1+1-2 = 0$, so the Factor Theorem informs us that $x-1$ is a factor of $P(x)$; however, the Factor Theorem does <u>not</u> tell us diddley-squat about the other factor that results when $x-1$ is factored out. In other words, we know $P(x) = (x-1)\,Q(x)$, but we don't know what $Q(x)$ is specifically.

This is why the synthetic division approach is so useful. If we suspect that 1 is a zero of P, then we can divide $P(x)$ by $x-1$ to not only confirm our suspicion (by getting a remainder of 0 & appealing to the Remainder Theorem), but also find out what specifically $Q(x)$ is from the row of numbers under the bar in the synthetic division set-up...

$$\begin{array}{r|rrrr}
1 & 1 & 0 & 1 & -2 \\
  &   & 1 & 1 & 2 \\
\hline
  & 1 & 1 & 2 & \boxed{0}
\end{array}$$

$\leftarrow$ Divide $1x^3+0x^2+1x-2$ by $x-1$

$\leftarrow$ Remainder $=0$ means $P(1)=0$ (Remainder Theorem)

Thus $x-1$ is a factor of x^3+x-2 (Factor Theorem)

Quotient $Q(x) = 1x^2+1x+2 = x^2+x+2$

Now, $P(x) = (x-1)\,Q(x) = (x-1)(x^2+x+2)$. Verify this from $\dfrac{P(x)}{x-k} = Q(x) + \dfrac{r}{x-k}$ if you have any doubts!

But we're not done yet. We need to <u>next</u> factor x^2+x+2 into a product of two linear binomials. It <u>can</u> be done! Since we're letting $Q(x) = x^2+x+2$, to find the factors of x^2+x+2 we need only find the zeros of Q.

$\longrightarrow$

That means finding the roots to the equation $Q(x)=0$. Plainly put, solve $x^2+x+2=0$ for x. But this is just a quadratic equation, so it can be solved using the venerable Quadratic Formula:

$$x=\frac{-1\pm\sqrt{(1)^2-4(1)(2)}}{2(1)}=\frac{-1\pm\sqrt{-7}}{2}=-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$$

That is, $Q\left(-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i\right)=0$. By the Factor Theorem, then, $x-\left(-\frac{1}{2}+\frac{\sqrt{7}}{2}i\right)$ & $x-\left(-\frac{1}{2}-\frac{\sqrt{7}}{2}i\right)$ are factors of $Q(x)$. Since Q has no <u>other</u> zeros, $Q(x)$ has no <u>other</u> factors. Therefore $Q(x)=x^2+x-2=\left[x-\left(-\frac{1}{2}+\frac{\sqrt{7}}{2}i\right)\right]\left[x-\left(-\frac{1}{2}-\frac{\sqrt{7}}{2}i\right)\right]$.

So finally, $P(x)=(x-1)Q(x)=(x-1)\left(x+\frac{1}{2}-\frac{\sqrt{7}}{2}i\right)\left(x+\frac{1}{2}+\frac{\sqrt{7}}{2}i\right)$, a <u>full</u> factorization! ✓

Here's a theorem that will help us find possible zeros for a polynomial function...

Rational Zeros Theorem If the polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ has integer coefficients, then every <u>rational</u> zero of P is of the form p/q, where p is a factor of a_0 & q is a factor of a_n.

Consider the polynomial $P(x)=3x^4-2x^3+6x-4$. The coefficients here are $3, -2, 6,$ & -4. According to the Rational Zeros Theorem, <u>if</u> P has any zero that happens to be a <u>rational number</u>, <u>then</u> that number will have to be of the form p/q, where p must be a factor of -4 (the constant term) & q must be a factor of 3 (the coefficient of the highest power of x — called the "leading coefficient")

Factors of -4 are: $1, -1, 2, -2, 4, -4$. So $p\in\{1,-1,2,-2,4,-4\}$
Factors of 3 are: $1, -1, 3, -3$. So $q\in\{1,-1,3,-3\}$

Thus $\frac{p}{q}$ can be any fraction that results from putting $1, -1, 2, -2, 4,$ or -4 in the numerator and $1, -1, 3,$ or -3 in the denominator.

In my own peculiar short-hand, $\frac{p}{q}=\frac{\pm1,\ \pm2,\ \pm4}{\pm1,\ \pm3}$, or...

$$\frac{p}{q}\in\left\{1,-1,\tfrac{1}{3},-\tfrac{1}{3},2,-2,\tfrac{2}{3},-\tfrac{2}{3},4,-4,\tfrac{4}{3},-\tfrac{4}{3}\right\}=\left\{\pm1,\pm2,\pm4,\pm\tfrac{1}{3},\pm\tfrac{2}{3},\pm\tfrac{4}{3}\right\}$$

Thus there are a dozen "candidate" rational zeros for the function P. As we'll discover later, since the highest power of x in $P(x)$ is 4, P can have <u>at most</u> as many as 4 zeros. It may turn out that <u>none</u> of the zeros for P are rational, but <u>if</u> any <u>are</u> rational, they'll be among the elements of the set $\{\pm1,\pm2,\pm4,\pm\tfrac{1}{3},\pm\tfrac{2}{3},\pm\tfrac{4}{3}\}$.

Example 5 Fully factor $x^5-2x^4-10x^3+16x^2+25x-30$

• Solution: Let $P(x)=x^5-2x^4-10x^3+16x^2+25x-30$ & find all values of x for which $P(x)=0$.

This means find all solutions to the equation $x^5-2x^4-10x^3+16x^2+25x-30=0$.

Possible rational zeros of P: $\dfrac{\text{Any factor of } -30}{\text{Any factor of } 1} = \dfrac{\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30}{\pm1}$

$= \pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$. Agony! Start with 1...

1	1	-2	-10	16	25	-30
		1	-1	-11	5	30
	1	-1	-11	5	30	0 ← So $P(1)=0$, meaning $x-1$ is a factor of $P(x)$.

Now, $P(x)=(x-1)(x^4-x^3-11x^2+5x+30)$

To factor $Q(x)=x^4-x^3-11x^2+5x+30$ is the next task. Note that any rational zero of Q will be of the form $\dfrac{\text{Factor of } 30}{\text{Factor of } 1}$, yielding the same list of candidates as above!

Try 1 again...

1	1	-1	-11	5	30
		1	0	-11	-6
	1	0	-11	-6	24 ← So $Q(1)\neq 0$, meaning $x-1$ is not a factor of $Q(x)$.

On your own, verify that 2 won't work. Try 3...

3	1	-1	-11	5	30
		3	6	-15	-30
	1	2	-5	-10	0 ← So $Q(3)=0$, meaning $x-3$ is a factor of $Q(x)$.

Now, $Q(x)=(x-3)(x^3+2x^2-5x-10)$
Then $P(x)=(x-1)(x-3)(x^3+2x^2-5x-10)$

Since $x^3+2x^2-5x-10$ has four terms, try factoring by grouping before resorting to running through the synthetic division process yet again:

$$x^3+2x^2-5x-10 = x^2(x+2)-5(x+2) = (x+2)(x^2-5) = (x+2)\underbrace{[x^2-(\sqrt{5})^2]}_{\text{Difference of Squares}}$$
$$= (x+2)(x-\sqrt{5})(x+\sqrt{5})$$

Thus, $P(x)=(x-1)(x-3)(x+2)(x-\sqrt{5})(x+\sqrt{5})$ ✓

In Example 5 notice that by the Factor Theorem P has zeros $1, 3, -2, \sqrt{5}, -\sqrt{5}$. Also notice that the rational zeros $1, 3, -2$ were all on the original list of candidates supplied by the Rational Zeros Theorem. The zeros $\sqrt{5}, -\sqrt{5}$ are irrational and therefore were absent from the list.

Is there a way to do the work in Example 5 *faster* & MORE EFFICIENTLY? **YES!**

Example 5.1 Fully factor $x^5 - 2x^4 - 10x^3 + 16x^2 + 25x - 30$

• Solution: Letting $P(x) = x^5 - 2x^4 - 10x^3 + 16x^2 + 25x - 30$, candidate rational zeros for P are: $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$.

