Math 102: Chapter 10 Supplement Solutions

1. The mathematical system consisting of the positive real numbers and the binary operation of multiplication.

   - Closure. For any positive real numbers a and b, A3 implies that $a \cdot b$ is a real number; and since we take for granted that the product of two positive numbers is positive, it follows that $a \cdot b$ must be a positive real number. Closure holds.
   - Associativity. Since positive real numbers are real numbers, A4 guarantees that associativity holds.
   - Identity. By A6 we have that 1 is the identity for real number multiplication, and fortunately for us 1 is a positive real number! The identity property holds.
   - Inverse. Let a be a positive real number (and so $a \neq 0$ in particular). Now, $a \cdot 1/a = 1/a \cdot a = 1$, where $1/a$ is itself a positive real number since we take for granted that the quotient of two positive numbers is positive, and also that a real divided by a nonzero real is real. The inverse property holds.
   - Commutativity. A7 guarantees that commutativity holds.

2. The system consisting of the set of nonnegative real numbers and the operation of addition.

   - Closure. A3 ensures that the sum of two reals is real, and we take for granted that if two numbers are nonnegative then their sum is nonnegative. Closure holds.
   - Associativity. Nonnegative reals are real, so A4 guarantees that associativity holds.
   - Identity. By A6 we have that 0 is the identity for real number addition, and fortunately for us 0 is a nonnegative real number! The identity property holds.
   - Inverse. 2 is a nonnegative real, but the solution to $2 + x = 0$ is -2, which is *not* a nonnegative real. Thus, 2 has no inverse that exists within the system. The inverse property fails.
   - Commutativity. A7 guarantees that commutativity holds.

3. The system consisting of the even integers and addition. (By definition an even integer is an integer n for which $n \div 2$ is an integer, so the set of even integers is $\{..., -4, -2, 0, 2, 4, ...\}$.)

   - Closure. Let a and b be even integers. Then by definition (see Exercise 3) $a/2$ is an integer and $b/2$ is an integer. It follows that $(a + b)/2$ is an integer since $(a + b)/2 = a/2 + b/2$, and $a/2 + b/2$ must be an integer by A1. Since $(a + b)/2$ is an integer (i.e. $a + b$ is divided evenly by 2), $a + b$ is an *even* integer. This confirms that the closure property holds.
   - Associativity. If a, b, c are even integers, then a, b, c, are real numbers and associativity follows by A4. Associativity holds.
   - Identity. If a is an even integer, then a is a real number and so $a + 0 = a = 0 + a$ follows from A6. Hence 0 is the identity element, and the identity property holds.
   - Inverse. Let a be an even integer. This means $a/2$ is an integer. Now, $-a/2 = -1 \cdot a/2$, which is a product of integers and must therefore be an integer by A1. Since $-a/2$ is an integer, it follows that $-a$ is an *even* integer. Finally, $a + (-a) = -a + a = 0$ shows that $-a$, an even

integer, is the *inverse* for a. Conclusion: every even integer has an inverse that is also an even integer. The inverse property holds.

- COMMUTATIVITY. Let a and b be even integers. Then a and b are real numbers and commutativity follows by A7. Commutativity holds.

4. The system consisting of the even integers and multiplication.

   - CLOSURE. Let a and b be even integers. Then $a/2$ and $b/2$ are integers. It follows that $(a \cdot b)/2$ is an integer since $(a \cdot b)/2 = a \cdot b/2$, and $a \cdot b/2$ must be an integer by A1. Since $(a \cdot b)/2$ is an integer, $a \cdot b$ is an *even* integer. Closure holds.
   - ASSOCIATIVITY. If a, b, c are even integers, then a, b, c, are real numbers and associativity follows by A5. Associativity holds.
   - IDENTITY. The identity for multiplication of even integers must be 1 by A6, but 1 is not an even integer! Identity property fails.
   - INVERSE. Without an identity element in our system to speak of inverses is silly talk. Inverse property fails.
   - COMMUTATIVITY. Even integers are real numbers, so commutativity follows by A8. Commutativity holds.

