Math 115 Exam #2 Key (Spring 2017)

1 By the Fundamental Counting Principle: $(3)(2)(4)(6)(5) = 720$ ways.

2 Probability of diamond is $\frac{13}{52} = \frac{1}{4}$, and probability of a face card is $\frac{12}{52} = \frac{3}{13}$. By the Multiplication Rule the probability of a diamond and then a face card is $\frac{1}{4} \cdot \frac{3}{13} = \frac{3}{52}$.

3a $\frac{414}{2152} \approx 0.192$.

3b $\frac{414}{2152} + \frac{46}{2152} = \frac{460}{2152} \approx 0.214$.

3c $\frac{502}{924} \approx 0.543$.

3d $\frac{208}{531} \approx 0.392$.

4 Let E_1 be the event of a puncture, and E_2 the event of a smashed corner. By the Addition Rule,

$$\mathsf{P}(E_1 \text{ or } E_2) = \mathsf{P}(E_1) + \mathsf{P}(E_2) - \mathsf{P}(E_1 \text{ and } E_2) = 0.05 + 0.08 - 0.004 = 0.126.$$

5 $${}_{27}P_4 = \frac{27!}{(27-4)!} = \frac{27!}{23!} = (24)(25)(26)(27) = 421{,}200.$$

6 There are $n = 15$ letters, with $n_1 = 1$ A, $n_2 = 2$ E's, $n_3 = 2$ G's, $n_4 = 5$ N's, $n_5 = 1$ O, $n_6 = 1$ R, $n_7 = 1$ S, $n_8 = 1$ T, and $n_9 = 1$ U. Number of ways the letters can be written:

$$\frac{n!}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!n_8!n_9!} = \frac{15!}{1!2!2!5!1!1!1!1!1!} = \frac{1{,}307{,}674{,}368{,}000}{480} = 2{,}724{,}321{,}600.$$

7 $$\frac{{}_3C_2 \cdot {}_{12}C_2 + {}_3C_3 \cdot {}_{12}C_1}{{}_{15}C_4} = \frac{(3)(66) + (1)(12)}{1365} = \frac{210}{1365} \approx 0.154.$$

8 The mean μ is:

$$\sum x\mathsf{P}(x) = 0(0.686) + 1(0.195) + 2(0.077) + 3(0.022) + 4(0.013) + 5(0.007) = 0.502.$$

The variance σ^2 is:

$$\begin{aligned}\sum (x-\mu)^2\mathsf{P}(x) &= \sum (x - 0.502)^2\mathsf{P}(x) \\ &= (0-0.502)^2(0.686) + (1-0.502)^2(0.195) + \cdots + (5-0.502)^2(0.007) \\ &= 0.831996 \approx 0.832.\end{aligned}$$

The standard deviation is $\sigma = \sqrt{0.831996} \approx 0.912$.

9a This is a binomial experiment with $n = 8$ independent trials, and if we define "success" to be "guessing the right answer," then the probability of success is $p = \frac{1}{5}$ for each trial. Thus the event of getting precisely 3 questions correct is the event of observing a total of $k = 3$ successes. The probability we seek is

$$\mathsf{P}(X = 3) = {}_8C_3\left(\frac{1}{5}\right)^3\left(1 - \frac{1}{5}\right)^{8-3} = \frac{8!}{(3!)(5!)}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^5 \approx 0.1468.$$

9b Only getting 6, 7, or 8 questions right results in a score of at least 70%, and so we must find the probability that $X \geq 6$. The event $X \geq 6$ is the union of the events $X = 6$, $X = 7$, and $X = 8$, and since these are mutually exclusive events, the Addition Rule implies that

$$\mathsf{P}(X \geq 6) = \mathsf{P}\big(X = 6 \text{ or } X = 7 \text{ or } X = 8\big) = \mathsf{P}(X = 6) + \mathsf{P}(X = 7) + \mathsf{P}(X = 8)$$

We calculate

$$\mathsf{P}(X = 6) = {}_8C_6\left(\frac{1}{5}\right)^6\left(\frac{4}{5}\right)^2 \approx 0.001147,$$

$$\mathsf{P}(X = 7) = {}_8C_7\left(\frac{1}{5}\right)^7\left(\frac{4}{5}\right)^1 \approx 8.192 \times 10^{-5},$$

and

$$\mathsf{P}(X = 8) = {}_8C_8\left(\frac{1}{5}\right)^8\left(\frac{4}{5}\right)^0 = \left(\frac{1}{5}\right)^8 \approx 2.560 \times 10^{-6}.$$

Therefore the probability of passing the quiz is

$$\mathsf{P}(X \geq 6) \approx 0.001147 + 8.192 \times 10^{-5} + 2.560 \times 10^{-6} \approx 0.00123.$$

10a This is a geometric experiment in which each trial consists of the drilling of a hole, "success" is defined to be discovery of a productive well, and the probability of success is $p = 0.21$ for each trial. Letting the random variable X denote the number of the trial in which the first success is observed, the probability of success on the 3rd trial is

$$\mathsf{P}(X = 3) = 0.21(1 - 0.21)^{3-1} = 0.21(0.79)^2 \approx 0.131.$$

10b We start by finding the probability the first success is observed upon performing one of the first six trials. With X defined as before, this means calculating the probability that $X \leq 6$. The event $X \leq 6$ is the union of the six events $X = 1, \ldots, X = 6$, and since these are mutually exclusive events the Addition Rule implies that

$$\begin{aligned}\mathsf{P}(X \leq 6) &= \sum_{k=1}^{6}\mathsf{P}(X = k) = \sum_{k=1}^{6} 0.21(1 - 0.21)^{k-1} = 0.21\sum_{k=1}^{6}(0.79)^{k-1} \\ &= 0.21(1 + 0.79 + 0.79^2 + 0.79^3 + 0.79^4 + 0.79^5) \approx 0.7569\end{aligned}$$

The probability of a failed search is the probability that no success occurs in the first 6 trials, which is $\mathsf{P}(X > 6)$. Since $\mathsf{P}(X \leq 6) + \mathsf{P}(X > 6) = 1$, we find that

$$\mathsf{P}(X > 6) = 1 - \mathsf{P}(X \leq 6) = 1 - 0.7569 \approx 0.2431.$$

11a This is a Poisson experiment with parameter $\lambda = 2$, and so if the random variable X counts the number of typos on a page, then

$$\mathsf{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{2^k e^{-2}}{k!}$$

for $k = 0, 1, 2, \ldots$. Thus the probability of exactly two typos on a randomly selected page is

$$\mathsf{P}(X = 2) = \frac{2^2 e^{-2}}{2!} \approx 0.2707.$$

11b We have

$$\begin{aligned}\mathsf{P}(X > 2) &= 1 - \mathsf{P}(X \le 2) = 1 - [\mathsf{P}(X = 2) + \mathsf{P}(X = 1) + \mathsf{P}(X = 0)] \\ &= 1 - \frac{2^2 e^{-2}}{2!} - \frac{2^1 e^{-2}}{1!} - \frac{2^0 e^{-2}}{0!} \\ &= 1 - 0.2707 - 0.2707 - 0.1353 = 0.3233.\end{aligned}$$