	1	-2	-10	16	25	-30	
1	1	-1	-11	5	30	0	← $x-1$ is a factor; new dividend: $x^4 - x^3 - 11x^2 + 5x + 30$
1	1	0	-11	-6	24		← There is no other $x-1$ factor
2	1	1	-9	-13	4		← $x-2$ is not a factor
3	1	2	-5	-10	0		← $x-3$ is a factor

Stop! So far we have $P(x) = (x-1)(x-3)(x^3 + 2x^2 - 5x - 10)$.

Factoring by grouping: $P(x) = (x-1)(x-3)(x+2)(x-\sqrt{5})(x+\sqrt{5})$ ✓ ka-ching!

That was fun — let's do it again. This time I'll add extra commentary on the right as the process unfolds.

Example 6 Fully factor $P(x) = 2x^5 - 17x^4 + 23x^3 + 88x^2 - 124x - 80$

• Solution: Possible rational zeros of P: $\dfrac{\pm 1, \pm 2, \pm 4, \pm 8, \pm 10, \pm 16, \pm 20, \pm 40, \pm 80}{\pm 1, \pm 2}$

$= \pm 1, \pm \frac{1}{2}, \pm 2, \pm 4, \pm 8, \pm 10, \pm 5, \pm 16, \pm 20, \pm 40, \pm 80$. As usual, start with 1 & go up...

	2	-17	23	88	-124	-80	
1	2	-15	8	96	-28	-108	← $x-1$ not a factor
2	2	-13	-3	82	40	0	← $x-2$ a factor; new dividend: $2x^4 - 13x^3 - 3x^2 + 82x + 40$
2	2	-9	-21	40	120		← $x-2$ not a factor of new dividend #1
4	2	-5	-23	-10	0		← $x-4$ a factor; new dividend #2: $2x^3 - 5x^2 - 23x - 10$. Notice 4 can't be a rational zero for n.d. #2! So don't try 4 again. Go on to 5...
5	2	5	2	0			← $x-5$ a factor; no need to make the quotient $2x^2 + 5x + 2$ n.d. #3 since it's easily factored.

So, $P(x) = (x-2)(x-4)(x-5)(2x^2 + 5x + 2)$

$= (x-2)(x-4)(x-5)(2x+1)(x+2)$ ✓

Another aid in finding (possible) rational zeros for any polynomial function P hinges on the eternal truth that P must be continuous. Thus, if $P(3)=8$ & $P(5)=12$, then for $3<x<5$ it must be that $P(x)$ assumes all the values between 8 & 12. That is, as x increases from 3 to 5, $P(x)$ will have to equal every real number between 8 & 12 at least once. An immediate consequence of this is the following.

Intermediate Value Theorem Let P be a polynomial function such that $P(x)$ has only real coefficients, and suppose that $a<b$. If $P(a)<0$ & $P(b)>0$ (or vice-versa), then there exists some number c such that $a<c<b$ and $P(c)=0$.

So if $P(2)=-5$ & $P(3)=1$, then for some value c between 2 & 3 we can be sure that $P(c)=0$. Put another way, there's no way a polynomial can go from being negative valued to positive valued (or vice-versa) without passing through 0.

Example 7 Fully factor $P(x)=6x^3-49x^2+88x+35$

• Solution: The possible rational zeros: $\dfrac{\pm1,\pm5,\pm7,\pm35}{\pm1,\pm2,\pm3,\pm6} = \pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm5, \pm\frac{5}{2},$ $\pm\frac{5}{3}, \pm\frac{5}{6}, \pm7, \pm\frac{7}{2}, \pm\frac{7}{3}, \pm\frac{7}{6}, \pm35, \pm\frac{35}{2}, \pm\frac{35}{3}, \pm\frac{35}{6}$.

There are few integer candidates, and the easiest are 1 & −1 ...

	6	−49	88	35	
1	6	−43	45	80	← So $P(1)=80>0$ } By I.V.T. $P(c)=0$ for some $-1<c<1$!
−1	6	−55	143	−108	← So $P(-1)=-108<0$ }
1/2	6	−46	65	67.5	← $P(-1)<0$ & $P(\frac{1}{2})>0$ implies $P(c)=0$ for some $-1<c<\frac{1}{2}$
−1/2	6	−52	114	−22	← $P(-\frac{1}{2})<0$ & $P(\frac{1}{2})>0$ implies $P(c)=0$ for $-\frac{1}{2}<c<\frac{1}{2}$
1/3	6	−47	$\frac{217}{3}$	$\frac{532}{9}$	← $P(-\frac{1}{2})<0$ & $P(\frac{1}{3})>0$ implies $P(c)=0$ for $-\frac{1}{2}<c<\frac{1}{3}$
−1/3	6	−51	105	0	← So $x+\frac{1}{3}$ is a factor.

Thus $P(x)=(x+\frac{1}{3})(6x^2-51x+105) = (x+\frac{1}{3})(3)(2x^2-17x+35)$
$= (3x+1)(2x^2-17x+35) = (3x+1)(2x-7)(x-5)$ ✓

Say that a polynomial function P is given by $P(x)=(x+2)(x-3)^2(x-5)^6$. It can be seen straightaway that the zeros of P are −2, 3, and 5. However because there are two $x-3$ factors, we say the zero 3 has multiplicity 2. Similarly, because there are six $x-5$ factors, the zero 5 has multiplicity 6. The stage is now set to state one more theorem that may raise eyebrows at tea parties but has little practical significance in a "real world" where stadiums are named after banks whose own names keep changing...

Zeros Theorem A polynomial function of degree $n\geq1$ has exactly n zeros, provided that a zero with multiplicity k is counted k times.

Example 8 Fully factor the nightmarish $P(x) = x^5 - 12x^4 + 58x^3 - 140x^2 + 168x - 80$

• Solution: Candidate rational zeros: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 16, \pm 20, \pm 40, \pm 80$

Mmm ikay, here goes...

	1	-12	58	-140	168	-80
1	1	-11	47	-93	75	-5
2	1	-10	38	-64	40	0
2	1	-8	22	-20	0	
2	1	-6	10	0		

So, $P(x) = (x-2)\overbrace{(x^4 - 10x^3 + 38x^2 - 64x + 40)}^{(A)}$. Proceed with (A) as the new dividend & note that 2 could also be a zero for (A).

So $P(x) = (x-2)(x-2)\overbrace{(x^3 - 8x + 22x - 20)}^{(B)}$. Let (B) be new dividend & note 2 could be a zero for (B).

So $P(x) = (x-2)(x-2)(x-2)(x^2 - 6x + 10)$

Next, solve $x^2 - 6x + 10 = 0$, which yields $x = \dfrac{6 \pm \sqrt{36 - 4(10)}}{2} = \dfrac{6 \pm \sqrt{-4}}{2} = 3 \pm i$

Factor Theorem then says $x^2 - 6x + 10 = [x - (3+i)][x - (3-i)]$

Finally, $P(x) = (x-2)^3 (x - 3 - i)(x - 3 + i)$ ✓ (As Mr. Burns would say, "Ex-cellent...")

Example 8 features a fifth-degree (or "quintic") polynomial, and it was found to have zeros 2, $3+i$, & $3-i$. The zero 2 has multiplicity 3. We could also call 2 a "triple root" for the equation $P(x) = 0$.

