6

The Riemann Integral

6.1 – Definition of the Riemann Integral

Definition 6.1. *Given an interval* $[a, b]$ *with* $a < b$*, a* ***partition*** P *of* $[a, b]$ *is a finite set of points* $\{x_0, x_1, \ldots, x_n\} \subseteq [a, b]$*, called* ***grid points****, such that* $x_0 = a$*,* $x_n = b$*, and* $x_{k-1} < x_k$ *for all* $k = 1, \ldots, n$*. The points* x_k *subdivide* $[a, b]$ *into* n *subintervals,*

$$[a, x_1],\ [x_1, x_2],\ \ldots\ , [x_{k-1}, x_k],\ \ldots\ , [x_{n-2}, x_{n-1}],\ [x_{n-1}, b],$$

with the length of the k*th subinterval being* $\Delta x_k = x_k - x_{k-1}$*. The* ***mesh*** *of* P*, denoted by* $\|P\|$*, is defined to be the length of the longest subinterval:*

$$\|P\| = \max_{1 \le k \le n} \Delta x_k.$$

The set of all possible partitions of $[a, b]$ *we denote by* $\mathcal{P}[a, b]$*.*

Notation. It is convenient to denote a partition $\{x_0, x_1, \ldots, x_n\}$ more compactly by $\{x_k\}_{k=0}^n$, and write $\{x_k\}_{k=0}^n \in \mathcal{P}[a, b]$ to specify that $\{x_k\}_{k=0}^n$ is a partition of the interval $[a, b]$.

Definition 6.2. *Let* $f : [a, b] \to \mathbb{R}$ *be bounded, and let* $P = \{x_k\}_{k=0}^n \in \mathcal{P}[a, b]$*. Define*

$$M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\} \quad \textit{and} \quad m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}$$

for each $1 \le k \le n$*. Then*

$$U(P, f) = \sum_{k=1}^{n} M_k \Delta x_k \quad \textit{and} \quad L(P, f) = \sum_{k=1}^{n} m_k \Delta x_k$$

are the ***upper sum of*** f ***with respect to*** P *and* ***lower sum of*** f ***with respect to*** P*, respectively.*

Note that since f is bounded on $[a, b,]$, it must be bounded on each $[x_{k-1}, x_k] \subseteq [a, b]$ and hence M_k and m_k must be real numbers as a consequence of the Completeness Axiom of $\mathbb{R}$.

Definition 6.3. *Let* $f : [a, b] \to \mathbb{R}$ *be bounded. The* ***upper Riemann integral of*** f ***over*** $[a, b]$ *is*

$$\overline{\int_a^b} f = \inf\{U(P, f) : P \in \mathcal{P}[a, b]\},$$

and the ***lower Riemann integral of*** f ***over*** $[a,b]$ *is*

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}.$$

Proposition 6.4. *If* $f : [a,b] \to \mathbb{R}$ *is bounded, then*

$$\overline{\int_a^b} f \quad \text{and} \quad \underline{\int_a^b} f$$

exist in $\mathbb{R}$.

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Then there exists some $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in [a,b]$. Let $P = \{x_k\}_{k=0}^n \in \mathcal{P}[a,b]$. For each $1 \leq k \leq n$ we have $f(x) \leq M$ for all $x \in [x_{k-1}, x_k]$, and so

$$m_k = \inf\{f(x) : x_{k-1} \leq x \leq x_k\} \leq M$$

for each $1 \leq k \leq n$. Thus

$$L(P,f) = \sum_{k=1}^n m_k \Delta x_k \leq \sum_{k=1}^n M \Delta x_k = M \sum_{k=1}^n \Delta x_k = M(b-a),$$

and since $P \in \mathcal{P}[a,b]$ is arbitrary we conclude that the real number $M(b-a)$ is an upper bound for $\{L(P,f) : P \in \mathcal{P}[a,b]\}$. Therefore by the Completeness Axiom the least upper bound of $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ is real-valued, which is to say

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}$$

exists in $\mathbb{R}$.

The proof of the statement concerning the upper Riemann integral of f over $[a,b]$ is similar and so left as an exercise. ■

Proposition 6.5. *If* $P_1, P_2 \in \mathcal{P}[a,b]$ *are such that* $P_1 \subseteq P_2$*, then*

$$L(P_1,f) \leq L(P_2,f) \quad \text{and} \quad U(P_2,f) \leq U(P_1,f).$$

Proposition 6.6. *If* $P_1, P_2 \in \mathcal{P}[a,b]$*, then*

$$L(P_1,f) \leq U(P_2,f).$$

Definition 6.7. *A bounded function* $f : [a,b] \to \mathbb{R}$ *is* ***Riemann integrable on*** $[a,b]$ *if*

$$\underline{\int_a^b} f = \overline{\int_a^b} f = I_f \in \mathbb{R}.$$

We call the real number I_f *the* ***Riemann integral of*** f ***over*** $[a,b]$*, and denote it by the symbol*

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx.$$

The set of all functions that are Riemann integrable on $[a,b]$ *is denoted by* $\mathcal{R}[a,b]$.

The Riemann integral, also known in these notes as the **definite integral**, is just one of many different kinds of integrals defined in mathematics. In the symbol $\int_a^b f$, which in practice may be read as "the integral of f from a to b," we call a the **lower limit of integration**, b the **upper limit of integration**, and f the **integrand**.

The x in the symbol

$$\int_a^b f(x)\,dx \tag{1}$$

given in Definition 6.7 is called the **variable of integration**. It is also called a **dummy variable**, since we could substitute other letters for x and the meaning of the symbol would be unchanged. Thus

$$\int_a^b f(x)\,dx, \quad \int_a^b f(t)\,dt, \quad \int_a^b f(u)\,du$$

and so on are all considered identical Riemann integrals. As the simpler symbol $\int_a^b f$ suggests, only the integrand f and the limits of integration a and b uniquely determine a Riemann integral.

Using the symbol (1) is preferred especially when the function $[a,b] \to \mathbb{R}$ in the integrand has no designation. Thus we may write

$$\int_0^9 x^2\,dx$$

to denote the Riemann integral $\int_0^9 f$ with integrand $f(x) = x^2$. The symbol (1) is also useful when there is more than one independent variable present in an analysis, as will often be the case from Chapter 13 onward.

