1.6 — OTHER EQUATIONS

We've seen linear & quadratic equations so far, and some cubic equations as well. That is, equations with standard forms $ax+b=0$, $ax^2+bx+c=0$, & even $ax^3+bx^2+cx+d=0$. Now we'll look at rational & radical equations, and equations having quadratic form.

A rational equation is an equation having at least one rational expression among its terms. Actually we've been solving rational equations since Section 1.1 since every polynomial is also a rational expression (the polynomial $x+1$, for example, is identical to the rational expression $\frac{x+1}{1}$). So more appropriately it should be said we'll now be looking at rational equations that are not also polynomial equations. Simply put: variables will now start appearing below fraction bars.

Example 1 Solve $\dfrac{2}{x+5}=7$

• Solution: Don't like fractions? Neither do most mathematicians, so blow away the denominator...

$$(x+5)\cdot\frac{2}{x+5}=7\cdot(x+5) \Rightarrow 2=7x+35 \Rightarrow \text{We're back to Section 1.1!}$$

So $7x=-33$, whence $x=-33/7$ obtains. Solution set is $\left\{-\frac{33}{7}\right\}$ ∎

Now ponder this little riddle: $\frac{1}{x}=1 \Rightarrow x^2\cdot\frac{1}{x}=x^2\cdot 1 \Rightarrow \frac{x^2}{x}=x^2 \Rightarrow x=x^2 \Rightarrow x^2-x=0 \Rightarrow x(x-1)=0 \Rightarrow x=0$ or $x-1=0 \Rightarrow x=0$ or $x=1$. But if $x=0$ then the original equation $\frac{1}{x}=1$ becomes $\frac{1}{0}=1$, which is horrifically wrong! (Recall division by 0 is undefined). So where is our error? Right at the beginning: when considering $\frac{1}{x}=1$, we should recognize right away that $x=0$ can't be allowed. By proceeding from $\frac{1}{x}=1$ to $x^2\cdot\frac{1}{x}=x^2\cdot 1$ we unwittingly signed a contract saying "We agree that x cannot be 0", because writing $\frac{1}{0}=1 \Rightarrow 0^2\cdot\frac{1}{0}=0^2\cdot 1$ is meaningless. The ravings of an unhinged propellerhead.

The best way to proceed, however, is to simply solve the equation as we did above, and then discard any values that result in division by 0 in the equation's original form. So we throw out $x=0$, which is a so-called "extraneous solution" and state $\underline{x=1}$ as the sole answer.

Example 2 Solve $\dfrac{x+5}{x-2}=\dfrac{5}{x+2}+\dfrac{28}{x^2-4}$

• Solution: 1st, factor all denominators: $\dfrac{x+5}{x-2}=\dfrac{5}{x+2}+\dfrac{28}{(x-2)(x+2)}$, (1)

Now it's fairly clear what we should do to annihilate all fractions: multiply (1) by $(x-2)(x+2)$...

$$\cancel{(x-2)}(x+2)\frac{x+5}{\cancel{x-2}}=(x-2)\cancel{(x+2)}\cdot\frac{5}{\cancel{x+2}}+\cancel{(x-2)}\cancel{(x+2)}\cdot\frac{28}{\cancel{(x-2)}\cancel{(x+2)}} \Rightarrow$$

$$(x+2)(x+5)=(x-2)\cdot 5+28 \Rightarrow x^2+7x+10=5(x-2)+28 \Rightarrow \text{We're back to Section 1.4!}$$

We now solve the quadratic equation $x^2+7x+10=5x+18$:
$x^2+2x-8=0 \Rightarrow (x+4)(x-2)=0 \Rightarrow x+4=0$ or $x-2=0 \Rightarrow$
$x=-4$ or $x=2$

Now, very important, we must check our two values for x to see if they really satisfy the original equation. The value -4 is a genuine solution since it does not result in division by 0: $\frac{-4+5}{-4-2}=\frac{5}{-4+2}+\frac{28}{(-4)^2-4} \Rightarrow \frac{1}{-6}=\frac{5}{-2}+\frac{28}{12}$ ✓

The value 2 is a disaster: $\frac{2+5}{2-2}=\frac{5}{2+2}+\frac{28}{2^2-4} \Rightarrow \frac{7}{0}=\frac{5}{4}+\frac{28}{0}$

HORRORS!

