2.2 — FUNCTIONS

A <u>relation</u> is a set of ordered pairs. Some simple examples:

(i) $\{(0,0)\}$

(ii) $\{(2,4), (-3,5), (3/8, \sqrt{2}), (\pi, 7)\}$

(iii) $\{(3,9), (-1,9), (8,0), (-8,8), (-13.72, 256)\}$

(iv) $\{(-7,2), (1,0.3), (2,-7), (1, 0.29)\}$

(v) $\{(4,4), (5,-6), (2,12), (5,-6), (8,-6), (-4,4)\}$

(vi) $\{(a,b), (x,w), (\star, \#), (\delta, \beta)\}$

Also defining a relation is any equation having two variables, like $2x-9y=14$ & $|y|+2=x$. In each case the relation is just the solution set of the equation. The relation defined by $2x-9y=14$ contains ordered pairs such as $(7,0)$, $(0,-14/9)$, and $(16,2)$, since these pairs of values (x value 1st, y value 2nd) are solutions to the equation. In fact $2x-9y=14$ is a relation containing an infinite number of ordered pairs!

A <u>function</u> is a relation such that no two distinct ordered pairs has the same first coordinate value. Above, the relations (i), (ii), (iii), (vi) are all functions. But (iv) is <u>not</u> a function since it contains $(1,0.3)$ & $(1,0.29)$, which are distinct pairs having 1 as a first coordinate value. In contrast (v) <u>is</u> a function despite the repeated occurrence of $(5,-6)$; the thing of it is that the two $(5,-6)$ pairs are not <u>distinct</u> — they're identical twins. In fact, the set $\{(4,4), (5,-6), (2,12), (8,-6), (-4,4)\}$ is considered equal to the set (v), since sets are blind to order and repetition by design. For example, $\{a,b,c\} = \{a,a,b,b,b,c,c,c,c\} = \{c,a,b\}$.

The equation $2x-9y=14$ above defines a function. But while all functions are relations not all relations are functions, as (v) above showed us. The equation $|y|+2=x$ is not a function if we adhere to the convention of expressing ordered pairs such that the value of x is 1st & the value of y is 2nd. Let $x=5$. Then $|y|+2=5 \Rightarrow |y|=3 \Rightarrow y=\pm 3$, so $(5,-3)$ & $(5,3)$ are both solutions to the equation, and therefore both pairs belong to the relation given by the solution set. Since $(5,-3)$ & $(5,3)$ are distinct ordered pairs with the same first coordinate value, the solution set is not a function.

The <u>domain</u> of a relation is the set of all first coordinate values found among its ordered pairs. For relation (ii) the domain is $\{2, -3, 3/8, \pi\}$; for (vi), the domain is $\{a, x, \star, \delta\}$.

The <u>range</u> of a relation is the set of all second coordinate values found among its ordered pairs. For (ii), range is $\{4, 5, \sqrt{2}, 7\}$; for (vi), the range is $\{b, w, \#, \beta\}$.

Let f denote the function defined by $2x-9y=14$. Since x is by convention the first coordinate value of each ordered pair belonging to the solution set of $2x-9y=14$, the domain of f (write $\mathrm{Dom}\, f$ for short) consists of all the possible values x can assume without wrecking the equation (e.g. by causing a division by zero to occur). It should be clear that any real number substituted for x in $2x-9y=14$ will result in a perfectly well-behaved equation that can be solved for y, and thus the domain of f consists of all real numbers. Write $\mathcal{D}om\, f = \mathbb{R}$ or $\mathcal{D}om\, f = (-\infty, \infty)$, where $\mathbb{R}$ & $(-\infty,\infty)$ are both symbols that represent the set of real numbers.

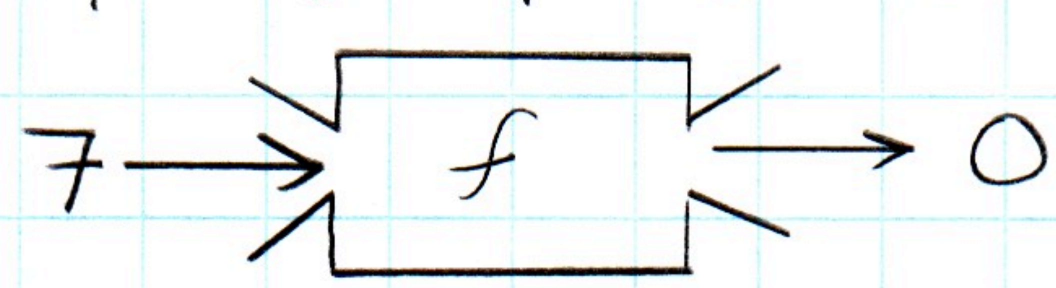
Let g denote the relation given by the solution set of $|y|+2=x$. Subtract 2 from both sides to get $|y| = x-2$, and ponder x's predicament. If $x=1$, say, we get $|y|=-1$, which is <u>impossible</u>! The absolute value of a number can never be negative, so $|y| \geq 0$ must always hold. This means we need x such that $x-2 \geq 0$, and hence $x \geq 2$ must always hold. Put succinctly, $\mathcal{D}om\, g = [2,\infty)$