Proposition 6.8. *Let $c \in \mathbb{R}$. If $f \equiv c$ on $[a,b]$, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = c(b-a)$.*

If $f \equiv c$ on $[a,b]$ then it's common practice to write

$$\int_a^b c = c(b-a) \quad \text{or} \quad \int_a^b c\,dx = c(b-a),$$

depending on one's preference.

Theorem 6.9. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}[a,b]$ if and only if for every $\epsilon > 0$ there exists some $P \in \mathcal{P}[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.*

6.2 – Riemann Sums

Given a partition $P = \{x_k\}_{k=0}^n$, a **sample point** from $[x_{k-1}, x_k]$ is any point x_k^* chosen from the interval, so that

$$x_{k-1} \leq x_k^* \leq x_k$$

for each $1 \leq k \leq n$.

Definition 6.10. *Given a function $f : [a, b] \to \mathbb{R}$, a partition $P = \{x_k\}_{k=0}^n \in \mathcal{P}[a, b]$, and sample points $x_k^* \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$, the sum*

$$S(P, f) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

*is called the **Riemann sum for** f **with respect to** P **on** $[a, b]$.*

In this definition as well as the next one it is important to bear in mind that the value of the integer n depends on the choice of partition P. We could write n_P instead of n to emphasize this, but will refrain from doing so to minimize clutter.

Definition 6.11. *Let $L \in \mathbb{R}$. Then we define*

$$\lim_{\|P\| \to 0} S(P, f) = L$$

to mean the following: for every $\epsilon > 0$ there exists some $\delta > 0$ such that if $P = \{x_k\}_{k=0}^n \in \mathcal{P}[a, b]$ with $0 < \|P\| < \delta$, then

$$|S(P, f) - L| < \epsilon$$

for all choice of sample points $x_k^* \in [x_{k-1}, x_k]$, $1 \leq k \leq n$.

The following theorem provides, at least theoretically, a means of calculating a Riemann integral by evaluating a limit. Further improvements are forthcoming.

Theorem 6.12. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a, b]$ if and only if*

$$\lim_{\|P\| \to 0} S(P, f) = L$$

for some $L \in \mathbb{R}$, in which case $\int_a^b f = L$.

6.3 – Properties of the Riemann Integral

In this section we establish various general properties of the Riemann integral. Most of these properties will make the task of evaluating Riemann integrals easier, and certainly all of them will be useful later on to prove further theoretical results.

Definition 6.13. *For any $f \in \mathcal{R}[a,b]$ we define*

$$\int_b^a f = -\int_a^b f.$$

For any function f for which $f(a) \in \mathbb{R}$ we define

$$\int_a^a f = 0.$$

Proposition 6.14 (**Linearity Properties of the Riemann Integral**)**.** *Suppose $f, g \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$. Then $f \pm g \in \mathcal{R}[a,b]$ and $cf \in \mathcal{R}[a,b]$ such that*

1. $\displaystyle\int_a^b cf = c\int_a^b f$
2. $\displaystyle\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$

Theorem 6.15. *Suppose $f, g \in \mathcal{R}[a,b]$. If $f \leq g$ on $[a,b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Theorem 6.16. *Suppose $c \in (a,b)$. If $f \in \mathcal{R}[a,b]$, then $f \in \mathcal{R}[a,c]$, $f \in \mathcal{R}[c,b]$, and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

A result amounting effectively to a converse of the statement of Theorem 6.16 is the following proposition.

Proposition 6.17. *Suppose $c \in (a,b)$. If $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$, then $f \in \mathcal{R}[a,b]$.*

Proof. Suppose $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$. Then f is bounded on $[a,c]$ and $[c,b]$, which implies that it is bounded on $[a,b]$. Let $\epsilon > 0$. By Theorem 6.9 there exist partitions $P_1 \in \mathcal{P}[a,c]$ and $P_2 \in \mathcal{P}[c,b]$ such that

$$U(P_1,f) - L(P_1,f) < \epsilon/2 \quad \text{and} \quad U(P_2,f) - L(P_2,f) < \epsilon/2$$

Define $P \in \mathcal{P}[a,b]$ by $P = P_1 \cup P_2$, so that

$$U(P,f) = U(P_1,f) + U(P_2,f) \quad \text{and} \quad L(P,f) = L(P_1,f) + L(P_2,f).$$

Recalling Proposition 6.5, we have

$$\begin{aligned}
U(P,f) - L(P,f) &= |U(P,f) - L(P,f)| \\
&= \big|[U(P_1,f) + U(P_2,f)] - [L(P_1,f) + L(P_2,f)]\big| \\
&\le |U(P_1,f) - L(P_1,f)| + |U(P_2,f) - L(P_2,f)| \\
&= U(P_1,f) - L(P_1,f) + U(P_2,f) - L(P_2,f) \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Thus there exists some partition in $\mathcal{P}[a,b]$ for which $U(P,f) - L(P,f) < \epsilon$. Therefore $f \in \mathcal{R}[a,b]$ by Theorem 6.9. ■

Theorem 6.18. *Suppose $f \in \mathcal{R}[a,b]$. If*

$$\{x \in [a,b] : g(x) \neq f(x)\}$$

is a finite set, then $g \in \mathcal{R}[a,b]$ and

$$\int_a^b g = \int_a^b f.$$

6.4 – Integrable Functions

In this section we develop tools to help determine whether a given function is Riemann integrable on an interval $[a, b]$.

Theorem 6.19. *If* $\varphi \in \mathcal{R}[a, b]$, $\operatorname{Ran}(\varphi) \subseteq [\alpha, \beta]$, *and* $\psi : [\alpha, \beta] \to \mathbb{R}$ *is continuous, then* $\psi \circ \varphi \in \mathcal{R}[a, b]$.

Proposition 6.20. *If* f *is continuous on* $[a, b]$, *then* $f \in \mathcal{R}[a, b]$.

