The Wirtinger Presentation

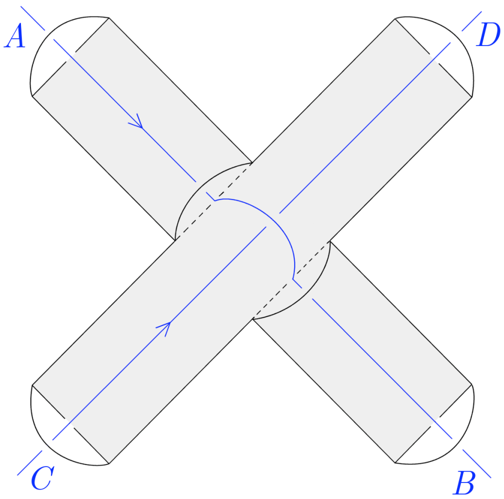

FIGURE 1. *The space Z is a planar region with attached 1-cells*

Let $K \subseteq \mathbb{R}^3$ be a smooth or piecewise linear knot. We examine the topology of the knot group $\mathbb{R}^3 - K$ by focusing first on a deformation retraction of the complement of one of the knot's crossings. Start first with a basic scaffolding, the space Z pictured in Figure 1. The knot crossing under consideration is depicted in the midst of Z, with the segment $\overline{AB}$ being the crossing's "overpass" and the segment $\overline{CD}$ being the "underpass". The space Z consists of the shaded areas lying below these segments, along with the six attached semicircular arcs that pass over $\overline{AB}$ and $\overline{CD}$.

In Figure 2 it can be clearly seen that $\pi_1(Z,p) \cong \langle x, y, z, x', y', z' \rangle$. Here we define $x = a_3a_1c_1b_1\bar{a}_1\bar{a}_3$, $x' = a'_1b'_1c'_1\bar{a}'_1$, $y = a_2c_2b_2\bar{a}_2$, $y' = a_2a'_2c'_2b'_2\bar{a}'_2\bar{a}_2$, $z = a_3b_3$, and $z' = a'_3b'_3c'_3\bar{a}'_3$.

Now, 2-cells R_1, R_2 and R_3 are attached to Z to create a new space Y as follows: attach R_1 via the homomorphism $\varphi_1 : \mathbb{S}^1 \to \mathbb{Z}$, where $\varphi_1 = a'_1b'_1d_1c_2\bar{d}_4\bar{c}_1\bar{a}_1\bar{a}_3$ and in particular we keep R_1 entirely *beneath* the overpass of segment $\overline{AB}$ in Figure 1; attach R_2 via $\varphi_2 = a'_2\bar{b}'_2d_2b_2$; and attach R_3 via $\varphi_3 = \bar{b}_3\bar{d}_3c'_3\bar{a}'_3$. Figure 3 gives an illustration of Y.

Proposition 1.26 implies that $\pi_1(Y,p) \cong \pi_1(Z,p)/N$, where N is the normal subgroup of $\pi_1(Z,p)$ that is generated by the set $\{\varphi_1, a_2\varphi_2\bar{a}_2, \varphi_3\}$. Now, letting

$$\begin{aligned} r_1 &= a'_1\bar{c}'_1d_1c_2\bar{d}_4b_1\bar{a}_1\bar{a}_3, \\ r_2 &= a_2a'_2c'_2d_2\bar{c}_2\bar{a}_2, \\ r_3 &= a_3\bar{d}_3\bar{b}'_3\bar{a}'_3, \end{aligned}$$

we find

$$\begin{aligned} x'r_1\bar{x} &= (a'_1b'_1c'_1\bar{a}'_1)(a'_1\bar{c}'_1d_1c_2\bar{d}_4b_1\bar{a}_1\bar{a}_3)(a_3a_1\bar{b}_1\bar{c}_1\bar{a}_1\bar{a}_3) = a'_1b'_1d_1c_2\bar{d}_4\bar{c}_1\bar{a}_1\bar{a}_3 = \varphi_1, \\ \bar{y}'r_2y &= (a_2a'_2\bar{b}'_2\bar{c}'_2\bar{a}'_2\bar{a}_2)(a_2a'_2c'_2d_2\bar{c}_2\bar{a}_2)(a_2c_2b_2\bar{a}_2) = a_2a'_2\bar{b}'_2\bar{b}'_2d_2b_2\bar{a}_2 = a_2\varphi_2\bar{a}_2, \\ \bar{z}r_3z' &= (\bar{b}_3\bar{a}_3)(a_3\bar{d}_3\bar{b}'_3\bar{a}'_3)(a'_3b'_3c'_3\bar{a}'_3) = \varphi_3. \end{aligned}$$

Since r_1, r_2 and r_3 are nullhomotopic (in particular homotopic to the constant path at p), it follows that $\varphi_1 \simeq x'\bar{x}$, $a_2\varphi_2\bar{a}_2 \simeq \bar{y}'y$ and $\varphi_3 \simeq \bar{z}z'$, and thus as a result

$$\pi_1(Y,p) \cong \langle x, y, z, x', y', z' \rangle / N \cong \langle x, y, z, x', y', z' \mid x'\bar{x}, \bar{y}'y, \bar{z}z' \rangle,$$

where N is now seen to be the normal closure of $\{x'\bar{x}, \bar{y}'y, \bar{z}z'\}$. Thus we have $x'\bar{x}N = N \Rightarrow x'N\bar{x} = N \Rightarrow x'N = Nx \Rightarrow x'N = xN$, and similarly $y'N = yN$ and $z'N = zN$. Hence $\pi_1(Y,p) \cong \langle xN, yN, zN\rangle \cong \langle x, y, z\rangle$, where Figure 3 provides a depiction of the generators x, y and z (we naturally identify xN with x and so on).