THE GRAPHS OF POLYNOMIAL FUNCTIONS

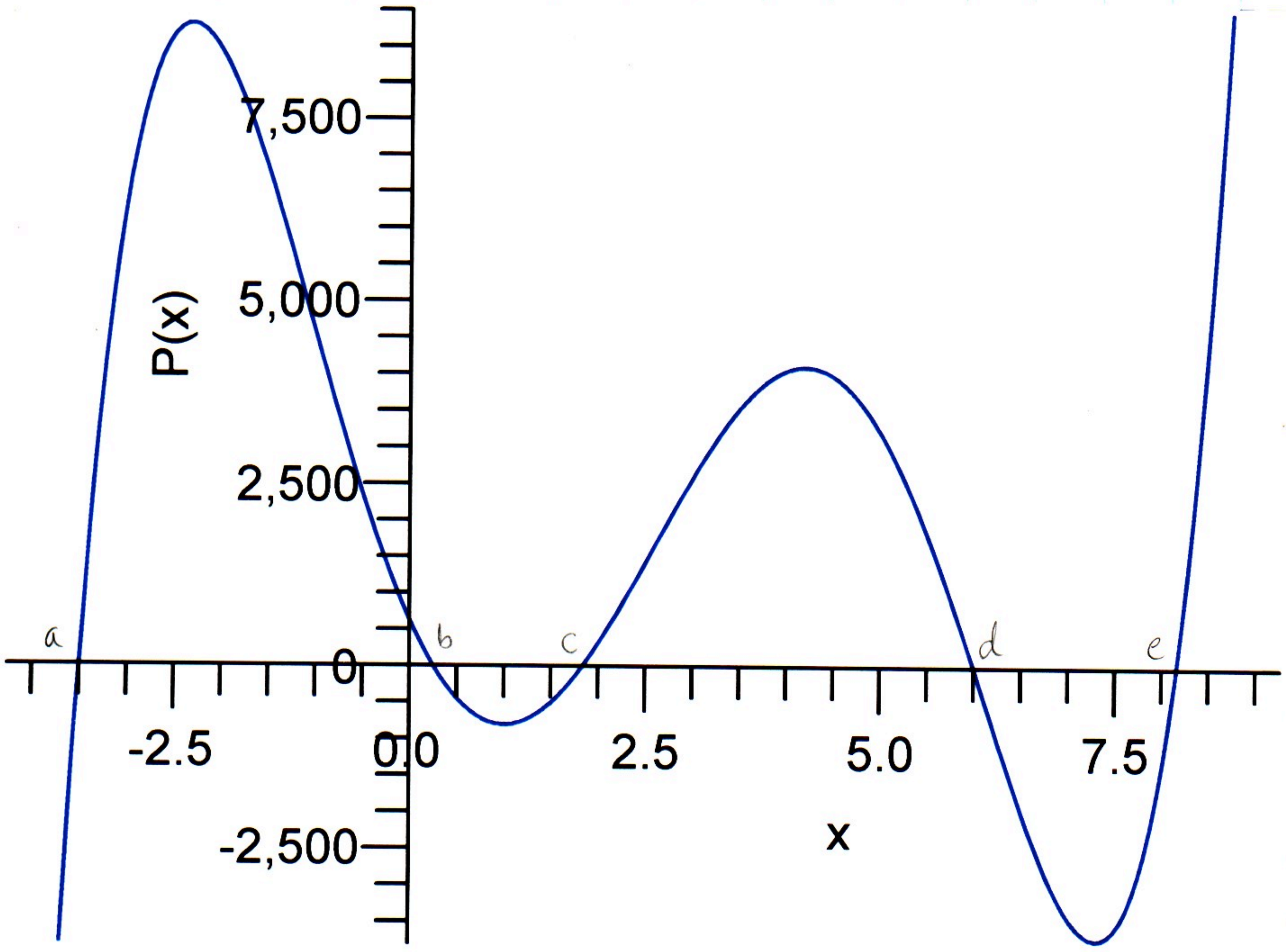
The Rational Zeros Theorem is sometimes a real business to do pleasure with. It often gives rise to dozens of possible rational zeros even in ideal situations, and it can turn out that none of the candidates pan out. On top of that it won't lift a finger to help turn up any irrational or complex zeros.

One way to find rational zeros with less trial and error is to consider the graph of a polynomial function. A graph will also lay bare any irrational zeros as well, although usually we'll have to settle for approximations rather than exact results.

We won't concern ourselves with the details of the numerical methods (read: algorithms) that are used to approximate real roots to nasty higher-degree polynomials. (One oldie but goodie is Newton's Method, very versatile but requiring some knowledge of calculus to develop.) Rather, we'll focus strictly on graphing using technology as an aid.

Example 9 Find the zeros of $P(x) = 8x^5 - 102x^4 + 177x^3 + 1342x^2 - 2865x + 630$

- Solution: So, anyone game to list all the possible rational zeros? Didn't think so. The graph is...



The zeros of P are at a, b, c, d, and e on the x-axis of the graph. Because $P(x)$ is a fifth degree polynomial, the Zeros Theorem tells us that P can have at most five distinct zeros, so we can be sure that no other zeros lie outside our graph's "window". Isn't that nice? The theorem has a use after all!

Any calculator can confirm that $a = -3.5$, $b = 0.25$, $c \approx 1.8377$, $d = 6$, and $e \approx 8.1623$. These, then, are our zeros. A complete factorization of $P(x)$ readily follows:

$P(x) = (x+3.5)(x-0.25)(x-1.8377)(x-6)(x-8.1623)$

But wait! More exact results could be achieved just by looking at the graph, making reasonable surmisals of some of the zeros, and then using synthetic divisions to confirm our guesses and corner the "less obvious" zeros. For instance, from the graph alone it looks like $a = -\frac{7}{2}$, $b = \frac{1}{4}$, and $d = 6$. (Note that the Rational Zeros Theorem would list these values among the candidates) Less clear are the values of c & e, but we'll deal with them later. Now...

	8	-102	177	1342	-2865	630	
$-\frac{7}{2}$	8	-130	632	-870	180	0	$P(-\frac{7}{2}) = 0$ confirmed. New divisor is $8x^4 - 130x^3 + 632x^2 - 870x + 180$
$\frac{1}{4}$	8	-128	600	-720	0		$P(\frac{1}{4}) = 0$ confirmed. New divisor: $8x^3 - 128x^2 + 600x - 720$
6	8	-80	120	0			$P(6) = 0$ confirmed. Quotient: $8x^2 - 80x + 120$

So $P(x) = (x+\frac{7}{2})(x-\frac{1}{4})(x-6)(8x^2-80x+120) = 8(x+\frac{7}{2})(x-\frac{1}{4})(x-6)(x^2-10x+15)$

Now, $x^2 - 10x + 15 = 0 \Rightarrow x = \dfrac{10 \pm \sqrt{100-4(15)}}{2} = \dfrac{10 \pm \sqrt{40}}{2} = \dfrac{10 \pm 2\sqrt{10}}{2} = 5 \pm \sqrt{10}$

So $P(x) = 8(x+\frac{7}{2})(x-\frac{1}{4})(x-6)[x-(5+\sqrt{10})][x-(5-\sqrt{10})]$

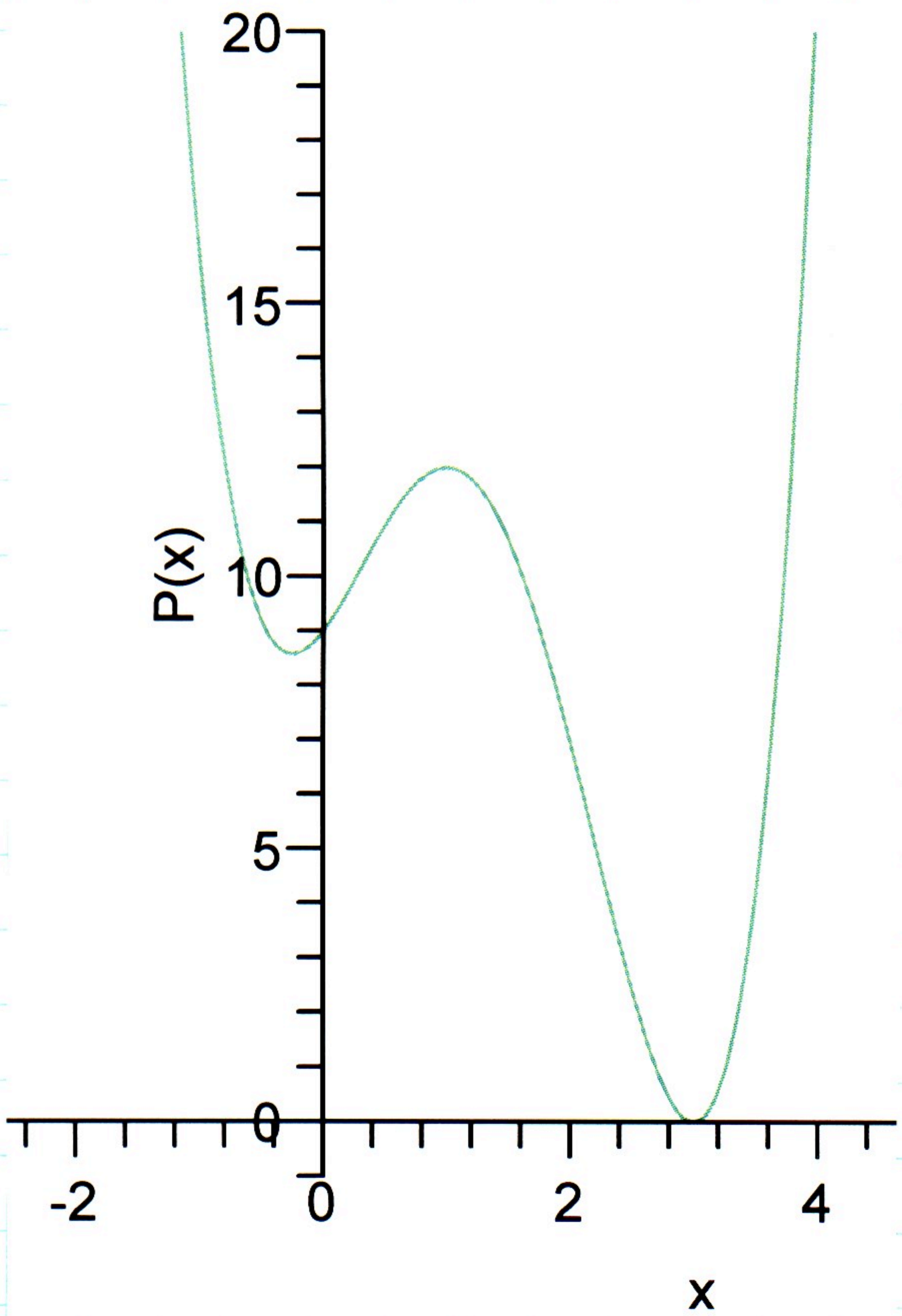
$= (2x+7)(4x-1)(x-6)[x-(5+\sqrt{10})][x-(5-\sqrt{10})]$ (so actually $c = 5-\sqrt{10}$ & $e = 5+\sqrt{10}$) ✓