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Let $\varphi : [a, b] \to \mathbb{R}$ be the identity function on $[a, b]$, so that $\varphi(x) = x$ for all $x \in [a, b]$. Since $\operatorname{Ran}(\varphi) = [a, b]$, and $\varphi \in \mathcal{R}[a, b]$ by Proposition **??**, it follows by Theorem 6.19 that $f \circ \varphi \in \mathcal{R}[a, b]$. Now, for any $x \in [a, b]$,

$$(f \circ \varphi)(x) = f(\varphi(x)) = f(x),$$

so we see that $f \circ \varphi = f$ and therefore $f \in \mathcal{R}[a, b]$. ■

Proposition 6.21. *If* $f, g \in \mathcal{R}[a, b]$, *then*

1. $f^2 \in \mathcal{R}[a, b]$,
2. $fg \in \mathcal{R}[a, b]$,
3. $|f| \in \mathcal{R}[a, b]$ *with* $\left|\int_a^b f\right| \le \int_a^b |f|$,
4. $f \vee g, f \wedge g \in \mathcal{R}[a, b]$.

Proof.
Proof of Part (1). Suppose that $f \in \mathcal{R}[a, b]$. Then $f : [a, b] \to \mathbb{R}$ is a bounded function, so that $\operatorname{Ran}(f) \subseteq [\alpha, \beta]$ for some $-\infty < \alpha < \beta < \infty$. Defining the function ψ by $\psi(x) = x^2$, it is clear that ψ is continuous on $[\alpha, \beta]$. By Theorem 6.19 we conclude that $\psi \circ f \in \mathcal{R}[a, b]$. Now, since

$$(\psi \circ f)(x) = \psi(f(x)) = [f(x)]^2 = f^2(x)$$

for any $x \in [a, b]$, we see that $\psi \circ f = f^2$ and therefore $f^2 \in \mathcal{R}[a, b]$.

Proof of Part (2). Suppose that $f, g \in \mathcal{R}[a, b]$. By Proposition 6.14 we have $f+g, f-g \in \mathcal{R}[a, b]$, and then by Part (1) we have $(f+g)^2, (f-g)^2 \in \mathcal{R}[a, b]$. Applying Proposition 6.14 once more, it follows that

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2] \in \mathcal{R}[a, b]$$

as was to be shown.

Proof of Part (3). Suppose that $f \in \mathcal{R}[a, b]$. Once again $\operatorname{Ran}(f) \subseteq [\alpha, \beta]$ for some $-\infty < \alpha < \beta < \infty$. If $\psi(x) = |x|$, then ψ is continuous on $[\alpha, \beta]$. By Theorem 6.19 we conclude that $\psi \circ f \in \mathcal{R}[a, b]$. Now, since

$$(\psi \circ f)(x) = \psi(f(x)) = |f(x)| = |f|(x)$$

for any $x \in [a, b]$, we see that $\psi \circ f = |f|$ and therefore $|f| \in \mathcal{R}[a, b]$. ■

6.5 – The Fundamental Theorem of Calculus

Recall that, according to the Extreme Value Theorem, a continuous function on a closed interval $[a, b]$ will attain an absolute maximum value and an absolute minimum value. That is, there will be some $x_1, x_2 \in [a, b]$ such that $f(x_1) \geq f(x)$ for all $x \in [a, b]$ and $f(x_2) \leq f(x)$ for all $x \in [a, b]$. Then we can write

$$f(x_1) = \max\{f(x) : x \in [a, b]\} \quad \text{and} \quad f(x_2) = \min\{f(x) : x \in [a, b]\}.$$

This is essential in what follows.

Lemma 6.22. *Let f be continuous on $[a, b]$ and $c \in [a, b]$. Define $\alpha_c, \beta_c : [a, b] \to \mathbb{R}$ by*

$$\alpha_c(x) = \begin{cases} \max\{f(t) : t \in [c, x]\}, & \text{if } x \geq c \\ \max\{f(t) : t \in [x, c]\}, & \text{if } x < c \end{cases}$$

and

$$\beta_c(x) = \begin{cases} \min\{f(t) : t \in [c, x]\}, & \text{if } x \geq c \\ \min\{f(t) : t \in [x, c]\}, & \text{if } x < c \end{cases}$$

Then $\lim_{x \to c} \alpha_c(x) = \lim_{x \to c} \beta_c(x) = f(c)$.

Remark. Note α_c and β_c are indeed real-valued functions for any $a \leq c \leq b$ since f is bounded on $[a, b]$ by the Extreme Value Theorem.

Proof. We will assume that $c \in (a, b)$. If $c = a$ or $c = b$ only a right- or left-hand limit, respectively, would need to be considered, but otherwise the argument would be the same.

Let $\epsilon > 0$. Since f is continuous at c, there exists some $\delta_1 > 0$ such that $c \leq x < c + \delta_1$ implies $|f(x) - f(c)| < \epsilon/2$. Suppose that $c < x < c + \delta_1$. For any $t \in [c, x]$ we have $|f(t) - f(c)| < \epsilon/2$, whence

$$f(c) - \epsilon/2 < f(t) < f(c) + \epsilon/2$$

obtains and thus

$$f(c) - \epsilon/2 < \alpha_c(x) = \max\{f(t) : t \in [c, x]\} \leq f(c) + \epsilon/2.$$

From this it can be seen that

$$|\alpha_c(x) - f(c)| \leq \epsilon/2 < \epsilon,$$

and so

$$\lim_{x \to c^+} \alpha_c(x) = f(c).$$

Next, continuity of f at c implies that there is some $\delta_2 > 0$ such that $|f(x) - f(c)| < \epsilon/2$ whenever $c - \delta_2 < x \leq c$. Suppose that $c - \delta_2 < x < c$. For any $t \in [x, c]$ we have $|f(t) - f(c)| < \epsilon/2$, which gives

$$f(c) - \epsilon/2 < f(t) < f(c) + \epsilon/2$$

and thus

$$f(c) - \epsilon/2 < \alpha_c(x) = \max\{f(t) : t \in [x, c]\} \leq f(c) + \epsilon/2.$$

Once again we're led to conclude that

$$|\alpha_c(x) - f(c)| \leq \epsilon/2 < \epsilon,$$

and so

$$\lim_{x \to c^-} \alpha_c(x) = f(c).$$

So $\lim_{x\to c} \alpha_c(x) = f(c)$ for any $c \in (a, b)$. The proof runs along similar lines for the function β_c and so it omitted. ∎

Theorem 6.23 (**The Fundamental Theorem of Calculus, Part 1**). *If f is continuous on $[a, b]$, then the function $\Phi : [a, b] \to \mathbb{R}$ given by*

$$\Phi(x) = \int_a^x f(t)\, dt, \quad a \leq x \leq b$$

is differentiable on $[a, b]$, with $\Phi'(x) = f(x)$ for each $a \leq x \leq b$.