Now what about the range of f, or $\mathcal{R}an\, f$? Since y is by convention the second coordinate value of each ordered pair belonging to the solution set of $2x-9y=14$, $\mathcal{R}an\, f$ will consist of all real numbers y can assume without wreaking havoc within the equation. You should be able to convince yourself that $2x-9y=14$ permits y to be any real number without reservations. Thus $\mathcal{R}an\, f = (-\infty,\infty)$.

What about $\mathcal{R}an\, g$, where g equals the solution set to $|y|+2=x$? Well if $y \geq 0$, we find that $|y|+2=x \Rightarrow y+2=x$, a nice equation that yields a nice value for x. If $y<0$, then $|y|+2=x \Rightarrow -y+2=x$, which also gives a nice value for x (here "nice" means real-valued). Since for any $y \geq 0$ or $y<0$ everything turns out roses, y is allowed to be any real number. Thus $\mathcal{R}an\, g = (-\infty,\infty)$.

<u>Functions as Machines</u>

A useful interpretation of a function is that of a "machine". Take f, given by $2x-9y=14$. Solve for y to get $y = \frac{2}{9}x - \frac{14}{9}$. One solution is $x=7$ & $y=0$, or $(7,0)$. Since the x value 7 comes <u>1st</u> & the y value 0 <u>2nd</u> in $(7,0)$, it's convenient to think of 7 as an <u>input</u> & 0 as an <u>output</u> for the function f. Here's the idea: 7 is put <u>into</u> the equation $y=\frac{2}{9}x-\frac{14}{9}$ in place of x, and 0 comes <u>out of</u> the equation as a value for y; but we've given the equation the designation f (that is, f is in essence the "name" of the equation), so what we're really doing is putting 7 <u>into f</u> and getting 0 out of it:



There's a notation that's used, called <u>function notation</u>, that captures the spirit of the "machine diagram" above. The function f is defined by the equation $y = \frac{2}{9}x - \frac{14}{9}$, and to indicate that an input of 7 yields an output of 0, we write $f(7)=0$. Note: $f(7)$ does not mean "f times 7", it means 7 is being put <u>inside f</u>; think of the parentheses as representing the outer walls of the machine f. In general we write: $f(x)=y$, meaning an x value is put into f (our equation), and a y value comes out.

In this way we can eliminate the need for y altogether. Since $f(x)=y$ & $y=\frac{2}{9}x-\frac{14}{9}$, we can cut out the middle-man and write $f(x)=\frac{2}{9}x-\frac{14}{9}$. This is ideal, since now but inputs and outputs of the function f are expressed in terms of the same variable x. Whatever x happens to be in $f(x)$, the same value appears in $\frac{2}{9}x-\frac{14}{9}$ ...

$$f(7) = \frac{2}{9}(7)-\frac{14}{9} = \frac{14}{9}-\frac{14}{9} = 0$$
$$f(0) = \frac{2}{9}(0)-\frac{14}{9} = 0-\frac{14}{9} = -\frac{14}{9}$$
$$f(-3) = \frac{2}{9}(-3)-\frac{14}{9} = -\frac{6}{9}-\frac{14}{9} = -\frac{20}{9}$$
$$f(c+2) = \frac{2}{9}(c+2)-\frac{14}{9} = \frac{2}{9}c+\frac{4}{9}-\frac{14}{9} = \frac{2}{9}c-\frac{10}{9}$$
$$f\left(\frac{a^2}{2}-5a+4\right) = \frac{2}{9}\left(\frac{a^2}{2}-5a+4\right)-\frac{14}{9} = \frac{a^2}{9}-\frac{10}{9}a+\frac{8}{9}-\frac{14}{9} = \frac{1}{9}a^2-\frac{10}{9}a-\frac{2}{3}$$

Recall set-builder notation, which is used to define sets precisely. The general structure is this:

$$\left\{ \boxed{} \;\Big|\; \boxed{} \right\}$$

Read as "The set of all [insert an algebraic expression in 1st box] such that [state conditions for membership in 2nd box]". The vertical bar, $|$, is read as "such that". Examples...