Conjugate Zeros Theorem Let P be a polynomial function such that $P(x)$ has only real coefficients. If the complex number $a+bi$ is a zero of P, then its conjugate $a-bi$ is also a zero.

So a polynomial function with real coefficients can have complex zeros (as already seen in examples 4 and 8), but they must always come in (conjugate) pairs.

Example 10 Fully factor $P(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$

• Solution: Let's peruse the graph...



It seems highly unlikely that the graph of the function does anything else but shoot into the stratosphere outside our window.

It also appears that $P(3) = 0$, which can easily confirmed by synthetic division (using the Remainder Theorem) or just calculating directly: $3^4 - 5(3)^3 + 4(3)^2 + 3(3) + 9 = 0$ ✓

But the Zeros Theorem tells us that P should have four zeros in all (since $P(x)$ is a fourth degree polynomial). Since the graph of P doesn't seem to cross the x-axis anywhere but 3, we would have to conclude that the other three zeros must be complex numbers (complex zeros are not visible on graphs done in real-valued coordinate systems).

However, the Conjugate Zeros Theorem says complex zeros must come in pairs. For a fourth-degree polynomial, then, there must be 0, 2, or 4 complex zeros (i.e. an even number of them).

No matter what, then, the zero 3 must have multiplicity 2 at the very least, and possibly multiplicity 4. Grind through the synthetic division process, then...

	1	−5	4	3	9
3	1	−2	−2	−3	0 ← 3 is a zero once...
3	1	1	1	0 ← 3 is a zero twice!	

So $P(x) = (x-3)^2(x^2+x+1)$, where $x^2+x+1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

Finally, $P(x) = (x-3)^2\left[x - \left(\frac{-1+i\sqrt{3}}{2}\right)\right]\left[x - \left(\frac{-1-i\sqrt{3}}{2}\right)\right]$ ✓

Example 11 Find a polynomial function P of degree 3 with real coefficients & zeros of $1, -1, 0$ such that $P(2) = 3$.

• <u>Solution:</u> By the Factor Theorem in order for P to have $1, -1, \& \ 0$ as zeros, $P(x)$ must have $x-1$, $x+1$, & x as factors (respectively).

So far, then, we have $P(x) = c\,x(x-1)(x+1)$, where c is some as-yet undetermined constant.

Now, we need $P(2) = 3$, meaning $c \cdot 2(2-1)(2+1) = 3 \Rightarrow c \cdot 2(1)(3) = 3 \Rightarrow 6c = 3 \Rightarrow c = 1/2$.

Therefore $P(x) = \frac{1}{2}x(x-1)(x+1)$ is our function. To verify that $P(x)$ really is of degree 3 and has only <u>real</u> coefficients, it's easy to multiply everything out to get: $P(x) = \frac{1}{2}x^3 - \frac{1}{2}x$.

Example 12 Find a polynomial function P of least degree having only real coefficients and zeros $3+2i$, -1, and 2.

• <u>Solution:</u> For P to have zeros $3+2i, -1, \& \ 2$, $P(x)$ must have factors $x-(3+2i)$, $x+1$, & $x-2$ by the Factor Theorem.

However, if $3+2i$ is a zero of P without its conjugate $3-2i$ also being a zero, by the Conjugate Zeros Theorem P will <u>not</u> have only real coefficients. Since we're told that P <u>must</u> have "only real coefficients", then, we have no choice but to add $3-2i$ as an additional zero. This means putting in the extra factor $x-(3-2i)$. Now...

$$
\begin{aligned}
P(x) &= (x+1)(x-2)[x-(3+2i)][x-(3-2i)] \\
&= (x^2-x-2)[x^2-(3-2i)x-(3+2i)x+(3+2i)(3-2i)] \\
&= (x^2-x-2)(x^2-3x+2ix-3x-2ix+9-6i+6i-4i^2) \\
&= (x^2-x-2)(x^2-6x+9+4) \qquad (\text{recall } i^2=-1, \text{ so } -4i^2=+4) \\
&= (x^2-x-2)(x^2-6x+13) \\
&= x^4-6x^3+13x^2-x^3+6x^2-13x-2x^2+12x-26 \\
&= x^4-7x^3+17x^2-x-26 \qquad (\text{which has only real coefficients as we hoped \& wanted}) \checkmark
\end{aligned}
$$

Nothing has been said about the <u>domain</u> of a polynomial function simply because there's little to say: it's always all reals, just as with linear & quadratic functions (which are themselves polynomial functions).

Symbolically, if P is a polynomial function, then $\mathcal{D}om\, P = (-\infty, \infty)$. If P is of <u>odd</u> degree, then $\mathcal{R}an\, P = (-\infty, \infty)$ also. If P is of <u>even</u> degree $\mathcal{R}an\, P$ will vary (think back to quadratic functions, which are of degree 2).

3.5 — RATIONAL FUNCTIONS

Definition A function f is a rational function if $f(x) = P(x)/Q(x)$, where P and Q are polynomial functions

Examples: $f(x) = \frac{1}{x}$, $g(x) = \frac{x^2+1}{2x+3}$, & $h(x) = \frac{5x^3-4x+8}{10x^4-x}$ are rational functions.

Unlike polynomial functions, rational functions do not always have domains consisting of all real numbers. Values for x that result in a division by zero must be excluded. More rigorously, if f is a rational function such that $f(x) = P(x)/Q(x)$, then $Dom(f) = \{x \mid Q(x) \neq 0\}$

Example 1 Find the domain of each rational function.

a) $f(x) = 1/x$

• Solution: $Dom(f) = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ ✓

b) $g(x) = \frac{5x}{2x-9}$

• Solution: $Dom(g) = \{x \mid 2x-9 \neq 0\} = \{x \mid x \neq 9/2\} = (-\infty, 9/2) \cup (9/2, \infty)$ ✓

c) $h(x) = \frac{7-13x^2}{6x^2+1}$

• Solution: $Dom(h) = \{x \mid 6x^2+1 \neq 0\} = \{x \mid x^2 \neq -1/6\} = \mathbb{R} = (-\infty, \infty)$ ✓

(Recall that when determining the domain of a function only real numbers are considered. Since any real number x will satisfy $x^2 \neq -1/6$, every real number belongs in the domain of h. Also note that the polynomial in the numerator is irrelevant.)

d) $r(x) = \frac{22x^4 - 11x^3}{x^3+x^2-8x-8}$

• Solution: $Dom(r) = \{x \mid x^3+x^2-8x-8 \neq 0\}$

So what values of x result in $x^3+x^2-8x-8=0$? They'll be the values that will need to be excluded from the domain of r. To solve the equation factor by grouping...