Remark. As is our custom, it is intended that $\Phi'(a)$ and $\Phi'(b)$ be taken as signifying the one-sided derivatives $\Phi'_+(a)$ and $\Phi'_-(b)$, respectively.

Proof. Suppose that f is continuous on $[a, b]$, so that $f \in \mathcal{R}[a, b]$ by Proposition 6.20. Let $c \in (a, b)$. Using Theorem 6.16, we have

$$\lim_{x\to c^+} \frac{\Phi(x) - \Phi(c)}{x - c} = \lim_{x\to c^+} \frac{1}{x - c}\left(\int_a^x f - \int_a^c f\right) = \lim_{x\to c^+} \frac{1}{x - c}\int_c^x f \tag{2}$$

Let $\alpha_c, \beta_c : [a, b] \to \mathbb{R}$ be as defined in Lemma 6.22. For any fixed $x \in (c, b)$ we have $\beta_c(x) \leq f(t) \leq \alpha_c(x)$ for all $t \in [c, x]$. Thus by Theorem 6.15 we obtain

$$\int_c^x \beta_c(x) \leq \int_c^x f \leq \int_c^x \alpha_c(x),$$

whence Proposition 6.5 gives

$$\beta_c(x) \cdot (x - c) \leq \int_c^x f \leq \alpha_c(x) \cdot (x - c)$$

and therefore

$$\beta_c(x) \leq \frac{1}{x - c}\int_c^x f \leq \alpha_c(x). \tag{3}$$

Since the inequality (3) holds for all $c < x < b$ and

$$\lim_{x\to c^+} \alpha_c(x) = \lim_{x\to c^+} \beta_c(x) = f(c)$$

by Lemma 6.22, by (2) and the Squeeze Theorem we obtain

$$\lim_{x\to c^+} \frac{\Phi(x) - \Phi(c)}{x - c} = \lim_{x\to c^+} \frac{1}{x - c}\int_c^x f = f(c).$$

A similar argument shows that

$$\lim_{x\to c^-} \frac{\Phi(x) - \Phi(c)}{x - c} = \lim_{x\to c^-} \frac{1}{x - c}\int_c^x f = f(c),$$

and thus

$$\Phi'(c) = \lim_{x\to c} \frac{\Phi(x) - \Phi(c)}{x - c} = f(c).$$

Since $a < c < b$ is arbitrary, we conclude that Φ is differentiable on (a, b) with $\Phi'(x) = f(x)$ for all $x \in (a, b)$.

Now let $c = a$. We have

$$\alpha_a(x) = \max\{f(t) : t \in [a, x]\} \quad \text{and} \quad \beta_a(x) = \min\{f(t) : t \in [a, x]\},$$

and Lemma 6.22 gives

$$\lim_{x\to a^+} \alpha_a(x) = \lim_{x\to a^+} \beta_a(x) = f(a). \tag{4}$$

Given any $x \in (a, b)$ we have $\beta_a(x) \le f(t) \le \alpha_a(x)$ for all $t \in [a, x]$, implying

$$\int_a^x \beta_a(x) \le \int_a^x f \le \int_a^x \alpha_a(x),$$

and thus

$$\beta_a(x) \le \frac{1}{x - a} \int_a^x f \le \alpha_a(x). \tag{5}$$

Since (5) holds for all $a < x < b$, by (4) and the Squeeze Theorem we obtain

$$\Phi'_+(a) = \lim_{x\to a^+} \frac{\Phi(x) - \Phi(a)}{x - a} = \lim_{x\to a^+} \frac{\Phi(x)}{x - a} = \lim_{x\to a^+} \frac{1}{x - a} \int_a^x f = f(a),$$

where $\Phi(a) = 0$ by Definition 6.15. A similar argument shows that $\Phi'_-(b) = f(b)$.

Therefore Φ is differentiable on $[a, b]$ with $\Phi'(x) = f(x)$ for all $a \le x \le b$. ■

Example 6.24. Given that

$$F(x) = \int_2^{\sin x} (1 - t^2)^7 \, dt,$$

find F'.

Solution. If we define $\Phi(x) = \int_2^x (1 - t^2)^7 \, dt$, then

$$F(x) = \Phi(\sin(x)) = (\Phi \circ \sin)(x)$$

and the Chain Rule gives

$$F'(x) = (\Phi \circ \sin)'(x) = \Phi'(\sin(x)) \cdot \sin'(x) = \Phi'(\sin(x)) \cdot \cos(x)$$

for all $x \in \mathbb{R}$. Now, by Theorem 6.23 we have $\Phi'(x) = (1 - x^2)^7$, and so

$$F'(x) = \Phi'(\sin x) \cdot \cos x = (1 - \sin^2 x)^7 (\cos x) = (\cos^2 x)^7 (\cos x) = \cos^{15}(x)$$

for all $x \in \mathbb{R}$. ■

Recalling Definition 4.31, what Theorem 6.23 says is that Φ is an antiderivative for f on $[a, b]$. We make use of this fact to prove the following.

Theorem 6.25 (**The Fundamental Theorem of Calculus, Part 2**). *Let f be continuous on $[a,b]$. If F is any antiderivative for f on $[a,b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

Proof. F be an antiderivative for f on $[a,b]$. By Theorem 4.33 there exists some constant c such that $F = \Phi + c$, where Φ is the antiderivative for f on $[a,b]$ that is given in Theorem 6.23. That is,

$$F(x) = \Phi(x) + c = \int_a^x f + c$$

for $x \in [a,b]$, which gives $F(a) = \int_a^a f + c = c$, and so

$$F(b) = \int_a^b f + c = \int_a^b f + F(a).$$

Therefore $\int_a^b f = F(b) - F(a)$. ∎

The first part of the Fundamental Theorem of Calculus shows how a definite integral can be used to determine an antiderivative for a function f on a closed interval $[a,b]$, while the second part shows how an antiderivative can be used to determine a definite integral for f on $[a,b]$. The symmetry is something to behold, the two parts taken together effectively uniting the differential and integral branches of calculus.

The following proposition is a modest generalization of Theorem 6.23 and will prove useful in the study of differential equations.