$\{x \mid 2<x<7\}$ is "The set of all x such that x is greater than 2 & less than 7", so this set contains all real numbers between 2 & 7. Note $\{x \mid 2<x<7\}$ is the same as the interval $(2,7)$, though interval notation is not very desirable anymore since we're working a lot with ordered pairs. Note $(2,7)$ could also mean "$x=2$ & $y=7$" rather than "all reals between 2 & 7", so we have to be careful here!

$\{n \mid n\in\mathbb{Z} \text{ \& } n>4\}$ is "The set of all n such that n is an element of $\mathbb{Z}$ and n is greater than 4." The symbol $\mathbb{Z}$ stands for the set of integers (i.e. $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$), so what we really have is "The set of all n such that n is an integer and $n>4$." Hence $\{n \mid n\in\mathbb{Z} \text{ \& } n>4\} = \{5,6,7,8,\ldots\}$, the set of all integers from 5 and up.

$\{2k \mid k\in\mathbb{Z} \text{ \& } k\geq 1\}$ is "The set of all $2k$ such that k is an integer and $k\geq 1$." Thus we have $k=1,2,3,\ldots$ (integers greater than or equal to 1), and therefore we have $2k=2,4,6,\ldots$ Simply put, $\{2k \mid k\in\mathbb{Z} \text{ \& } k\geq 1\}=\{2,4,6,\ldots\}$, the set of even counting numbers.

With function notation & set-builder notation we are in a position to define function, domain, and range from a "machine" standpoint more precisely. Recall $\mathbb{R}$ represents the set of real numbers...

For any relation f, the domain of f is given by $\mathcal{D}om\, f = \{x \mid x\in\mathbb{R} \text{ \& } f(x)\in\mathbb{R}\}$.

For any relation f, the range of f is given by $\mathcal{R}an\, f = \{y \mid y=f(x) \text{ for some } x\in\mathcal{D}om\, f\}$.

Since it will be assumed throughout this section (2.2) and the next (2.7) that x & y represent real numbers, we can shorten the definition of $\mathcal{D}om\, f$ to $\mathcal{D}om\, f = \{x \mid f(x) \in \mathbb{R}\}$. That is, $\mathcal{D}om\, f$ consists of all (real) values which, when substituted for x, result in $f(x)$ being real valued.

The definition of $\mathcal{R}an\, f$ can also be shortened by cutting out the middle-man y and writing $\mathcal{R}an\, f = \{f(x) \mid x \in \mathcal{D}om\, f\}$; that is, $\mathcal{R}an\, f$ equals the set of all values $f(x)$ takes on for $x \in \mathcal{D}om\, f$.

So truly, $\mathcal{D}om\, f$ consists of all possible inputs for f, and $\mathcal{R}an\, f$ consists of all possible outputs. This holds for any relation f, whether f is a function or not.

Finally, a relation f is a <u>function</u> if, for each $x \in \mathcal{D}om\, f$, $f(x)$ assumes a single value. That is, there is only one real number that $f(x)$ can possibly equal.

Example 1 Let f be the relation defined by the equation $y^2 - x - 5 = 0$. Find the domain & range of f. Is f a function?

- <u>Solution:</u> Solve for y first: $y^2 = x+5 \Rightarrow \sqrt{y^2} = \sqrt{x+5} \Rightarrow |y| = \sqrt{x+5} \Rightarrow y = \pm\sqrt{x+5}$.

Replace y by $f(x)$, so we have $f(x) = \pm\sqrt{x+5}$.

Now, $\mathcal{D}om\, f = \{x \mid f(x) \in \mathbb{R}\} = \{x \mid \pm\sqrt{x+5} \in \mathbb{R}\}$. In order for $\pm\sqrt{x+5}$ to be <u>real valued</u>, we need $x+5$ to not be negative. Thus $\mathcal{D}om\, f = \{x \mid x+5 \geq 0\} = \{x \mid x \geq -5\} = [-5, \infty)$ ✓

Next, $\mathcal{R}an\, f = \{f(x) \mid x \in \mathcal{D}om\, f\} = \{\pm\sqrt{x+5} \mid x \in [-5,\infty)\} = \{\pm\sqrt{x+5} \mid x \geq -5\}$. This set actually equals $\mathbb{R}$ (i.e. it includes all real numbers). Here's how to verify this: let r be any real number. Can we find some $x \geq -5$ such that $r = \sqrt{x+5}$ or $r = -\sqrt{x+5}$? If $r \geq 0$, let $r = \sqrt{x+5}$; then $r^2 = x+5 \Rightarrow x = r^2 - 5$, which is a value ≥ -5 that will work: $\sqrt{x+5} = \sqrt{(r^2-5)+5} = \sqrt{r^2} = |r| = r$. If $r < 0$, let $r = -\sqrt{x+5}$; then $r^2 = (-\sqrt{x+5})^2 = x+5 \Rightarrow x = r^2 - 5$ again, and when we put this into $-\sqrt{x+5}$ we get: $-\sqrt{(r^2-5)+5} = -\sqrt{r^2} = -|r| = -(-r) = r$ (recall $r < 0$ means $|r| = -r$). So we can always find some $x \geq -5$ for which either $\sqrt{x+5}$ or $-\sqrt{x+5}$ will equal r, no matter what real number r happens to be. Hence $\mathcal{R}an\, f = \{\pm\sqrt{x+5} \mid x \geq -5\} = \mathbb{R}$. ✓