$x^2(x+1) - 8(x+1) = 0 \Rightarrow (x+1)(x^2-8) = 0 \Rightarrow (x+1)(x-\sqrt{8})(x+\sqrt{8}) = 0 \Rightarrow$
$(x+1)(x-2\sqrt{2})(x+2\sqrt{2}) = 0 \Rightarrow x = -1, 2\sqrt{2}, -2\sqrt{2}$. Then...

$Dom(r) = \{x \mid x \neq -2\sqrt{2}, -1, 2\sqrt{2}\} = (-\infty, -2\sqrt{2}) \cup (-2\sqrt{2}, -1) \cup (-1, 2\sqrt{2}) \cup (2\sqrt{2}, \infty)$ ✓

(Note: I personally wouldn't recommend using interval notation in part d.)

A natural question arises: what does the graph of a rational function do when it encounters a value of x that is not in its domain? Sometimes it doesn't do anything but exhibit a "hole" in the curve...

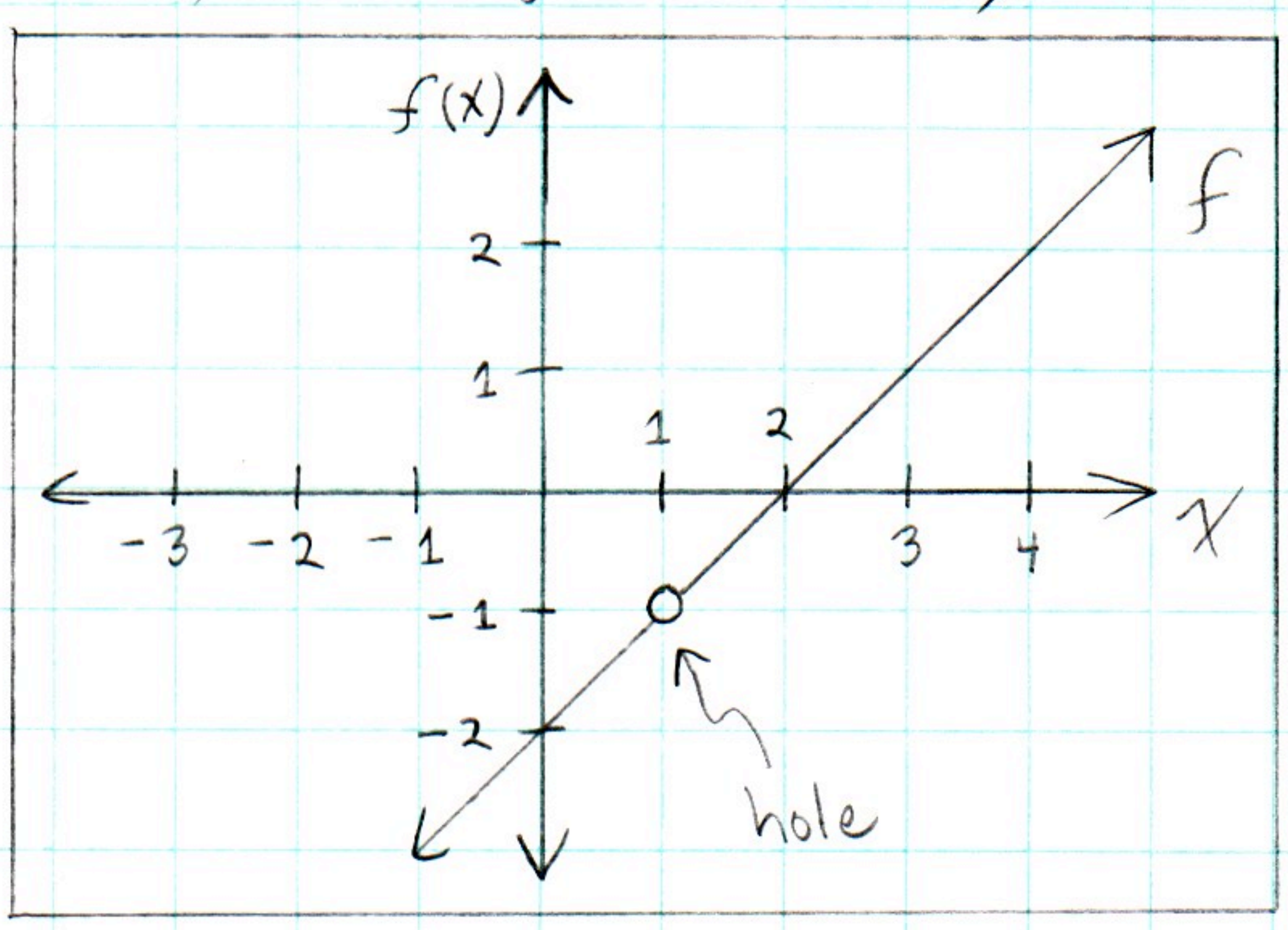
Example 2 Graph $f(x) = \dfrac{x^2-3x+2}{x-1}$

• Solution: $Dom(f) = \{x \mid x \neq 1\}$, so we cannot plug 1 into the function f or we'll break it! In other words when $x = 1$, $f(x) = f(1)$ simply does not exist because division by 0 is undefined.

But when $x \neq 1$ we have $x - 1 \neq 0$, so we can divide $x^2 - 3x + 2$ by $x - 1$ with carefree abandon. In fact things turn out rather nicely...

$$f(x) = \frac{x^2-3x+2}{x-1} = \frac{(x-2)(x-1)}{x-1} = x - 2 \quad \text{for} \quad x \neq 1.$$

Thus, so long as $x \neq 1$, we can graph f by graphing $y = x - 2$. At $x = 1$ leave a "hole"...



Now, if we graphed instead the function $g(x) = x - 2$, whose domain is all reals, the graph would be identical to the graph of f except that the "hole" would be filled in.

Moral of the story: the domain of a rational function should be ascertained from the function's original <u>unreduced</u> form!