Proposition 6.26. *If f is continuous on (a,b) and $c \in (a,b)$, then the function $\Phi : (a,b) \to \mathbb{R}$ given by*

$$\Phi(x) = \int_c^x f(t)\,dt, \quad a < x < b$$

is differentiable on (a,b), with $\Phi'(x) = f(x)$ for each $a < x < b$.

Proof. Let $d \in [c,b)$. Since f is continuous on $[c,d]$, by Theorem 6.23 the function Φ is differentiable on $[c,d]$ such that $\Phi'(x) = f(x)$ for all $x \in [c,d]$. Since $d \in [c,b)$ is arbitrary we conclude that $\Phi'(x) = f(x)$ for all $x \in (c,b)$, with $\Phi'_+(c) = f(c)$ in particular.

Now let $d \in (a,c]$. Since f is continuous on $[d,c]$, by Theorem 6.23 the function $\Psi : [d,c] \to \mathbb{R}$ given by

$$\Psi(x) = \int_d^x f(t)\,dt$$

for all $x \in [d,c]$ is differentiable such that $\Psi'(x) = f(x)$ for each $d \le x < c$, and $\Psi'_-(c) = f(c)$. Now, by Theorem 6.16 and Definition 6.15 we obtain

$$\int_d^c f(t)\,dt = \int_d^x f(t)\,dt + \int_x^c f(t)\,dt = \int_d^x f(t)\,dt - \int_c^x f(t)\,dt = \Psi(x) - \Phi(x),$$

and thus

$$\Phi(x) = \Psi(x) - \int_d^c f(t)\,dt.$$

for each $d \leq x \leq c$. Since $\int_d^c f$ is a constant we have $\Phi'(x) = \Psi'(x) = f(x)$ for $x \in [d, c]$, and since $d \in (a, c]$ is arbitrary we conclude that $\Phi'(x) = f(x)$ for all $x \in (a, c)$, with $\Phi'_-(c) = f(c)$ in particular.

Finally, from $\Phi'_+(c) = f(c) = \Phi'_-(c)$ we have $\Phi'(c) = f(c)$, and therefore $\Phi'(x) = f(x)$ for all $a < x < b$. ■

6.6 – Improper Riemann Integrals

Suppose f is Riemann integrable on $[a, x]$ for all $x \geq a$. By definition we have

$$\int_a^x f \in \mathbb{R}$$

for each $x \geq a$. This observation leads us to ask whether $\int_a^x f$ tends to some limiting value $L \in \mathbb{R}$ as $x \to \infty$; that is, does the limit

$$\lim_{x \to \infty} \int_a^x f$$

exist in $\mathbb{R}$? Such a question arises frequently in applications, and so motivates the following definition.

Definition 6.27. *If $f \in \mathcal{R}[a, b]$ for all $b \geq a$, then we define*

$$\int_a^\infty f = \lim_{b \to \infty} \int_a^b f$$

*and say that $\int_a^\infty f$ **converges to L** if $\int_a^\infty f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_a^\infty f$ **diverges**.*

If $f \in \mathcal{R}[a, b]$ for all $a \leq b$, then we define

$$\int_{-\infty}^b f = \lim_{a \to -\infty} \int_a^b f$$

*and say that $\int_{-\infty}^b f$ **converges to L** if $\int_{-\infty}^b f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_{-\infty}^b f$ **diverges**.*

If an improper integral converges to some real number L then it is customary to say simply that the integral "converges" or is "convergent." An integral that "diverges" is also said to be "divergent." Any integral of the form $\int_a^\infty f$, $\int_{-\infty}^b f$, or $\int_{-\infty}^\infty f$ (see below) is called an **improper integral of the first kind**.

The next proposition establishes linearity properties specifically for integrals of the form $\int_a^\infty f$ that are identical in form to the linearity properties of the Riemann integral given in §5.3. There are similar linearity properties for all types of improper integrals.

Proposition 6.28. *If $\int_a^\infty f$ and $\int_a^\infty g$ are convergent and $c \in \mathbb{R}$, then $\int_a^\infty cf$ and $\int_a^\infty (f \pm g)$ are convergent such that*

1. $\displaystyle\int_a^\infty cf = c \int_a^\infty f$
2. $\displaystyle\int_a^\infty (f \pm g) = \int_a^\infty f \pm \int_a^\infty g$

The proof is a routine application of relevant laws of limits established back in Chapter 2, and so left as an exercise.

Example 6.29. Determine whether

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. It will be easier to first determine the indefinite integral

$$\int \frac{\ln(x)}{x^2}\,dx.$$

We start with a substitution: let $w = \ln(x)$, so that $dw = (1/x)dx$ and $e^w = e^{\ln(x)} = x$; now,

$$\int \frac{\ln(x)}{x^2}\,dx = \int we^{-w}\,dw.$$

Next, we employ integration by parts, letting $u'(w) = e^{-w}$ and $v(w) = w$ to obtain

$$\int we^{-w}\,dw = -we^{-w} + \int e^{-w}\,dw = -we^{-w} - e^{-w} + C.$$

Hence,

$$\int \frac{\ln(x)}{x^2}\,dx = -\ln(x)\cdot\frac{1}{x} - \frac{1}{x} + C = -\frac{\ln(x)+1}{x} + C.$$

Now we turn to the improper integral,

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln(x)+1}{x}\right]_1^b \\ &= \lim_{b\to\infty}\left[-\frac{\ln(b)+1}{b} + \frac{\ln(1)+1}{1}\right] = \lim_{b\to\infty}\left(\frac{b-\ln(b)+1}{b}\right) \\ &\overset{LR}{=} \lim_{b\to\infty}\left(\frac{1-1/b}{1}\right) = 1,\end{aligned}$$

using L'Hôpital's Rule where indicated.

Therefore the improper integral is convergent, and its value is 1. ∎

Proposition 6.30. *Suppose that $f \in \mathcal{R}[s,t]$ for all $-\infty < s < t < \infty$. If $\int_{-\infty}^c f$ and $\int_c^\infty f$ converge for some $c \in \mathbb{R}$, then for any $\hat{c} \neq c$ the integrals $\int_{-\infty}^{\hat{c}} f$ and $\int_{\hat{c}}^\infty f$ also converge, and*

$$\int_{-\infty}^{\hat{c}} f + \int_{\hat{c}}^\infty f = \int_{-\infty}^c f + \int_c^\infty f$$

Proof. Suppose $\int_{-\infty}^c f$ and $\int_c^\infty f$ converge for some $c \in \mathbb{R}$, meaning the limits

$$\lim_{a\to-\infty}\int_a^c f \quad\text{and}\quad \lim_{b\to\infty}\int_c^b f$$

both exist. Let $\hat{c} < c$.