Is f a function? No! Notice that $f(4) = \pm\sqrt{4+5} = \pm\sqrt{9} = \pm 3$. So here we have $4 \in \mathcal{D}om\, f = [-5, \infty)$, and yet $f(4)$ can equal <u>two</u> different real numbers: 3 & -3. ∎

In general finding the range of a relation is a serious pain — as Example 1 clearly shows. Do not be too concerned if you did not fully understand how the range in Example 1 was determined.

Example 2 Let g be the relation defined by $g(x) = x^2 - 7$. Find $\mathscr{Dom}\, g$, $\mathscr{Ran}\, g$, and determine whether g is a function.

- Solution: $\mathscr{Dom}\, g = \{x \mid g(x) \in \mathbb{R}\} = \{x \mid x^2 - 7 \in \mathbb{R}\} = \mathbb{R}$, since it's clear that whenever x is real, x^2-7 must be real also. (This stems from one of the axioms which define the real numbers.)

As for $\mathscr{Ran}\, g$, note that $x^2 \geq 0$ for any $x \in \mathbb{R}$, and thus $g(x) = x^2 - 7 \geq -7$; that is, $g(x)$ can only assume values greater than or equal to -7. Can $g(x)$ attain <u>any</u> value ≥ -7, or just some? Well, let r be any real number ≥ -7. We wish to find some x in the **domain** of g for which $g(x) = r$ (that is, an input x which yields an output r). But behold: $g(x) = r \Rightarrow x^2 - 7 = r \Rightarrow x^2 = r + 7 \Rightarrow x = \pm\sqrt{r+7}$. Oh ho: since $r \geq -7$ holds <u>by assumption</u>, we know that $r + 7 \geq 0$ and therefore $\sqrt{r+7}$ is <u>real</u> valued. Put $\sqrt{r+7}$ in for x in $g(x) = x^2 - 7$: $g(\sqrt{r+7}) = (\sqrt{r+7})^2 - 7 = (r+7) - 7 = r$, just as we desired! That is, $\sqrt{r+7}$ is a real number, so $\sqrt{r+7} \in \mathscr{Dom}\, g$, & $g(\sqrt{r+7}) = r$. Conclusion: for any $r \geq -7$ there exists some $x \in \mathscr{Dom}\, g$ such that $g(x) = r$. It follows that g can yield as output <u>any</u> real number from -7 on up. $\mathscr{Ran}\, g = \{y \mid y \geq -7\} = [-7, \infty)$.

g is a function since, for any x value we put into it, only <u>one</u> value for $g(x)$ results: $x^2 - 7$. ∎

By similar reasoning we find that h given by $h(x) = x^4 + \frac{2}{3}$ is a function with $\mathscr{Dom}\, h = \mathbb{R}$ and $\mathscr{Ran}\, h = [2/3, \infty)$. Note how the range is figured: x goes <u>into</u> h, and $x^4 + \frac{2}{3}$ comes <u>out</u>; but $x^4 \geq 0$ implies that $x^4 + 2/3 \geq 2/3$ must be the case. Once you can convince yourself that $x^4 + 2/3$ can attain <u>any</u> value from $2/3$ on up, you can bypass much of the long-winded argumentation presented in Example 2 to conclude that $\mathscr{Ran}\, h = [2/3, \infty)$.

Usually it's more important to determine the domain of a function than the range, so that's where we'll concentrate our efforts in the next few examples.

Example 3 Let φ be given by $\varphi(x) = \sqrt{\sqrt{x} - 9}$. Find $\mathscr{Dom}\, \varphi$.

- Solution: $\mathscr{Dom}\, \varphi = \{x \mid \varphi(x) \in \mathbb{R}\} = \{x \mid \sqrt{\sqrt{x}-9} \in \mathbb{R}\} = \{x \mid \sqrt{x} - 9 \geq 0\}$, since in order for $\sqrt{\sqrt{x}-9}$ to be real valued, the radicand $\sqrt{x} - 9$ must <u>not</u> be negative. Now, $\sqrt{x} - 9 \geq 0 \Rightarrow \sqrt{x} \geq 9 \Rightarrow (\sqrt{x})^2 \geq 9^2 \Rightarrow x \geq 81$.