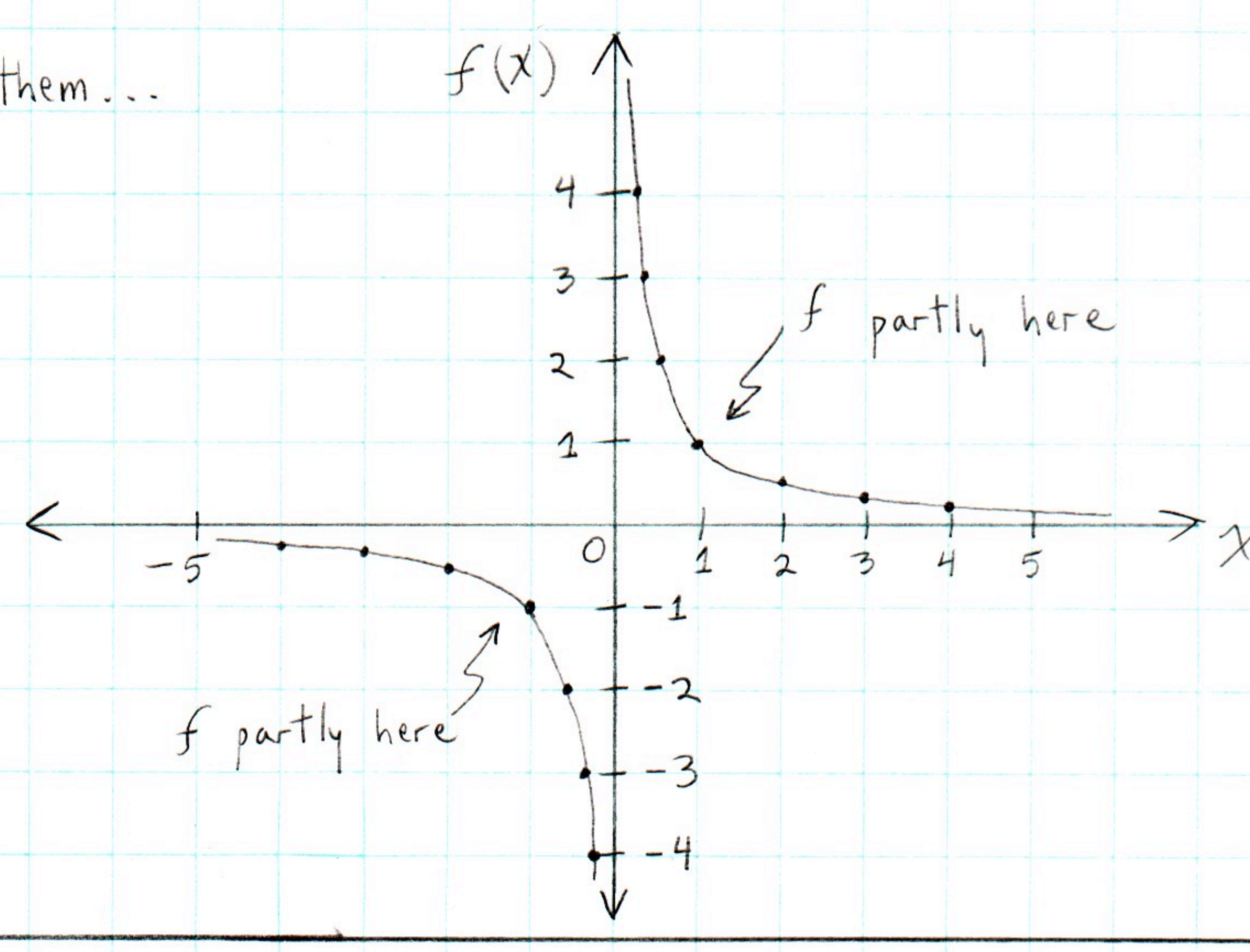
Henceforth we shall only consider rational functions that come to us already in reduced form. So if a rational function has no common factor in its numerator & denominator, what does its graph do when it encounters an x value that's not in its domain? Consider $f(x) = \frac{1}{x}$ ...

Example 3 Graph $f(x) = \dfrac{1}{x}$

• Solution: Gather up some points & plot them...

x	$f(x)$
3	1/3
2	1/2
1	1
1/2	2
1/3	3
1/10	10
1/100	100

x	$f(x)$
-3	-1/3
-2	-1/2
-1	-1
-1/2	-2
-1/3	-3
-1/10	-10
-1/100	-100



As we can see, as x approaches the "forbidden value" 0, some crazy stuff happens. Specifically as x approaches 0 from the right (i.e. goes down from 3 to 2, 1, ½, ⅓, etc.), which we can write symbolically as $x \to 0^+$, $f(x)$ shoots up toward infinity, which we can write as $f(x) \to \infty$.

And as x approaches 0 from the left (i.e. goes up from −3 to −2, −1, −½, −⅓, etc.), written $x \to 0^-$, we see that $f(x)$ plunges down toward negative infinity, written $f(x) \to -\infty$.

$f(0)$ is itself undefined, since $f(0) = \frac{1}{0}$. This means the graph of f never intersects the vertical line given by $x = 0$ (which is the y-axis). We call $x = 0$ a vertical asymptote.

But there's something else afoot. As x moves away from 0 in either the positive or negative direction, f peters out toward the horizontal line $y = 0$. We call this the end behavior of f, and the line $y = 0$ is a horizontal asymptote.

More specifically, as x goes up in value, the value of $f(x)$ approaches 0 from the right (symbolically: as $x \to \infty$, $f(x) \to 0^+$); and as x goes down in value, the value of $f(x)$ approaches 0 from the left (as $x \to -\infty$, $f(x) \to 0^-$). Refer to the graph.

A fairly decent sketch of a rational function can be made just by finding its asymptotes, determining its behavior in the vicinity of the asymptotes, and finding any x- & y-intercepts.

Example 4 Sketch $f(x) = \dfrac{x}{2x-7}$

• Solution: $\dfrac{x}{2x-7}$ is already a reduced fraction, so there'll be no surprises. Proceed...

i) Find any vertical asymptotes: Since $Dom(f) = \{x \,/\, 2x-7 \neq 0\} = \{x \mid x \neq 7/2\}$, the vertical line $x = 7/2$ will be a vertical asymptote (V.a.)

ii) Find any horizontal asymptote: Here's a trick: divide num. & den. by highest power of x in den., which is x itself, to get $f(x) = \dfrac{x}{2x-7} \cdot \dfrac{1/x}{1/x} = \dfrac{1}{2 - 7/x}$. Now as $x \to \pm\infty$, we can see that $7/x \to 0$.
Thus $f(x) = \dfrac{1}{2-7/x} \to \dfrac{1}{2-0} = \dfrac{1}{2}$ as $x \to \pm\infty$ (the end behavior of f), telling us that $y = 1/2$ will be a horizontal asymptote.

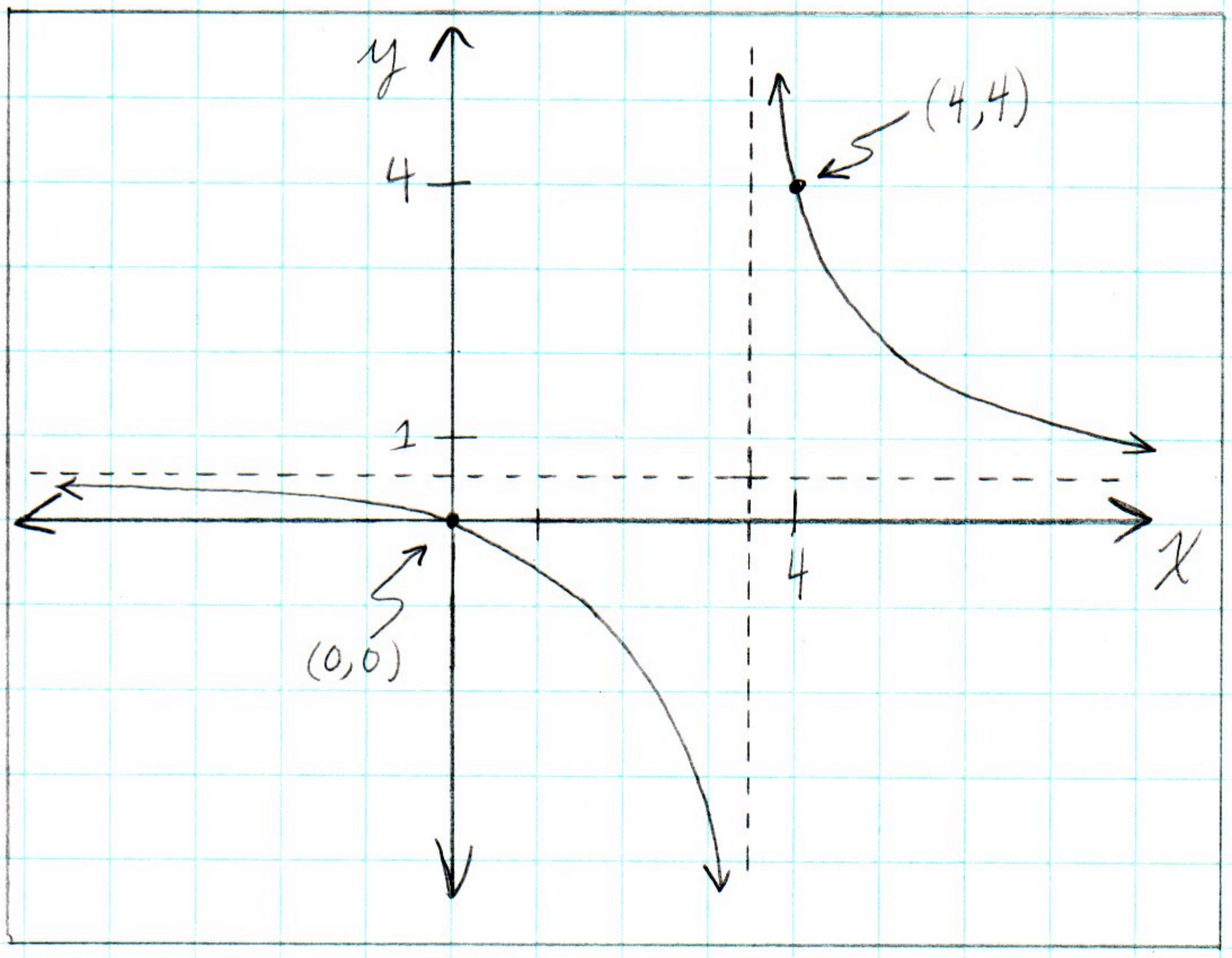
iii) Find any x-intercepts: Setting $y = \dfrac{x}{2x-7}$, the only way we'll get $y = 0$ is if the num. $x = 0$. That is, when $x = 0$ we obtain $y = \dfrac{0}{2(0)-7} = 0$, and so $(0,0)$ is an x-intercept.

iv) Find any y-intercept: A function can only ever have at most one y-intercept, & $(0,0)$ is it.