For all $b > c$ we have

$$\int_{\hat{c}}^b f = \int_{\hat{c}}^c f + \int_c^b f,$$

where $\int_{\hat{c}}^{c} f, \int_{c}^{b} f \in \mathbb{R}$ since f is integrable on $[\hat{c}, c]$ and $[c, b]$, and so

$$\int_{\hat{c}}^{\infty} f = \lim_{b\to\infty} \int_{\hat{c}}^{b} f = \lim_{b\to\infty} \left(\int_{\hat{c}}^{c} f + \int_{c}^{b} f \right) = \int_{\hat{c}}^{c} f + \lim_{b\to\infty} \int_{c}^{b} f = \int_{\hat{c}}^{c} f + \int_{c}^{\infty} f. \tag{6}$$

Observing that $\int_{\hat{c}}^{c} f, \int_{c}^{\infty} f \in \mathbb{R}$, we readily conclude that $\int_{\hat{c}}^{\infty} f \in \mathbb{R}$ and hence $\int_{\hat{c}}^{\infty} f$ converges.

For all $a < \hat{c}$ we have

$$\int_{a}^{\hat{c}} f = \int_{a}^{c} f - \int_{\hat{c}}^{c} f,$$

where $\int_{a}^{c} f, \int_{\hat{c}}^{c} f \in \mathbb{R}$ since f is integrable on $[a, c]$ and $[\hat{c}, c]$, and so

$$\int_{-\infty}^{\hat{c}} f = \lim_{a\to-\infty} \int_{a}^{\hat{c}} f = \lim_{a\to-\infty} \left(\int_{a}^{c} f - \int_{\hat{c}}^{c} f \right) = \lim_{a\to-\infty} \int_{a}^{c} f - \int_{\hat{c}}^{c} f = \int_{-\infty}^{c} f - \int_{\hat{c}}^{c} f. \tag{7}$$

Observing that $\int_{-\infty}^{c} f, \int_{\hat{c}}^{c} f \in \mathbb{R}$, we readily conclude that $\int_{-\infty}^{\hat{c}} f \in \mathbb{R}$ and hence $\int_{-\infty}^{\hat{c}} f$ converges.

Finally, combining (6) and (7), we obtain

$$\int_{-\infty}^{\hat{c}} f + \int_{\hat{c}}^{\infty} f = \left(\int_{-\infty}^{c} f - \int_{\hat{c}}^{c} f \right) + \left(\int_{\hat{c}}^{c} f + \int_{c}^{\infty} f \right) = \int_{-\infty}^{c} f + \int_{c}^{\infty} f,$$

as desired. ■

Due to Proposition 6.30 we can unambiguously define an improper integral of the first kind whose interval of integration is $(-\infty, \infty)$.

Definition 6.31. *Suppose that $f \in \mathcal{R}[s, t]$ for all $-\infty < s < t < \infty$. If $\int_{-\infty}^{c} f$ and $\int_{c}^{\infty} f$ both converge for some $-\infty < c < \infty$, then we define*

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{c} f + \int_{c}^{\infty} f.$$

*and say that $\int_{-\infty}^{\infty} f$ **converges**. Otherwise we say $\int_{-\infty}^{\infty} f$ **diverges**.*

It should be stressed that $\int_{-\infty}^{\infty} f$ can not be reliably evaluated simply by computing the limit

$$\lim_{b\to\infty} \int_{-b}^{b} f,$$

as the next example illustrates.

Example 6.32. Show that

$$\int_{-\infty}^{\infty} \frac{2x}{1 + x^2}\, dx$$

diverges, and yet

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{1 + x^2}\, dx = 0.$$

Solution. Letting $u = 1 + x^2$ gives $du = 2x\,dx$. Then

$$\int_0^b \frac{2x}{1+x^2}\,dx = \int_1^{1+b^2} \frac{1}{u}\,du = [\ln|u|]_1^{1+b^2} = \ln(1+b^2) - \ln(1) = \ln(1+b^2),$$

and so

$$\int_0^\infty \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty} \ln(1+b^2) = \infty.$$

Thus

$$\int_0^\infty \frac{2x}{1+x^2}\,dx$$

diverges, and therefore

$$\int_{-\infty}^\infty \frac{2x}{1+x^2}\,dx$$

diverges as well.

On the other hand, again employing the substitution $u = 1 + x^2$ we find that

$$\int_{-b}^b \frac{2x}{1+x^2}\,dx = \int_{1+b^2}^{1+b^2} \frac{1}{u}\,du = 0,$$

and so

$$\lim_{b\to\infty} \int_{-b}^b \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty} (0) = 0.$$

∎

An **improper integral of the second kind** is an integral of the form

$$\int_a^b f,$$

where $-\infty < a < b < \infty$, for which there exists some $p \in [a,b]$ such that $p \notin \mathrm{Dom}(f)$. The following definition establishes how such an integral is to be evaluated, if it can be evaluated at all, in the case when $p = a$ or $p = b$.

Definition 6.33. *If $f \in \mathcal{R}[c,b]$ for all $c \in (a,b]$ and $a \notin \mathrm{Dom}(f)$, then we define*

$$\int_a^b f = \lim_{c\to a^+} \int_c^b f$$

and say that $\int_a^b f$ ***converges to L*** *if $\int_a^b f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_a^b f$* ***diverges****.*

If $f \in \mathcal{R}[a,c]$ for all $c \in [a,b)$ and $b \notin \mathrm{Dom}(f)$, then we define

$$\int_a^b f = \lim_{c\to b^-} \int_a^c f$$

and say that $\int_a^b f$ ***converges to L*** *if $\int_a^b f = L$ for some $L \in \mathbb{R}$. Otherwise we say that $\int_a^b f$* ***diverges****.*