So 2 is extraneous & must be committed to the flames. The solution set is $\{-4\}$ ∎

Whenever solving any rational equation, always:
1) Factor all denominators
2) Multiply the equation by an expression that will cancel out <u>all</u> denominators (this will be the Least Common Multiple of the denominators, ideally).
3) Solve the resultant linear or quadratic equation
4) Check all values, and discard any that result in division by 0 in the original equation.
5) State the solution set.

Next, a <u>radical equation</u> is an equation having at least one radical expression among its terms. When encountering one, the strategy is once again to manipulate symbols until a polynomial (usually linear or quadratic) equation results.

Example 3 Solve $\sqrt{2x+5}=7$

• <u>Solution:</u> Square both sides, and presto: a polynomial equation of fair complexion emerges...

$(\sqrt{2x+5})^2=7^2 \Rightarrow 2x+5=49 \Rightarrow 2x=44 \Rightarrow x=22$. Solution set: $\{22\}$

Note: we have to assume the radicand $2x+5$ is nonnegative — for how else would $\sqrt{2x+5}$ equal a nice real number like 7? And why care? Well, seeing as $\sqrt{2x+5}$ is real-valued, the property of radicals that says $\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$ applies when we square:

$$(\sqrt{2x+5})^2=\sqrt{2x+5}\cdot\sqrt{2x+5}=\sqrt{(2x+5)(2x+5)}=\sqrt{(2x+5)^2}=|2x+5|=2x+5 \quad (\text{since } 2x+5\geq 0)$$

Recall that $\sqrt{a^2}=|a|$, which is a direct consequence of the definition of a square root: $\sqrt{a}=b$, where b is such that $b^2=a$ <u>and</u> $b\geq 0$. Indeed $\sqrt[n]{a^n}=|a|$ for any even integer n.

When solving radical equations we will only be looking for <u>real-valued</u> solutions, not complex values.

As with rational equations we have to be on the look-out for extraneous solutions when solving radical equations, but only when the radicals into the equation have an even index!

Here's a recipe for solving radical equations:

1) If there's one radical term, isolate it on one side of the equation; if two, get them on opposite sides
2) Raise both sides of the equation to a power equal to the index of the radical(s)
3) If a radical term remains, repeat steps 1 & 2 until all radicals are eliminated.
4) Solve the resultant polynomial equation
5) If the power in Step 2 was even, check for extraneous solutions; if odd, go to Step 6.
6) State the solution set.

Example 4 Solve $x = 1 - \sqrt{2 - \frac{x}{2}}$

• Solution: 1) Isolate the radical term $\sqrt{2 - x/2}$: $\sqrt{2 - \frac{x}{2}} = 1 - x$

2) Index of a square-root radical is 2, so square both sides of the equation:

$$\left(\sqrt{2 - x/2}\right)^2 = (1-x)^2 \Rightarrow 2 - \frac{x}{2} = 1 - 2x + x^2$$

3) No radicals remain, so proceed to Step 4...

4) We have a quadratic equation to solve: $4 - x = 2 - 4x + 2x^2 \Rightarrow 2x^2 - 3x - 2 = 0 \Rightarrow$ $(2x+1)(x-2) = 0 \Rightarrow x = -1/2, 2$

5) The power in Step 2 was 2, and thus even. We check each value of x obtained above by substituting it into the original equation...

∘ Checking $-1/2$: $-\frac{1}{2} = 1 - \sqrt{2 - \frac{1}{2}\left(-\frac{1}{2}\right)} \Rightarrow -\frac{1}{2} = 1 - \sqrt{2 + \frac{1}{4}} \Rightarrow -\frac{1}{2} = 1 - \sqrt{\frac{9}{4}}$ $\Rightarrow -\frac{1}{2} = 1 - \frac{3}{2} \Rightarrow -\frac{1}{2} = -\frac{1}{2}$, which is true, so $-\frac{1}{2}$ is a valid solution.

∘ Checking 2: $-\frac{1}{2} = 1 - \sqrt{2 - \frac{2}{2}} \Rightarrow -\frac{1}{2} = 1 - \sqrt{2-1} \Rightarrow -\frac{1}{2} = 1 - \sqrt{1}$ $\Rightarrow -\frac{1}{2} = 1 - 1 \Rightarrow -\frac{1}{2} = 0$, which is false, so 2 is not a solution.