5. The system consisting of the odd integers and addition. (The set of odd integers is the set of integers that are not even: $\{..., -5, -3, -1, 1, 3, 5, ...\}$.).

   - CLOSURE. 3 and 5 are odd integers, but behold the tragedy: $3 + 5 = 8$, and 8 ain't an odd integer! Closure fails.
   - ASSOCIATIVITY. If a, b, c are odd integers, then a, b, c, are real numbers and associativity follows by A4. Associativity holds.
   - IDENTITY. The identity for addition of odd integers must be 0 by A6, but 0 is not an odd integer! Identity property fails.
   - INVERSE. Without an identity element in our system the inverse property must fail.
   - COMMUTATIVITY. Odd integers are real numbers, so commutativity follows by A7. Commutativity holds.

6. The system consisting of the odd integers and multiplication. Assume that the product of two odd integers is odd.

   - CLOSURE. We're given that the product of two odds is odd. Closure holds.
   - ASSOCIATIVITY. If a, b, c are odd integers, then a, b, c, are real numbers and associativity follows by A5. Associativity holds.
   - IDENTITY. The identity for multiplication of odd integers must be 1 by A6, and 1 is an odd integer! Identity property holds.
   - INVERSE. Let a be an odd integer. It should be clear that $-a$ must also be odd, since otherwise it would have to be even and then its opposite value would have to be even as well (see the investigation of the inverse property in #3 above). Now, since $a + (-a) = -a + a = 0$, we conclude that every odd integer has an inverse that is also odd. Inverse property holds.

- Commutativity. Odd integers are real numbers, so commutativity follows by A8. Commutativity holds.

7. The system consisting of the integers and the binary operation $\ominus$ defined as follows: $a \ominus b = |a - b|$ (i.e. the absolute value of $a - b$). Assume that the absolute value of an integer is an integer.

   - Closure. Let $a, b \in \mathbb{Z}$. Then $-b \in \mathbb{Z}$ since $-b = -1 \cdot b$, and the product of two integers is an integer by A1. Next, $a - b \in \mathbb{Z}$ since $a - b = a + (-b)$, an the sum of two integers is an integer by A1. We're given that the absolute value of an integer is an integer (actually an easy thing to prove using our axioms), so $|a - b| \in \mathbb{Z}$ obtains at last. Closure holds.
   - Associativity. Note that $1 \ominus (2 \ominus 3) = 1 \ominus 1 = 0$, but $(1 \ominus 2) \ominus 3 = 1 \ominus 3 = 2$. Associativity fails.
   - Identity. Here 0 is our only viable candidate, however notice that $-2 \ominus 0 = 2 \neq -2$, which disqualifies 0. Identity property fails.
   - Inverse. There is no identity element, so the inverse property is dead on arrival.
   - Commutativity. For any $a, b \in \mathbb{Z}$, $a \ominus b = |a - b| = |b - a| = b \ominus a$. Commutativity holds.

8. The system consisting of the set of whole numbers $\{0, 1, 2, 3, ...\}$ and the binary operation $\ominus$. Assume that the absolute value of an integer is an integer.

   - Closure. Let a and b be whole numbers. Then a and b are integers, and so $|a - b|$ is an integer as argued in #7. But the absolute value of any real number is nonnegative, so $|a - b|$ must be an integer that's greater than or equal to 0; that is, $|a - b|$ must be a whole number. Closure holds.
   - Associativity. $1 \ominus (2 \ominus 3) = 1 \ominus 1 = 0$, but $(1 \ominus 2) \ominus 3 = 1 \ominus 3 = 2$. Associativity fails.
   - Identity. For any whole number a we have $a \ominus 0 = |a - 0| = |a| = a$ and $0 \ominus a = |0 - a| = |-a| = a$, so the identity element is 0. Identity property holds!
   - Inverse. For any whole number a we have $a \ominus a = |a - a| = |0| = 0$, so a is its own inverse. It's a miracle![1] Inverse property holds.
   - Commutativity. For any whole numbers a and b, $a \ominus b = |a - b| = |b - a| = b \ominus a$. Commutativity holds.

[1] "No," Mr. Spock replies flatly with arched eyebrow. "It's simple logic."