Very often if f is continuous on, say, $(a,b]$ and $a \notin \mathrm{Dom}(f)$, then f has a vertical asymptote at a; that is, $\lim_{x\to a^+} f(x) = \pm\infty$. However, it could just be that a value for f is simply not specified at a by construction. For example for the function

$$\varphi(x) = \begin{cases} 3x^2, & \text{if } x < 5 \\ 4 - 8x, & \text{if } x > 5 \end{cases}$$

it's seen that $\varphi(5)$ is left undefined, and so the integral $\int_5^9 \varphi$ is an improper integral of the second kind. By Definition 6.33 we obtain

$$\begin{aligned}\int_5^9 \varphi &= \lim_{c\to 5^+} \int_c^9 (4-8x)\,dx = \lim_{c\to 5^+} \left[4x - 4x^2\right]_c^9 = \lim_{c\to 5^+} \left[\left(4(9) - 4(9)^2\right) - \left(4c - 4c^2\right)\right] \\ &= \left(4(9) - 4(9)^2\right) - \left(4(5) - 4(5)^2\right) = -208,\end{aligned}$$

which shows that $\int_5^9 \varphi$ is convergent.

Example 6.34. Determine whether

$$\int_{-1}^0 \frac{1}{x^2}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. The function $f(x) = 1/x^2$ being integrated has a vertical asymptote at $x = 0$, which is the right endpoint of the interval of integration $[-1, 0]$. By Definition 6.33 we obtain

$$\int_{-1}^0 \frac{1}{x^2}\,dx = \lim_{c\to 0^-} \int_{-1}^c \frac{1}{x^2}\,dx = \lim_{c\to 0^-} \left[-\frac{1}{x}\right]_{-1}^c = \lim_{c\to 0^-} \left(-\frac{1}{c} - 1\right) = \infty,$$

which shows that the improper integral is divergent. ■

Example 6.35. Determine whether

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. Here $x/\sqrt{4-x^2}$ has a vertical asymptote at $x = 2$, the right endpoint of the interval of integration $[0, 2]$. By Definition 6.33

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{c\to 2^-} \int_0^c \frac{x}{\sqrt{4-x^2}}\,dx,$$

and so, letting $u = 4 - x^2$ so that $x\,dx = -\frac{1}{2}du$, we obtain

$$\begin{aligned}\lim_{c\to 2^-} \int_0^c \frac{x}{\sqrt{4-x^2}}\,dx &= \lim_{c\to 2^-} \int_4^{4-c^2} \frac{-1/2}{\sqrt{u}}\,du = \lim_{c\to 2^-} \left(-\frac{1}{2}\left[2\sqrt{u}\,\right]_4^{4-c^2}\right) \\ &= \lim_{c\to 2^-} \left(2 - \sqrt{4-c^2}\right) = 2 - \sqrt{4-2^2} = 2.\end{aligned}$$

Hence the improper integral is convergent, and its value is 2. ■

The next definition addresses the circumstance when a function f is not defined at some point p in the interior of an interval of integration. Again, this is commonly due to f having a vertical asymptote at p, so that

$$\lim_{x\to p^+} |f(x)| = \infty \quad \text{or} \quad \lim_{x\to p^-} |f(x)| = \infty,$$

but other scenarios are possible.

Definition 6.36. *Suppose that $f \in \mathcal{R}[a,c]$ for all $c \in [a,p)$, $f \in \mathcal{R}[c,b]$ for all $c \in (p,b]$, and $p \notin \mathrm{Dom}(f)$. If $\int_a^p f$ and $\int_p^b f$ both converge, then we define*

$$\int_a^b f = \int_a^p f + \int_p^b f.$$

*and say that $\int_a^b f$ **converges**. Otherwise we say $\int_a^b f$ **diverges**.*

Example 6.37. Determine whether

$$\int_{-2}^{3} \frac{1}{x^4}\,dx$$

converges or diverges. Evaluate if convergent.

Solution. Here $1/x^4$ has a vertical asymptote at $x = 0$, an interior point of the interval of integration $[-2,3]$. Now, by Definition 6.33

$$\int_0^3 \frac{1}{x^4}\,dx = \lim_{c\to 0^+} \int_c^3 \frac{1}{x^4}\,dx = \lim_{c\to 0^+} \left[-\frac{1}{x^3}\right]_c^3 = \lim_{c\to 0^+} \left(-\frac{1}{27} + \frac{1}{c^3}\right) = \infty,$$

which shows that $\int_0^3 x^{-4}\,dx$ is divergent. Thus, since

$$\int_0^3 x^{-4}\,dx \quad \text{and} \quad \int_{-2}^0 x^{-4}\,dx$$

cannot both be convergent, by Definition 6.36 it's concluded that $\int_{-2}^3 x^{-4}\,dx$ is divergent. ■

The integral treated in Example 6.37, like all improper integrals of the second kind, does not look improper at first glance. If one is careless and undertakes to evaluate the integral by conventional means, one is likely to arrive at a reasonable-looking answer without ever suspecting that something is amiss:

$$\int_{-2}^3 \frac{1}{x^4}\,dx = \left[-\frac{1}{x^3}\right]_{-2}^3 = -\frac{1}{27} + \frac{1}{-8} = -\frac{35}{216},$$

which is incorrect! So, before attempting to evaluate a definite integral, it is necessary to check that the integral is not improper in some way.

It is possible to have an integral that is improper in more than one sense, such as

$$\int_0^\infty \frac{1}{x^2}\,dx.$$

Here we have an integral of f over an unbounded interval $[0, \infty)$, so it's an improper integral of the first kind, and also f is undefined at 0, so it's an improper integral of the second kind. Such an integral is called a **mixed improper integral**.

Definition 6.38. *If $f \in \mathcal{R}[s,t]$ for all $a < s < t < \infty$, $a \notin \mathrm{Dom}(f)$, and $\int_a^c f$ and $\int_c^\infty f$ both converge for some $c \in (a, \infty)$, then we define*

$$\int_a^\infty f = \int_a^c f + \int_c^\infty f$$

*and say $\int_a^\infty f$ **converges**. Otherwise we say $\int_a^\infty f$ **diverges**.*

If $f \in \mathcal{R}[s,t]$ for all $-\infty < s < t < b$, $b \notin \mathrm{Dom}(f)$, and $\int_{-\infty}^c f$ and $\int_c^b f$ both converge for some $c \in (-\infty, b)$, then we define

$$\int_{-\infty}^b f = \int_{-\infty}^c f + \int_c^b f$$

*and say $\int_{-\infty}^b f$ **converges**. Otherwise we say $\int_{-\infty}^b f$ **diverges**.*

Example 6.39. Determine whether the mixed improper integral

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\, dx$$

converges or diverges. Evaluate if convergent.