Thus $\mathscr{Dom}\, \varphi = \{x \mid x \geq 81\} = [81, \infty)$. ∎

In general, when finding the domain of a function, we start with $\mathbb{R}$ (all reals) and "weed out" any numbers that, when put into the function, results in division by zero or an even root of a negative number.

Example 4 Find $\mathcal{D}om\ s$ if: $s(x) = \dfrac{x}{x^2+5x-24}$

• <u>Solution:</u> Note $s(x) = \dfrac{x}{(x+8)(x-3)}$. The only input s can't handle are -8 & 3, which would result in division by 0.

$\mathcal{D}om\ s = \{x \mid s(x) \in \mathbb{R}\} = \{x \mid \frac{x}{(x+8)(x-3)} \in \mathbb{R}\} = \{x \mid (x+8)(x-3) \neq 0\} = \{x \mid x \neq -8, 3\}$.

Also acceptable: $\mathcal{D}om\ s = (-\infty,-8) \cup (-8,3) \cup (3,\infty)$. ∎

Example 5 Find $\mathcal{D}om\ F$ if $F(x) = \sqrt{x^2+5x-24}$

• <u>Solution:</u> $\mathcal{D}om\ F = \{x \mid F(x) \in \mathbb{R}\} = \{x \mid \sqrt{x^2+5x-24} \in \mathbb{R}\} = \{x \mid x^2+5x-24 \geq 0\}$

So the quadratic inequality $x^2+5x-24 \geq 0$ must be solved, something done in Section 1.7.

$x^2+5x-24 \geq 0 \Rightarrow (x+8)(x-3) \geq 0 \Rightarrow$

Case 1: $x+8 \geq 0$ & $x-3 \geq 0$ OR Case 2: $x+8 \leq 0$ & $x-3 \leq 0$

$\underbrace{x \geq -8 \ \& \ x \geq 3}_{x \geq 3}$ $\qquad$ $\underbrace{x \leq -8 \ \& \ x \leq 3}_{x \leq -8}$

So $x^2+5x-24 \geq 0$ holds if $x \geq 3$ <u>or</u> $x \leq -8$. Now we have:

$\mathcal{D}om\ F = \{x \mid x \leq -8 \text{ or } x \geq 3\} = (-\infty,-8] \cup [3,\infty)$. ∎

Anyone hoping quadratic inequalities would go away is going to be sorely disappointed.

Example 6 Let p be given by $p(x) = 2x-5$ for $-1 \leq x \leq 8$. Find $\mathcal{D}om\,p$ & $\mathcal{R}an\,p$.

• <u>Solution:</u> Here we're essentially told what the allowed inputs for p are. Normally if we were told $p(x) = 2x-5$, we'd conclude that the domain of p consists of all reals (i.e. $\mathcal{D}om\,p = \mathbb{R}$). But here p is rather limited: we are given that $p(x) = 2x-5$ only when x is such that $-1 \leq x \leq 8$. We have no idea what, if anything, $p(x)$ equals for $x<-1$ or $x>8$. Since $p(x)$ is not defined for $x<-1$ or $x>8$, we must assume that p simply cannot take such numbers as inputs. That is, $\mathcal{D}om\,p = \{x \mid -1 \leq x \leq 8\} = [-1,8]$ is the domain.

Note that $p(-1) = 2(-1)-5 = -7$ & $p(8) = 2(8)-5 = 11$, and when x is between -1 & 8 we get a value for $p(x)$ that's between -7 & 11. Thus $p(x)$ attains all values from -7 to 11, and nothing else. So $\mathcal{R}an\,p = \{f(x) \mid x \in \mathcal{D}om\,p\} = \{f(x) \mid -1 \leq x \leq 8\} = [-7,11]$. ∎

2.7 — COMBINING FUNCTIONS

Let f be a relation. For our purposes we only allow f to have real number "inputs" & "outputs", so the following are our definitions for <u>domain</u>, <u>range</u>, & <u>function</u>:

- $\mathcal{D}om\, f = \{x \mid f(x) \in \mathbb{R}\}$
- $\mathcal{R}an\, f = \{f(x) \mid x \in \mathcal{D}om\, f\}$
- f is a <u>function</u> if & only if for each $x \in \mathcal{D}om\, f$ there exists a unique y such that $f(x) = y$.

This latest version of the definition of a function comes closest to the "official" wording found in more rigorous texts. It's suitable for <u>all</u> functions, not just those that deal in real numbers.