6) Solution set: $\{-1/2\}$ ∎

To see how a bogus solution can arise when raising the two sides of an equation to an even power, consider something simple like $x = 3$.* Squaring yields $x^2 = 9$, which in turn leads to two "solutions": $x = 3$ & $x = -3$. Obviously $x = -3$ does not satisfy the original equation (*). The problem is that while $x = 3$ does imply that $x^2 = 9$, the converse does not follow: $x^2 = 9$ does not imply that $x = 3$. The squaring operation can not be reliably reversed!

Example 5 Solve $\sqrt{12x+9} - \sqrt{24x} = -3$

• Solution: 1) There are two radical terms; things will generally be easier if they're separated:

$$\sqrt{12x+9} = \sqrt{24x} - 3$$

2) Square both sides: $\left(\sqrt{12x+9}\right)^2 = \left(\sqrt{24x} - 3\right)^2 \Rightarrow 12x+9 = \underbrace{(\sqrt{24x} - 3)(\sqrt{24x} - 3)}_{\text{FOIL}} \Rightarrow$
$12x+9 = 24x - 6\sqrt{24x} + 9$

3) A radical term remains, so isolate it & square again: $6\sqrt{24x} = 12x \Rightarrow$
$\left(6\sqrt{24x}\right)^2 = (12x)^2 \Rightarrow 36(24x) = 144x^2$

4) We now have a quadratic equation to solve. Fun!
$864x = 144x^2 \Rightarrow 144x^2 - 864x = 0 \Rightarrow x(144x - 864) = 0 \Rightarrow$
$x = 0$ or $144x - 864 = 0 \Rightarrow x = 0$ or $x = 6$.

5) Step 2 featured an even power, so check for phony solutions:

∘ Check $x=0$: $\sqrt{12(0)+9} - \sqrt{24(0)} = -3 \Rightarrow \sqrt{9} - \sqrt{0} = -3 \Rightarrow 3 - 0 = -3$
$\Rightarrow 3 = -3$, which is false, so 0 is not a solution.

∘ Check $x=6$: $\sqrt{12(6)+9} - \sqrt{24(6)} = -3 \Rightarrow \sqrt{81} - \sqrt{144} = -3 \Rightarrow 9 - 12 = -3$
$\Rightarrow -3 = 3$, which is true, so 6 is a solution.

6) Solution set: $\{6\}$ ∎

Example 6 Solve $\sqrt{x} + \sqrt{2x+7} - 2\sqrt{2x-2} = 0$

• Solution: 1) Three radicals! The best strategy is to move one to the right side...

$$\sqrt{x} + \sqrt{2x+7} = 2\sqrt{2x-2}$$

2) Square: $\left(\sqrt{x} + \sqrt{2x+7}\right)^2 = \left(2\sqrt{2x-2}\right)^2 \Rightarrow x + \overbrace{2\sqrt{x}\sqrt{2x+7}}^{2\sqrt{2x^2+7x}} + (2x+7) = 4(2x-2)$

3) Isolate the radical term & square again: $2\sqrt{2x^2+7x} = 5x - 15 \Rightarrow$
$\left(2\sqrt{2x^2+7x}\right)^2 = (5x-15)^2 \Rightarrow 4(2x^2+7x) = 25x^2 - 150x + 225$

4) Solve the quadratic equation: $17x^2 - 178x + 225 = 0 \Rightarrow (x-9)(17x-25) = 0 \Rightarrow$
$x = 9, 25/17$ (rather than factor it might be easier to use the Quadratic Formula).

5) Time for a reality check...

∘ Check $x=9$: $\sqrt{9}+\sqrt{2(9)+7}-2\sqrt{2(9)-2}=0 \Rightarrow \sqrt{9}+\sqrt{25}-2\sqrt{16}=0 \Rightarrow$
$3+5-2(4)=0 \Rightarrow 0=0$, which is true, so 9 is a solution.

∘ Check $x=\frac{25}{17}$: $\sqrt{25/17}+\sqrt{2(25/17)+7}-2\sqrt{2(25/17)-2}=0 \Rightarrow$

$$\sqrt{\frac{25}{17}}+\sqrt{\frac{169}{17}}-2\sqrt{\frac{16}{17}}=0 \Rightarrow \frac{5}{\sqrt{17}}+\frac{13}{\sqrt{17}}-2\cdot\frac{4}{\sqrt{17}}=0 \Rightarrow$$

$$\frac{18}{\sqrt{17}}-\frac{8}{\sqrt{17}}=0 \Rightarrow \frac{10}{\sqrt{17}}=0$$ — false!