Solution. We start by determining the indefinite integral

$$\int \frac{1}{\sqrt{x}(1+x)}\, dx.$$

Let $u = \sqrt{x}$, so that $1 + u^2 = 1 + x$ and we replace dx with $2u\, du$ to obtain

$$\begin{aligned}\int \frac{1}{\sqrt{x}(1+x)}\, dx &= \int \frac{2u}{u(u^2+1)}\, du = 2\int \frac{1}{u^2+1}\, du \\ &= 2\arctan(u) + c = 2\arctan(\sqrt{x}\,) + c.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}(1+x)}\, dx &= \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}(1+x)}\, dx = \lim_{a \to 0^+} \left[2\arctan(\sqrt{x}\,)\right]_a^1 \\ &= \lim_{a \to 0^+} 2[\arctan(1) - \arctan(a)] = 2[\arctan(1) - \arctan(0)] \\ &= 2\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2},\end{aligned}$$

and

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\, dx &= \lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x}(1+x)}\, dx = \lim_{b \to \infty} \left[2\arctan(\sqrt{x}\,)\right]_1^b \\ &= \lim_{b \to \infty} 2\Big[\arctan(\sqrt{b}\,) - \arctan(1)\Big] = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2}.\end{aligned}$$

Since

$$\int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

both converge, we conclude that

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

by Definition 6.38. ■

6.7 – Comparison Tests for Integrals

It can be difficult to determine by direct means whether an improper integral is convergent or not, largely because definite integrals themselves can be difficult to evaluate. One tool to remedy this is a comparison test. Before stating the first such test we need to establish a lemma that will be needed for its proof.

Lemma 6.40. *Suppose the function f is monotone increasing on (a,∞). If*

$$\lim_{x\to\infty} f(x) = M,$$

then $f(x) \leq M$ for all $x > a$.

Proof. Suppose there exists some $x_0 > a$ such that $f(x_0) > M$. Thus $f(x_0) = M + \epsilon$ for some $\epsilon > 0$. Now, for any $\beta > 0$ we can let $x_1 = \max\{x_0, \beta\} + 1$. Since $x_1 > x_0$ and f is monotone increasing, we have

$$f(x_1) \geq f(x_0) = M + \epsilon \;\;\Rightarrow\;\; f(x_1) - M \geq \epsilon \;\;\Rightarrow\;\; |f(x_1) - M| \geq \epsilon.$$

Observing that $x_1 > \beta$ also, we conclude that for any $\beta > 0$ there exists some $x > \beta$ for which $|f(x) - M| \geq \epsilon$, and therefore

$$\lim_{x\to\infty} f(x) \neq M.$$

The contrapositive of the statement of the lemma is proven. ■

Theorem 6.41. *Suppose $f \in \mathcal{R}[a,x]$ for all $x \geq a$, and $0 \leq f \leq g$ on $[a,\infty)$. If $\int_a^\infty g$ is convergent, then $\int_a^\infty f$ is convergent.*

Proof. Suppose $\int_a^\infty g$ is convergent. By definition it follows that $g \in \mathcal{R}[a,x]$ for all $x \geq a$, and so we may define $\psi : [a,\infty) \to \mathbb{R}$ by $\psi(x) = \int_a^x g$. Similarly we define $\varphi : [a,\infty) \to \mathbb{R}$ by $\varphi(x) = \int_a^x f$.

Now, $g \geq 0$ on $[a,\infty)$ implies that

$$\int_x^y g \geq 0$$

for any $a \leq x < y$. Thus, for any $x, y \in [a,\infty)$ such that $x < y$ we have

$$\psi(y) = \int_a^y g = \int_a^x g + \int_x^y g \geq \int_a^x g = \psi(x),$$

which shows that ψ is monotone increasing on $[a,\infty)$. Since $f \geq 0$ on $[a,\infty)$, a similar argument establishes that φ also is monotone increasing on $[a,\infty)$.

Since $\int_a^\infty g$ converges, there exists some $M \in \mathbb{R}$ such that

$$\lim_{x\to\infty} \int_a^x g = M.$$

That is, ψ is monotone increasing on (a,∞) and

$$\lim_{x\to\infty} \psi(x) = M,$$

so $\psi(x) \leq M$ for all $x > a$ by Lemma 6.40.

Finally, from the hypothesis that $f \leq g$ on $[a, \infty)$, we have

$$\varphi(x) = \int_a^x f \leq \int_a^x g = \psi(x) \leq M$$

for all $x > a$. It is now established that φ is both monotone increasing and bounded above on (a, ∞), and therefore by Proposition 2.20 $\lim_{x\to\infty} \varphi(x)$ exists in $\mathbb{R}$. Since

$$\int_a^\infty f = \lim_{x\to\infty} \int_a^x f = \lim_{x\to\infty} \varphi(x),$$

it follows that $\int_a^\infty f$ is convergent. ∎

Proposition 6.42. *Suppose $f \in \mathcal{R}[a, x]$ for all $x \geq a$. If $\int_a^\infty |f|$ is convergent, then $\int_a^\infty f$ is convergent.*

Proof. Suppose that $\int_a^\infty |f|$ is convergent. Then $|f| \in \mathcal{R}[a, x]$ for all $x \geq a$, and since the same holds true for f by hypothesis, we have $f + |f| \in \mathcal{R}[a, x]$ for all $x \geq a$ by Proposition 5.15. Now, since $0 \leq f + |f| \leq 2|f|$ on $[a, \infty)$, and

$$\int_a^\infty 2|f|$$

is convergent by Proposition 6.28, Theorem 6.41 implies that

$$\int_a^\infty (f + |f|)$$

is convergent. Then, because $\int_a^\infty -|f|$ is convergent by Proposition 6.28, it follows by Proposition 6.28 that

$$\int_a^\infty \big[(f + |f|) + (-|f|)\big]$$

is convergent. Of course $(f + |f|) + (-|f|) = f$ on $[a, \infty)$, and thus we conclude that $\int_a^\infty f$ is convergent. ∎