Let f & g be functions. There are all sorts of ways we can combine f & g to construct new functions. Here's a few:

$f+g$ is the function that works as follows: $(f+g)(x) = f(x)+g(x)$.
$f-g$ is the function given by: $(f-g)(x) = f(x)-g(x)$.
fg is the function given by: $(fg)(x) = f(x)\cdot g(x)$.

Consider $f+g$ to be a symbol denoting a function that takes an input x & returns an output $f(x)+g(x)$. Notice that since x winds up being put into both f <u>and</u> g, so of necessity x must be something that belongs to the domain of f <u>and</u> the domain of g. That is...

$$\mathcal{D}om\,(f+g) = \{x \mid x \in \mathcal{D}om\, f \;\&\; x \in \mathcal{D}om\, g\} = \mathcal{D}om\, f \cap \mathcal{D}om\, g$$

Recall that if A & B are sets, the $A \cap B$ represents the <u>intersection</u> of A & B. By definition $A \cap B = \{x \mid x \in A \;\&\; x \in B\}$. Do not confuse $A \cap B$ with $A \cup B$, which represents the <u>union</u> of A & B, and is defined thus: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. If $A = \{-2,3,8\}$ & $B = \{0,3,8\}$, then $A \cap B = \{3,8\}$ & $A \cup B = \{-2,0,3,8\}$. If $A = (-2,3]$ & $B = [0,8)$, then $A \cap B = [0,3]$ & $A \cup B = (-2,8)$.

Example 1 Let $f(x) = \sqrt[10]{3-x}$ & $g(x) = \sqrt[10]{2x+7}$. Find $f+g$ & its domain.

- <u>Solution:</u> By "find $f+g$" is meant "for input x, find an algebraic expression (in terms of x) that determines the output $(f+g)(x)$". That's straightforward:

$$(f+g)(x) = f(x)+g(x) = \sqrt[10]{3-x} + \sqrt[10]{2x+7} \checkmark$$ (note: in general $\sqrt[n]{a} + \sqrt[n]{b} \neq \sqrt[n]{a+b}$)

Now, $\mathcal{D}om\, f = \{x \mid 3-x \geq 0\} = \{x \mid x \leq 3\} = (-\infty, 3]$
$\mathcal{D}om\, g = \{x \mid 2x+7 \geq 0\} = \{x \mid x \geq -7/2\} = [-7/2, \infty)$

Hence $\mathcal{D}om\,(f+g) = \mathcal{D}om\, f \cap \mathcal{D}om\, g = (-\infty,3] \cap [-7/2, \infty) = [-7/2, 3]$. ∎

For the functions $f-g$ & fg the domain is also $\mathcal{D}om\, f \cap \mathcal{D}om\, g$. It is very important to adhere to this! The following example will illustrate the fallacy of not doing so...

Example 2 Let $f(x) = \sqrt[10]{3-x}$ & $g(x) = \sqrt[10]{5-x}$ Find fg & $\mathcal{D}om\,(fg)$

• Solution: We find fg: $(fg)(x) = f(x) \cdot g(x) = \sqrt[10]{3-x} \cdot \sqrt[10]{5-x} = \sqrt[10]{(3-x)(5-x)} \Rightarrow$ $(fg)(x) = \sqrt[10]{15-8x+x^2}$. ✓

Next, $\mathcal{D}om\, f = (-\infty, 3]$ & $\mathcal{D}om\, g = (-\infty, 5]$, so $\mathcal{D}om\,(fg) = \mathcal{D}om\, f \cap \mathcal{D}om\, g =$ $(-\infty, 3] \cap (-\infty, 5] = (-\infty, 3]$ ✓ ∎

So where's the fallacy? Consider that, after some manipulation, we obtained $(fg)(x) = \sqrt[10]{15-8x+x^2}$. What if we attempted to find $\mathcal{D}om\,(fg)$ from this manipulated form? We'd proceed as follows: $\mathcal{D}om\,(fg) = \{x \mid (fg)(x) \in \mathbb{R}\} = \{x \mid \sqrt[10]{15-8x+x^2} \in \mathbb{R}\} = \{x \mid 15-8x+x^2 \geq 0\}$, where $15-8x+x^2 \geq 0 \Rightarrow (3-x)(5-x) \geq 0$ and there are two cases:

Case I: $3-x \geq 0$ & $5-x \geq 0 \Rightarrow x \leq 3$ & $x \leq 5 \Rightarrow x \leq 3$

Case II: $3-x \leq 0$ & $5-x \leq 0 \Rightarrow x \geq 3$ & $x \geq 5 \Rightarrow x \geq 5$