6) Solution set: $\{9\}$ ■

An equation is said to have quadratic form if a choice substitution puts it into the form $ax^2+bx+c=0$. Example: $(x+5)^{4/3}+(x+5)^{2/3}-20=0$. If we let $u=(x+5)^{2/3}$, the equation becomes $u^2+u-20=0$, which fits the standard form of a quadratic equation. Once we solve for u using the techniques of Section 1.5, it becomes straightforward to solve for x next.

Example 7 Solve $(x+5)^{4/3}+(x+5)^{2/3}-20=0$

• Solution: We have $[(x+5)^{2/3}]^2+(x+5)^{2/3}-20=0$, so let $u=(x+5)^{2/3}$ so equation becomes $u^2+u-20=0$.

Now, factoring yields $(u+5)(u-4)=0$, and thus $u=-5$ or $u=4$.

Reversing the substitution (since we want to solve for x): $(x+5)^{2/3}=-5$ or $(x+5)^{2/3}=4$.

$(x+5)^{2/3}=-5 \Rightarrow [(x+5)^{2/3}]^3=(-5)^3 \Rightarrow (x+5)^2=-125 \Rightarrow \sqrt{(x+5)^2}=\sqrt{-125} \Rightarrow$
$|x+5|=5i\sqrt{5} \Rightarrow x+5=\pm 5i\sqrt{5} \Rightarrow x=-5\pm 5i\sqrt{5}$ ✓

$(x+5)^{2/3}=4 \Rightarrow [(x+5)^{2/3}]^3=4^3 \Rightarrow (x+5)^2=64 \Rightarrow x+5=\pm\sqrt{64} \Rightarrow x=-5\pm 8$
$\Rightarrow x=-13, 3$ ✓

Solution set: $\{-5\pm 5i\sqrt{5}, -13, 3\}$ ■

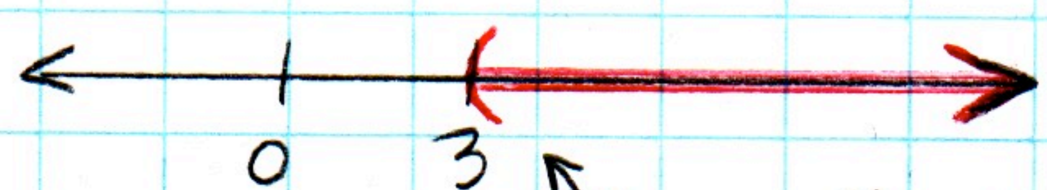
Some comments concerning Example 7 are in order. First, $(x+5)^{2/3}=-5$ is really just a radical equation in disguise: $\sqrt[3]{(x+5)^2}=-5$, & it was said earlier that when solving radical equations we're only looking for real-valued solutions. So, since $(x+5)^{2/3}=-5$ has no real solutions, it's permitted to pass over it & fixate solely on $(x+5)^{2/3}=4$. Therefore the solution set we really want is simply $\{-13, 3\}$. The text follows this convention, and henceforth so will I. Notice there's no need to check the values obtained from $(x+5)^{2/3}=4$ since we raised to an odd power.

1.7 — INEQUALITIES

An <u>inequality</u> is a mathematical statement that relates two (or more) mathematical expressions using one (or more) conceptions of "non-equality" symbolized by $>, <, \geq, \leq$ (called <u>inequality symbols</u>).

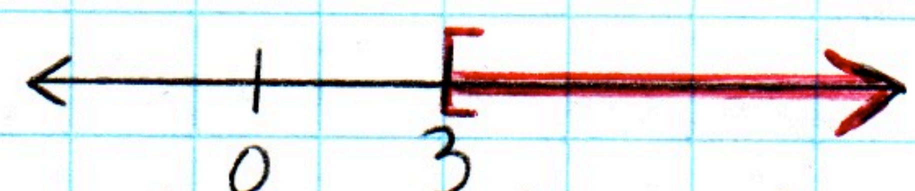
Examples: $x > 3$ ("x is greater than 3")
$x \leq 2$ ("x is less than or equal to 2")
$-1 < x < 4$ ("x is greater than -1 <u>and</u> less than 4")

There are a couple of common ways to express the idea "x is greater than 3" besides $x > 3$:

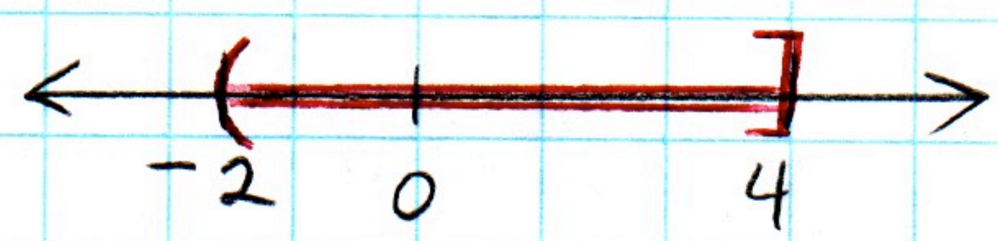
1) <u>A number-line graph:</u> 

parenthesis means 3 itself is <u>not</u> included

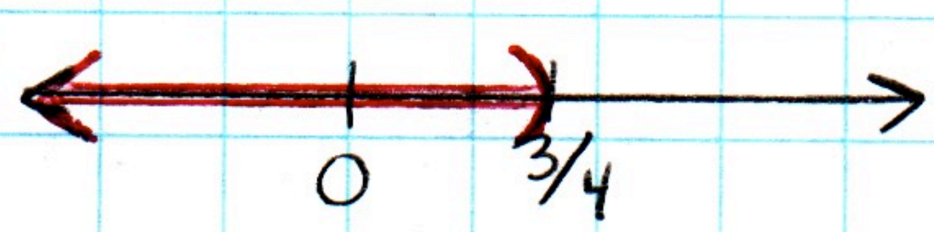
2) <u>Interval notation:</u> $(3, \infty)$, which can be read as "the interval of numbers from 3 to infinity", where the parentheses next to 3 & ∞ mean that 3 & ∞ are <u>not</u> included.

Similarly, $x \geq 3$ can be represented by 

or $[3, \infty)$, where in each case the bracket [next to the 3 means that now the number 3 <u>is</u> included.

We can write $-2 < x \leq 4$ as 

or $(-2, 4]$.

Finally, $x < 3/4$ can be conveyed as 

or $(-\infty, 3/4)$

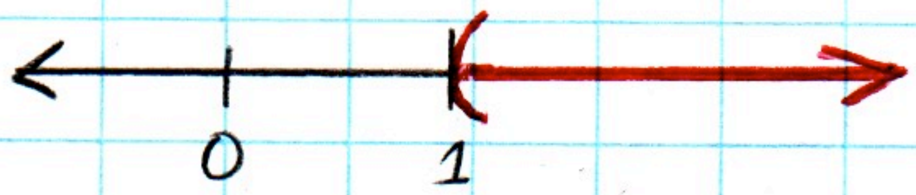
Note: $-\infty$ & ∞ are not numbers, but rather symbols that say "keep going down" & "keep going up" forever and ever, respectively. A bracket never goes next to either one, ever.

To <u>solve</u> an inequality such as $3(x-2)+5 > 2x$ means the same thing as solving an equation like $3(x-2)+5 = 2x$: namely, isolate x to determine what values it can assume to make the statement true.

Example 1 Solve $3(x-2)+5 > 2x$. Express the solution set as a graph & as an interval.

• <u>Solution:</u> $3(x-2)+5 > 2x \Rightarrow 3x-6+5 > 2x \Rightarrow 3x - 1 > 2x$

$$\begin{array}{r} 3x - 1 > 2x \\ \underline{-2x \qquad -2x} \\ x - 1 > 0 \end{array} \Rightarrow x > 1$$

∘ <u>Graph:</u> 

∘ <u>Interval:</u> $(1, \infty)$

Recall that when multiplying or dividing the two sides of an inequality by a negative number the inequality symbol must <u>reverse direction</u>: $-1<5 \Rightarrow (-1)(-1) > (5)(-1) \Rightarrow 1>-5$.

reverse

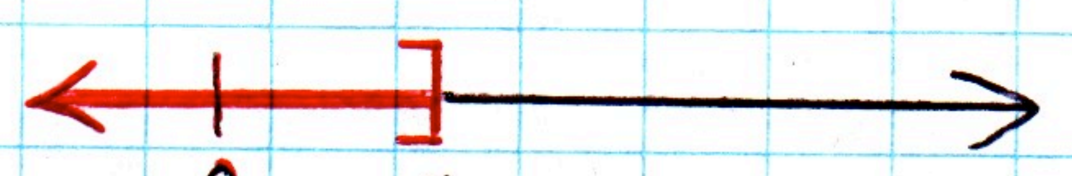
Example 2 Solve $2-5(x-4) \geq 2(x+1)-8$