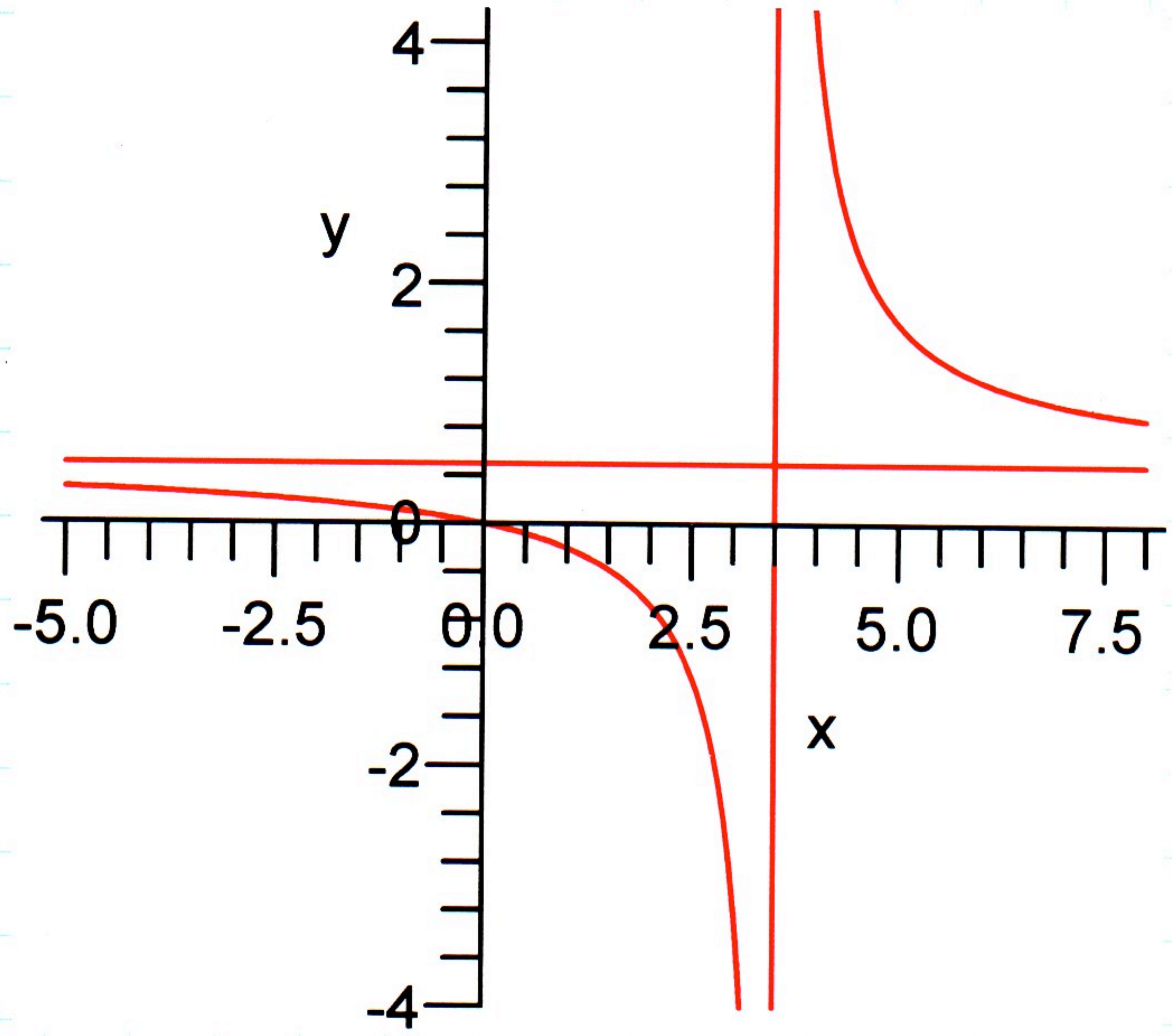
v) Does f ever cross the h.a.? Since the h.a. is $y = 1/2$ & f is given by $y = \frac{x}{2x-7}$, we have to find the x value at which $\frac{x}{2x-7} = \frac{1}{2}$. Thus... $2x = 2x - 7$, leading to $0 = -7$. As this is impossible, f does not cross the h.a.

vi) Where's f w.r.t. h.a.? To the left of v.a. $x = 7/2$ we know f passes through $(0,0)$, which is below the h.a. $y = 1/2$. To the right of v.a. $f(4) = \frac{4}{2(4)-7} = 4$, so f passes through $(4,4)$ which is above h.a.

We have enough for a sketch...



Actual graph...



Example 5 Sketch $f(x) = \frac{3x-1}{x^2-x-6}$

• Solution:

i) Find v.a.'s: $Dom(f) = \{x \mid x^2-x-6 \neq 0\} = \{x \mid (x+2)(x-3) \neq 0\} = \{x \mid x \neq -2, 3\}$, so v.a.'s are $x = -2$ & $x = 3$.

ii) Find h.a.: Divide num. & den. by x^2 (the highest power of x in den.):

$f(x) = \frac{3x-1}{x^2-x-6} \cdot \frac{1/x^2}{1/x^2} = \frac{3/x - 1/x^2}{1 - 1/x - 6/x^2}$, so as $x \longrightarrow \pm\infty$ we find that

$f(x) \rightarrow \frac{0-0}{1-0-0} = \frac{0}{1} = 0$, meaning $y = 0$ is h.a.

iii) x-intercepts: For $y = \frac{3x-1}{x^2-x-6}$, let $y = 0$ & solve $\frac{3x-1}{x^2-x-6} = 0$. This implies that $3x - 1 = 0$, and so $x = 1/3$. Hence $(1/3, 0)$ is an x-intercept.