Then $\mathcal{D}om\,(fg) = \{x \mid 15-8x+x^2 \geq 0\} = \{x \mid x \leq 3$ or $x \geq 5\} = (-\infty, 3] \cup [5, \infty)$, which is wrong! The interval $[5, \infty)$ is not part of $\mathcal{D}om\,(fg)$. Look what happens when 6 is put into fg: $(fg)(6) = f(6) \cdot g(6) = \sqrt[10]{3-6} \cdot \sqrt[10]{5-6} = \sqrt[10]{-3} \cdot \sqrt[10]{-1}$, but $\sqrt[10]{-3}$ & $\sqrt[10]{-1}$ are not real numbers! A-U-U-UGH!!! In Example 2, when we "simplified" $(fg)(x) = \sqrt[10]{3-x} \cdot \sqrt[10]{5-x}$ and wrote $(fg)(x) = \sqrt[10]{15-8x+x^2}$, we "hid" weaknesses in the machinery: it looks like 6 is an allowable input (seemingly $(fg)(6) = \sqrt[10]{15-8(6)+6^2} = \sqrt[10]{3} \in \mathbb{R}$), but it is not allowable since any x put into fg is actually put into f and g separately. 6 "breaks" f & g.

Now, by definition, $(f/g)(x) = \dfrac{f(x)}{g(x)}$, but $\mathcal{D}om\,(f/g)$ is not $\mathcal{D}om\, f \cap \mathcal{D}om\, g$; rather, we have $\mathcal{D}om\,(f/g) = \{x \mid x \in \mathcal{D}om\, f$ & $x \in \mathcal{D}om\, g$ & $g(x) \neq 0\} = \{x \mid x \in \mathcal{D}om\, f \cap \mathcal{D}om\, g$ & $g(x) \neq 0\}$. Thus $\mathcal{D}om\,(f/g)$ must not include any values of x for which $g(x) = 0$, since then $\dfrac{f(x)}{g(x)}$ would lead to division by 0.

Example 3 Let $f(x) = \sqrt{x+4}$ & $g(x) = x^2 - 1$. Find f/g & $\mathcal{D}om\,(f/g)$

• Solution: Find f/g: $(f/g)(x) = \dfrac{f(x)}{g(x)} = \dfrac{\sqrt{x+4}}{x^2-1}$ ✓

Now, $\mathcal{D}om\, f = \{x \mid x+4 \geq 0\} = \{x \mid x \geq -4\} = [-4, \infty)$ & $\mathcal{D}om\, g = (-\infty, \infty) = \mathbb{R}$.

Then $\mathcal{D}om\,(f/g) = \{x \mid x \in [-4, \infty) \cap (-\infty, \infty)$ & $g(x) \neq 0\} = \{x \mid x \in [-4, \infty)$ & $x^2 - 1 \neq 0\}$ $= \{x \mid x \in [-4, \infty)$ & $x^2 \neq 1\} = \{x \mid x \in [-4, \infty)$ & $x \neq \pm 1\} = [-4, -1) \cup (-1, 1) \cup (1, \infty)$. ∎

To recap: If $\star$ represents $+, -,$ or $\cdot$, then $f \star g$ is given by $(f\star g)(x) = f(x) \star g(x)$, and $\mathcal{D}om\,(f\star g) = \mathcal{D}om\, f \cap \mathcal{D}om\, g$. Similarly f/g is given by $(f/g)(x) = f(x)/g(x)$, however $\mathcal{D}om\,(f/g) = \{x \mid x \in \mathcal{D}om\, f \cap \mathcal{D}om\, g \ \& \ g(x) \neq 0\}$.

These ideas can be extended: If f, g, h are functions, then $(fgh)(x) = f(x)\cdot g(x) \cdot h(x)$. If $n \in \mathbb{Z}$, then f^n is the function for which $f^n(x) = \underbrace{f(x)\cdot f(x) \cdot \cdots \cdot f(x)}_{n \text{ factors}} = [f(x)]^n$.

Perhaps more important than all of the function combinations above is the operation known as function composition. The <u>composition of f with g</u>, written $f\circ g$, is a new function that operates as follows: $(f\circ g)(x) = f(g(x))$. It's typical to read $f\circ g$ as "f composed with g" or even "f circle g". As the diagram below shows, $f\circ g$ is essentially an "assembly line" consisting of the machines f & g: to evaluate $(f\circ g)(x)$, x first goes into g to yield $g(x)$, and then $g(x)$ in turn is inserted into f...

$$x \longrightarrow \boxed{\ g\ } \longrightarrow g(x) \longrightarrow \boxed{\ f\ } \longrightarrow f(g(x)) = (f\circ g)(x)$$

Function composition, then, links functions in series ("serial processing", as opposed to "parallel processing"). To determine the domain of $f\circ g$ simply follow the sequence of events in the diagram above: When x is put into $f\circ g$, $f(g(x))$ results; that is, x goes into g — so we need $x \in \mathcal{D}om\, g$ to hold — and then $g(x)$ goes into f, which requires that $g(x)$ be an element of the domain of f. That is: $\mathcal{D}om\,(f\circ g) = \{x \mid x \in \mathcal{D}om\, g \ \& \ g(x) \in \mathcal{D}om\, f\}$. There is no guesswork involved in finding $\mathcal{D}om\,(f\circ g)$, only careful reasoning using algebra.