- <u>Solution:</u> $2-5x+20 \geq 2x+2-8 \Rightarrow -5x+22 \geq 2x-6 \Rightarrow$
  $\quad -2x \qquad\qquad -2x$

  $-7x+22 \geq -6 \Rightarrow -7x \geq -28 \Rightarrow \dfrac{-7x}{-7} \leq \dfrac{-28}{-7} \Rightarrow x \leq 4.$ ✓
  $\quad -22 \qquad -22$

  reverse

  - <u>Graph:</u>



  - <u>Interval:</u> $(-\infty, 4]$

A <u>compound inequality</u> is an inequality that contains two or more inequality symbols, such as $-3 \leq 4x+9 < 17$. To solve, isolate x in the middle part...

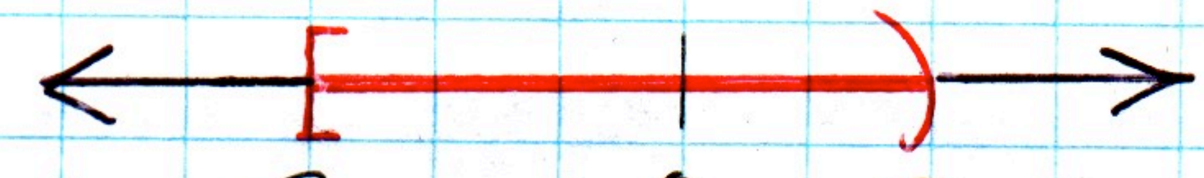
Example 3 Solve $-3 \leq 4x+9 < 17$

- <u>Solution:</u> Isolate x in the middle, but remember that whatever is done to one part must be done to all parts (left, middle, <u>and</u> right).

  $-3 \leq 4x+9 < 17 \Rightarrow -12 \leq 4x < 8 \Rightarrow \dfrac{-12}{4} \leq \dfrac{4x}{4} < \dfrac{8}{4} \Rightarrow$
  $\quad -9 \qquad -9 \quad -9$

  $-3 \leq x < 2$ ✓

  - <u>Graph:</u>



  - <u>Interval:</u> $[-3, 2)$

We come at last to something a little different (i.e. not generally encountered in intermediate algebra): <u>nonlinear</u> inequalities. To start, let's look specifically at quadratic inequalities, which generally can be molded into the "standard form" $ax^2+bx+c>0$ (or ≥ 0, <0, ≤ 0). To solve such a thing I will present a technique very much different from the book's rather mindless "cookbook recipe" approach. Initially "my way" will seem harder, but in the long run it will hone highly valuable logical thinking skills that will be absolutely necessary in a calculus course. In contrast the "book's way" depends heavily on the idea of a continuous function — something not fully developed until calculus.

So, here we go...

Example 4 Solve $x^2+x<12$

• **Solution:** As when solving the equation $x^2+x=12$, the approach will be to get 0 on the right side & factor the left side...

$$x^2+x<12 \Rightarrow x^2+x-12<0 \Rightarrow (x+4)(x-3)<0$$

Now it is our time to become pointy-eared Vulcans & apply some basic logic. If $A\cdot B<0$, then $A\cdot B$ has a negative value; but if $A\cdot B$ is negatively valued, that leaves open two distinct possibilities: (i) A is positive & B is negative
(ii) A is negative & B is positive

Absent any additional information about A & B, both possibilities are equally valid and need to be investigated. We can express the cases (i) & (ii) above using inequalities:

(i) $A>0$ & $B<0$

(ii) $A<0$ & $B>0$

In this example we have $(x+4)(x-3)<0$ instead of $A\cdot B<0$, and so our cases become:

(i) $x+4>0$ & $x-3<0$

(ii) $x+4<0$ & $x-3>0$

These are the possibilities open to us given that $(x+4)(x-3)<0$. Examine each case separately.

Case (i): $x+4>0$ & $x-3<0 \Rightarrow x>-4$ & $x<3 \Rightarrow -4<x<3$, and so all values in the interval $(-4,3)$ will give rise to Case (i).