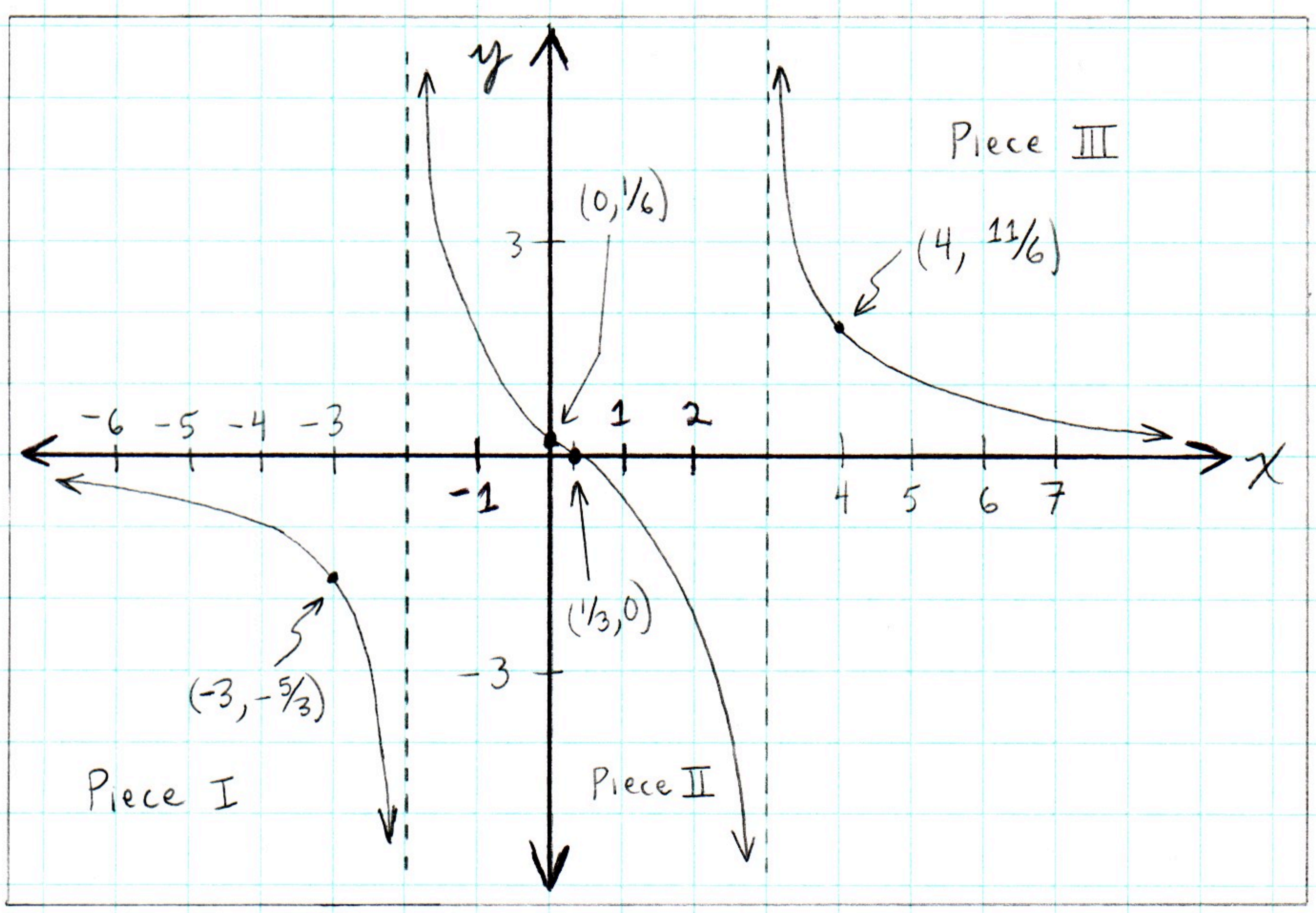
iv) y-intercept: For $y = \frac{3x-1}{x^2-x-6}$, let $x = 0$ & solve $y = \frac{3(0)-1}{0^2-0-6} = \frac{1}{6}$. Hence $(0, 1/6)$ is the y-intercept.

v) <u>Does f cross h.a.?:</u> Since h.a. is $y=0$, we already know it does. After all, $y=0$ is the x-axis, & we know that $(1/3, 0)$ is an x-intercept.

vi) <u>Where's f w.r.t. h.a.?:</u> The two v.a.'s break our graph into three pieces: Piece I is to the left of v.a. $x=-2$, Piece II is between v.a.'s $x=-2$ & $x=3$, & Piece III is to the right of v.a. $x=3$.

- Piece I: $f(-3) = \dfrac{3(-3)-1}{(-3+2)(-3-3)} = \dfrac{-10}{(-1)(-6)} = \dfrac{-10}{6} = -\dfrac{5}{3}$, <u>below</u> h.a. $y=0$
- Piece III: $f(4) = \dfrac{3(4)-1}{(4+2)(4-3)} = \dfrac{11}{(6)(1)} = \dfrac{11}{6}$, <u>above</u> h.a. $y=0$
- Piece II: Beware, since f <u>intersects</u> h.a. $y=0$ at $x=1/3$. Pick a value for x such that $-2 < x < 1/3$ (so x will be between v.a. $x=-2$ & the pt. $(1/3, 0)$ where f meets h.a.). Nice value: 0. We already know from (iv) that $f(0) = 1/6$, <u>above</u> h.a. $y=0$. Next, pick a value for x such that $1/3 < x < 3$ like 2, & note that $f(2) = -5/4$, <u>below</u> h.a. $y=0$.

We know enough for a sketch...



Comments:

- Piece I must stay below $y=0$ & left of $x=-2$, yet it must taper toward both lines. So ↖↘ is all that's possible.
- Piece II must stay above $y=0$ for $x < 1/3$, yet taper toward $x=-2$ without crossing. Piece II must stay below $y=0$ for $x > 1/3$, yet taper toward $x=3$.
- Piece III must stay above $y=0$ & right of $x=3$. Only ↖↘ will work.

Graphs of rational functions <u>never</u> touch a vertical asymptote, but as we've already seen they <u>sometimes</u> intersect with a horizontal asymptote. Also, there can be any number of x-intercepts (none, one, or many), but there's never more than one y-intercept (and sometimes none).

In general, if $f(x) = \dfrac{P(x)}{Q(x)}$ & the degree of $P(x)$ is less than the degree of $Q(x)$, then $y=0$ will be the h.a. If degree of $P(x)$ = degree of $Q(x)$, $y = \dfrac{\text{leading coefficient of } P(x)}{\text{leading coefficient of } Q(x)}$ will be the h.a.

If deg. of $P(x)$ is <u>1 greater than</u> deg. of $Q(x)$, the graph of f will have a <u>slant asymptote</u> instead of a horizontal asymptote.

Example 6 Sketch $f(x) = \dfrac{2x^2+x-6}{x-1}$

• Solution:

i) Find V.a.'s: $\mathcal{D}om(f) = \{x \mid x \neq 1\}$, so $x = 1$ is a V.a.

ii) Find slant asymptote: Since $deg(2x^2+x-6) = 2$ & $deg(x-1) = 1$, we see that the degree of the polynomial in the numerator is 1 greater than the degree of the poly. In the denominator, there will be a slant asymptote instead on a h.a. To find it, use long division:

$$\begin{array}{r} 2x+3 \\ x-1 \overline{\smash{)}\,2x^2+x-6} \\ \underline{2x^2-2x} \\ 3x-6 \\ \underline{3x-3} \\ 3 \end{array} \Longrightarrow f(x) = 2x+3+\frac{3}{x-1}.$$

Now, notice that as $x \to \pm\infty$, we have $\frac{3}{x-1} \to 0$

So, as $x \to \pm\infty$, $f(x) \to 2x+3$.

Slant asymptote is $y = 2x+3$.

iii) x-intercepts: When $y=0$ we get $\frac{2x^2+x-6}{x-1} = 0$, leading to $2x^2+x-6=0$ and finally $x = -2, 3/2$. So x-intercepts are at $(-2,0)$ & $(3/2, 0)$.

iv) y-intercepts: When $x=0$ we get $y = \frac{2(0)^2+0-6}{0-1} = \frac{-6}{-1} = 6$. So $(0,6)$ is the y-intercept.

v) Does f cross slant asymptote? The function is $y = \frac{2x^2+x-6}{x-1}$ while the s.a. is $y=2x+3$. The question is this: is the a value for x for which we have $\frac{2x^2+x-6}{x-1} = 2x+3$? Solve: $2x^2+x-6 = (2x+3)(x-1) \Rightarrow 2x^2+x-6 = 2x^2+x-3 \Rightarrow -6=-3$. Impossible! Our answer: no.

vi) Where's f w.r.t. s.a.? The v.a. $x=1$ will break the graph of f into two pieces. To the left of $x=1$ we have the test point $(-2,0)$, meaning $f(x)=0$ when $x=-2$. As for the s.a. given by $y=2x+3$, we get $y=-1$ when $x=-2$. So, when $x=-2$ the graph of f is at $y=0$ while the s.a. is at $y=-1$, meaning f is above s.a.

To the right of the v.a. $x=1$ we have the test point $(3/2, 0)$, meaning $f(x)=0$ when $x=3/2$. Meanwhile on the s.a. we get $y=6$ when $x=3/2$. Conclusion: f is below s.a.

Notice that the six steps above are nearly identical to the six steps of previous examples, except any mention of an h.a. has been replaced by mention of a s.a.

We have enough for a sketch (next page).

Since I have some extra space down here, I'd just like to say the weather is way too bitterly cold for my liking.

Graph must approach asymptotes without (in this example) crossing them. Make sure your sketch passes the "Vertical Line Test"!