Example 4 Let $f(x) = \sqrt{x-2}$ & $g(x) = \sqrt[4]{x}$. Find $f\circ g$ & $\mathcal{D}om\,(f\circ g)$.

- <u>Solution:</u> Find $f\circ g$: $(f\circ g)(x) = f(g(x)) = f(\sqrt[4]{x}) = \sqrt{\sqrt[4]{x} - 2}$, (1) ✓

Now, $\mathcal{D}om\, f = [2, \infty)$ & $\mathcal{D}om\, g = [0, \infty)$, so...

$$\begin{aligned}
\mathcal{D}om\,(f\circ g) &= \{x \mid x \in \mathcal{D}om\, g \ \& \ g(x) \in \mathcal{D}om\, f\} \\
&= \{x \mid x \in [0,\infty) \ \& \ \sqrt[4]{x} \in [2, \infty)\} \\
&= \{x \mid x \geq 0 \ \& \ \sqrt[4]{x} \geq 2\}, \quad \text{where } \sqrt[4]{x} \geq 2 \Rightarrow (\sqrt[4]{x})^4 \geq 2^4 \Rightarrow x \geq 16, \text{ so...} \\
&= \{x \mid x \geq 0 \ \& \ x \geq 16\} \\
&= \{x \mid x \geq 16\} \\
&= [16, \infty) \quad \blacksquare
\end{aligned}$$

A few simple evaluations using f & g in Example 4:

- To find $(f\circ g)(81)$, either use (1) to get $(f\circ g)(81) = \sqrt{\sqrt[4]{81} - 2} = \sqrt{3-2} = \sqrt{1} = 1$, or run through the process: $(f\circ g)(81) = f(g(81)) = f(\sqrt[4]{81}) = f(3) = \sqrt{3-2} = 1$.
- Watch this: $(g\circ f)(81) = g(f(81)) = g(\sqrt{81-2}) = g(\sqrt{79}) = \sqrt[4]{\sqrt{79}} = \sqrt[8]{79}$.

So $(f\circ g)(81) \neq (g\circ f)(81)$.

Definition Let f & g be functions such that $\mathrm{Dom}\, f = \mathrm{Dom}\, g = D$. If $f(x) = g(x)$ for every $x \in D$, then we say that f & g are <u>equal</u> and write $f = g$

Since functions are sets of ordered pairs, $f = g$ if and only if f & g contain the same ordered pairs. From the observation at the bottom of the last page, it can be seen that, in general, $f \circ g \neq g \circ f$.

Example 5 Let $W(x) = \dfrac{3}{(x-2)^5}$. Find functions f, g, h such that $f, g, h \neq W$, but $f \circ g \circ h = W$.

• <u>Solution</u>: So we must find three "little machines" that together do the job of the "big machine" W. Specifically, we want f, g, h such that $(f \circ g \circ h)(x) = W(x)$. A diagram may help here:

$$x \longrightarrow \boxed{h} \longrightarrow h(x) \longrightarrow \boxed{g} \longrightarrow g(h(x)) \longrightarrow \boxed{f} \longrightarrow f(g(h(x))) = (f \circ g \circ h)(x) = \frac{3}{(x-2)^5}$$

Consider the procedure we would follow we're we to evaluate $\dfrac{3}{(x-2)^5}$ for a given value of x: By the usual Order of Operations we would:

1) Subtract 2 from x
2) Raise the result of Step 1 to the power of 5
3) Divide 3 by the result of Step 2.

This recommends to us the appropriate definitions of h, g, & f:

1) Let h be the "subtract 2" function: $h(x) = x - 2$
2) Let g be the "raise to the 5th power" function: $g(x) = x^5$
3) Let f be the "divide 3" function: $f(x) = 3/x$

Behold, the Power of ~~the Dark Side of the Force~~ Function Composition:

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-2)) = f((x-2)^5) = \frac{3}{(x-2)^5} = W(x). \quad \blacksquare$$

Another look at the diagram in Example 5 should clarify how to find the domain of a composition of <u>three</u> functions:

$$\mathrm{Dom}\,(f \circ g \circ h) = \{x \mid x \in \mathrm{Dom}\, h,\ h(x) \in \mathrm{Dom}\, g, \text{ and } g(h(x)) \in \mathrm{Dom}\, f\}$$