Case (ii): $x+4<0$ & $x-3>0 \Rightarrow x<-4$ & $x>3$. But wait: how can x be both less than -4 <u>and</u> greater than 3!? Case (ii), we discover, is asking for the impossible. *No* value of x will give rise to Case (ii).

So to satisfy $(x+4)(x-3)<0$, the only solutions for x lie in the interval $(-4,3)$. The solution set for $x^2+x<12$, therefore, is $(-4,3)$. ∎

Example 5 Solve $2x^2>-7x-3$

• **Solution:** $2x^2>-7x-3 \Rightarrow 2x^2+7x+3>0 \Rightarrow (2x+1)(x+3)>0$.

Generally if $A\cdot B>0$ there are two possibilities: (i) $A>0$ & $B>0$
(ii) $A<0$ & $B<0$

In short, the product of two positive numbers is positive, and <u>also</u> the product of two negative numbers is positive. In this example Cases (i) & (ii) boil down to...

(i) $2x+1>0$ & $x+3>0$

(ii) $2x+1<0$ & $x+3<0$

Now to employ some impeccable logic to unravel this little puzzle...

Case (i): $2x+1>0$ & $x+3>0$

$$\underbrace{x>-1/2 \quad \& \quad x>-3}_{\boxed{x>-1/2}}$$

Case (ii): $2x+1<0$ & $x+3<0$

$$\underbrace{x<-1/2 \quad \& \quad x<-3}_{\boxed{x<-3}}$$

Notice how each case was able to be simplified. In Case (i) it's found that x must be a number that's greater than $-1/2$ <u>and</u> greater than -3. But if x is greater than $-1/2$, then it will automatically be greater than -3 since $-3<-1/2$; thus to satisfy $x>-1/2$ <u>and</u> $x>-3$, we only need to satisfy $x>-1/2$.

In Case (ii) it's required that x be less than -3 <u>and</u> less than $-1/2$; but if $x<-3$, it will always follow that $x<-1/2$ since $-1/2$ is bigger than -3. As a result Case (ii) can be simplified to $x<-3$.

So, in order to satisfy $2x^2>-7x-3$, we can have either $x>-1/2$ or $x<-3$; that is, x can be a value in $(-1/2,\infty)$ or $(-\infty,-3)$. Therefore the solution set for our inequality $2x^2>-7x-3$ is the union of the two intervals: $(-\infty,-3)\cup(-1/2,\infty)$. ∎

Example 6 Solve $x^3-3x^2-x+3\geq 0$.

• Solution: Factor by grouping. The 1st & 3rd terms have x in common while the 2nd & 4th terms have 3 in common, so group thusly: $(x^3-x)+(-3x^2+3)\geq 0$.

Factoring, we get $x(x^2-1)+3(-x^2+1)\geq 0$. Now pull -1 out of the second group to get the same expression between the two pairs of parentheses: $x(x^2-1)-3(x^2-1)\geq 0$.

Now: $(x^2-1)(x-3)\geq 0 \Rightarrow \underbrace{(x-1)}_{\text{1st factor}}\overbrace{(x+1)}^{\text{2nd factor}}\underbrace{(x-3)}_{\text{3rd factor}}\geq 0$, (A)

There are <u>four</u> ways the product on the left can turn out to be positive (i.e. greater than 0):

i) $+++$ (all three factors are positive)
ii) $+--$ (the 1st factor is positive, the other two are negative)
iii) $-+-$ (the 2nd factor is positive, the other two are negative)
iv) $--+$ (the 3rd factor is positive, the others negative)

Each case must be examined. Note in (A) that we have $\geq$, not $>$; so in all cases we attach equality to every inequality. Below, read "positive" as "greater than or equal to zero".

Case (i): $x-1\geq 0$ & $x+1\geq 0$ & $x-3\geq 0$ (all factors are positive)

$$\underbrace{x\geq 1 \quad \& \quad x\geq -1 \quad \& \quad x\geq 3}_{x\geq 3}$$

($x\geq 3$ is one of the requirements, but when it's true, $x\geq 1$ & $x\geq -1$ are automatic)

Case (ii): $x-1 \geq 0$ & $x+1 \leq 0$ & $x-3 \leq 0$ ⟵ (pos. & neg. & pos.)

$\underbrace{x \geq 1 \quad \& \quad x \leq -1}_{\text{Impossible!}}$ & $x \leq 3$

Case (iii): $x-1 \leq 0$ & $x+1 \geq 0$ & $x-3 \leq 0$ ⟵ (neg. & pos. & neg.)

