Math 102: Chapter 10 Supplementary Exercises

In doing the exercises in section 10.1 in the textbook and in this supplement, there are some very basic facts that you can take for granted without comment such as: (1) The product or quotient of two positive real numbers is positive; (2) the product or quotient of a positive and negative number is negative; (3) the product or quotient of two negative numbers is positive; and the sum of two negative numbers is negative whilst the sum of two positive numbers is positive. If we don't take some basic things for granted we can't get anywhere. On top of these facts are nine additional results that for our purposes are taken to be axioms,[1] and they should be cited in the course of doing the homework problems whenever they are being employed. Here they are:

A1)	For any $a, b \in \mathbb{Z}$, $a + b \in \mathbb{Z}$ and $a \cdot b \in \mathbb{Z}$
A2)	For any $a, b \in \mathbb{Q}$, $a + b \in \mathbb{Q}$ and $a \cdot b \in \mathbb{Q}$
A3)	For any $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$
A4)	For any $a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$
A5)	For any $a, b, c \in \mathbb{R}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
A6)	For any $a \in \mathbb{R}$, $a + 0 = a = 0 + a$ and $a \cdot 1 = a = 1 \cdot a$
A7)	For any $a, b \in \mathbb{R}$, $a + b = b + a$
A8)	For any $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$
A9)	For any $a, b \in \mathbb{R}$, if $a \cdot b = 0$ then either $a = 0$ or $b = 0$

This table will be supplied on exam day so that everyone can refer to them using the designations I've given them (A1, A2, and so on). Recall that $\mathbb{R}$ is the set of real numbers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ the set of integers. Unlike the axioms, you *must* know the five properties of a commutative group with set of elements S and binary operation $*$ by heart:

Closure	For any $a, b \in S$, $a * b \in S$.
Associativity	For any $a, b, c \in S$, $a * (b * c) = (a * b) * c$.
Identity	There exists some $i \in S$ such that, for any $a \in S$, $i * a = a * i = a$.
Inverse	For any $a \in S$ there exists some $\bar{a} \in S$ such that $a * \bar{a} = \bar{a} * a = i$.
Commutativity	For any $a, b \in S$, $a * b = b * a$.

In addition to the exercises assigned in section 10.1, do also the exercises below. Instructions for doing the section 10.1 problems as well as the problems below are as follows: *For each mathematical system, determine which of the five properties of a commutative group holds. If a property fails, give a counterexample. If a property holds, explain why by referring to axioms A1 through A9 and other reasoning.*

1. The mathematical system consisting of the positive real numbers and the binary operation of multiplication.
2. The system consisting of the set of nonnegative real numbers and the operation of addition.

[1] An *axiom* is just a statement that is taken to be true without proof. In contrast a *theorem* is a statement that must be proved, usually using given axioms or other theorems that have already been established.

3. The system consisting of the even integers and addition. (By definition an even integer is an integer n for which $n \div 2$ is an integer, so the set of even integers is $\{..., -4, -2, 0, 2, 4, ...\}$.)
4. The system consisting of the even integers and multiplication.
5. The system consisting of the odd integers and addition. (The set of odd integers is the set of integers that are not even: $\{..., -5, -3, -1, 1, 3, 5, ...\}$.)
6. The system consisting of the odd integers and multiplication. Assume that the product of two odd integers is odd.
7. The system consisting of the integers and the binary operation $\ominus$ defined as follows: $a \ominus b = |a - b|$ (i.e. the absolute value of $a - b$). Assume that the absolute value of an integer is an integer.
8. The system consisting of the set of whole numbers $\{0, 1, 2, 3, ...\}$ and the binary operation $\ominus$. Assume that the absolute value of an integer is an integer.

EXAMPLE. Find which of the five properties of a commutative group hold for the mathematical system consisting of the set of even integers and the binary operation of addition.

Examine each property one at a time. Except for the requirement that the identity property be confirmed before looking at the inverse property, the order in which the properties are addressed is not important.

- CLOSURE. Let a and b be even integers. Then by definition (see Exercise 3) $a/2$ is an integer and $b/2$ is an integer. It follows that $(a + b)/2$ is an integer since $(a + b)/2 = a/2 + b/2$, and $a/2 + b/2$ must be an integer by A1. Since $(a + b)/2$ is an integer (i.e. $a + b$ is divided evenly by 2), $a + b$ is an *even* integer. This confirms that the closure property holds.
- ASSOCIATIVITY. If a, b, c are even integers, then a, b, c, are real numbers and associativity follows by A4. Associativity holds.
- IDENTITY. If a is an even integer, then a is a real number and so $a + 0 = a = 0 + a$ follows from A6. Hence 0 is the identity element, and the identity property holds.
- INVERSE. Let a be an even integer. This means $a/2$ is an integer. Now, $-a/2 = -1 \cdot a/2$, which is a product of integers and must therefore be an integer by A1. Since $-a/2$ is an integer, it follows that $-a$ is an *even* integer. Finally, $a + (-a) = (-a) + a = 0$ shows that $-a$, an even integer, is the *inverse* for a. Conclusion: every even integer has an inverse that is also an even integer. The inverse property holds.
- COMMUTATIVITY. Let a and b be even integers. Then a and b are real numbers and commutativity follows by A7. Commutativity holds.

So all the properties hold here, making the system consisting of addition of even integers a commutative group. ■