$x \leq 0$ & $x \geq -1$ & $x \leq 3$

$x \leq 0$ & $x \geq -1$ ⟵ ($x \leq 0$ & $x \leq 3$ reduce to $x \leq 0$)

$-1 \leq x \leq 0$

Case (iv): $x-1 \leq 0$ & $x+1 \leq 0$ & $x-3 \geq 0$ ⟵ (neg. & neg. & pos.)

$x \leq 1$ & $\underbrace{x \leq -1 \quad \& \quad x \geq 3}_{\text{Impossible!}}$

So, Case (i) yields the interval of values $[3,\infty)$ & Case (iii) yields the interval $[-1,0]$. Cases (ii) & (iv) turn out to be impossible, meaning they can never arise & so contribute nothing to the solution set of (A).

Therefore the solution set is $[-1,0] \cup [3,\infty)$. ■

Next, a <u>rational inequality</u> is an inequality that has at least one rational expression among its terms. To solve such a thing requires a certain measure of delicacy, because we cannot multiply the two sides of an inequality by an expression involving a variable if the <u>sign</u> of the expression (i.e. whether the expression is <u>positive</u> or <u>negative</u>) depends on the value of the variable. After all, if we multiply the two sides of an inequality by a negative quantity, the direction of the inequality symbol must be reversed!

Example 7 Solve $\dfrac{3}{x-2} \leq 1$

• <u>Solution</u>: You <u>can't</u> do this: $(x-2)\cdot\dfrac{3}{x-2} \leq 1\cdot(x-2) \Rightarrow 3 \leq x-2 \Rightarrow x \geq 5$. $[5,\infty)$ is <u>not</u> the solution set! The problem: $x-2$ could be negative, in which case the inequality symbol indicated by the arrow above would need to reverse. If we assume $x-2$ is negative we get $x \leq 5$; but $(-\infty,5]$ isn't the solution set either, and neither is $(-\infty,5] \cup [5,\infty)$ (which encompasses all real numbers).

Here's the way to proceed...

$$\frac{3}{x-2} - 1 \leq 0 \Rightarrow \frac{3}{x-2} - \frac{x-2}{x-2} \leq 0 \Rightarrow \frac{3-(x-2)}{x-2} \leq 0 \Rightarrow \frac{5-x}{x-2} \leq 0.$$

Now, how does a quotient like $\frac{A}{B}$ turn out negative (i.e. less than zero)? There are two mutually exclusive ways: $A<0$ & $B>0$ (A negative, B positive) <u>or</u> $A>0$ & $B<0$ (A pos., B neg.)

So, for $\dfrac{5-x}{x-2}$ to be negative (<0), either: (i) $5-x<0$ & $x-2>0$

or: (ii) $5-x>0$ & $x-2<0$

But wait! $\frac{5-x}{x-2} \leq 0$ allows for the possibility that $\frac{5-x}{x-2} = 0$. This suggests that are two cases are actually:

(i) $5-x \leq 0$ & $x-2 \geq 0$ or (ii) $5-x \geq 0$ & $x-2 \leq 0$

But wait! If $x-2=0$, then our fraction becomes $\frac{5-x}{0}$, and division by zero is a mathematical felony! We must amend our two cases one last time to get the correct versions:

(i) $5-x \leq 0$ & $x-2 > 0$ or (ii) $5-x \geq 0$ & $x-2 < 0$

Lose the = sign

NOW we are ready to don our tin-foil propeller beanies and get cracking on a solution...

Case (i):

$$\underbrace{\begin{array}{ccc} 5-x \leq 0 & \& & x-2>0 \\ -x \leq -5 & \& & x>2 \\ x \geq 5 & \& & x>2 \end{array}}_{x \geq 5}$$

($x \geq 5$ is stricter than $x>2$)

Case (ii):

$$\underbrace{\begin{array}{ccc} 5-x \geq 0 & \& & x-2<0 \\ -x \geq -5 & \& & x<2 \\ x \leq 5 & \& & x<2 \end{array}}_{x < 2}$$

($x<2$ is stricter than $x \leq 5$)

So Case (i) yields the values $[5,\infty)$ (Look familiar? See the beginning of this example in red). Case (ii) yields $(-\infty,2)$ (something that never arose from the faulty analyses perused earlier).

Solution set: $(-\infty,2) \cup [5,\infty)$ ∎