Next, to the space Y we attach a 2-cell S_1 as shown in Figure 4: along the inward-facing "short edges" of R_2 and R_3, and along two paths in R_1 that connect endpoints of the aforementioned short edges. This process can be formalized by defining a homomorphism $\psi_1 = \psi_{11}\psi_{12}\psi_{13}\psi_{14} : \mathbb{S}^1 \to Y$, where the paths ψ_{1i} are illustrated in Figure 4. It's important to see that S_1 should lie entirely *above* the overpass of segment $\overline{AB}$ in Figure 1. Comparing Figures 3 and 4, it can be seen that $x = hij\bar{h}$, $y = aegf\bar{e}\bar{a}$ and $z = klm\bar{k}$. Moreover, note that $\psi_{11}\bar{a} \simeq hij\bar{h}$, $a\psi_{12}cd \simeq a\psi_{12}\bar{b}\bar{a} \simeq a(e\bar{f}\bar{g}\bar{e})\bar{a}$, $\bar{d}\bar{c}\psi_{13}d \simeq \bar{d}(dh\bar{j}\bar{\imath}\bar{h}\bar{d})d \simeq h\bar{j}\bar{\imath}\bar{h}$, and $\bar{d}\psi_{14} \simeq klm\bar{k}$. Then

$$\begin{aligned}\psi_1 &\simeq \psi_{11}(\bar{a}a)\psi_{12}(cd\bar{d}\bar{c})\psi_{13}(d\bar{d})\psi_{14}\\ &\simeq (\psi_{11}\bar{a})(a\psi_{12}cd)(\bar{d}\bar{c}\psi_{13}d)(\bar{d}\psi_{14})\\ &\simeq (hij\bar{h})(a(e\bar{f}\bar{g}\bar{e})\bar{a})(h\bar{j}\bar{\imath}\bar{h})(klm\bar{k})\\ &= x\bar{y}\bar{x}z.\end{aligned}$$

Let X be the space that results from attaching S_1 to Y. According to Proposition 1.26 the path $x\bar{y}\bar{x}z$ is nullhomotopic in X, and so $\pi_1(X,p) \cong \pi_1(Y,p)/M$, where M is the normal subgroup generated by $\{x\bar{y}\bar{x}z\}$; that is, $\pi_1(X,p) \cong \langle x, y, z \mid x\bar{y}\bar{x}z\rangle$. Since $\bar{z}xy\bar{x}$ is the inverse of $x\bar{y}\bar{x}z$, it also generates M. So if we let $x_1 = x$, $x_2 = y$ and $x_3 = z$, we find that

$$\pi_1(X,p) \cong \langle x_1, x_2, x_3 \mid x_3^{-1}x_1x_2x_1^{-1}\rangle = \langle x_1, x_2, x_3 \mid x_1x_2x_1^{-1} = x_3\rangle$$

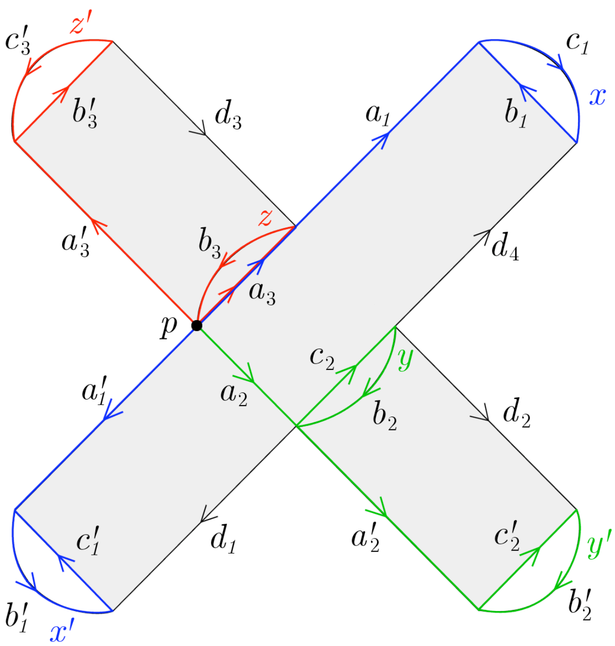

FIGURE 2

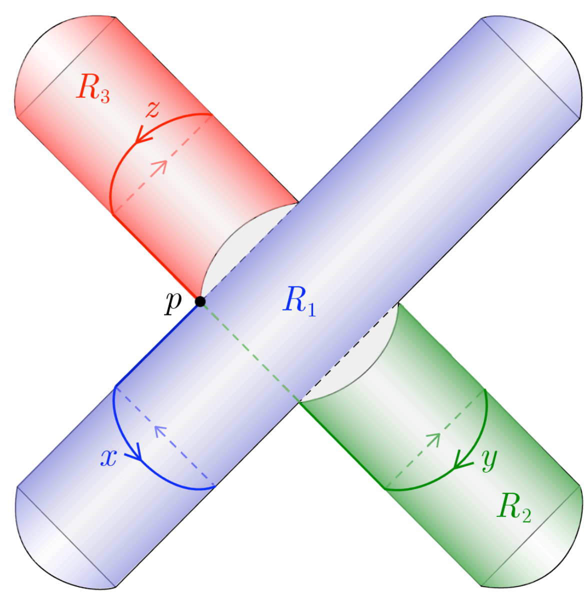

FIGURE 3

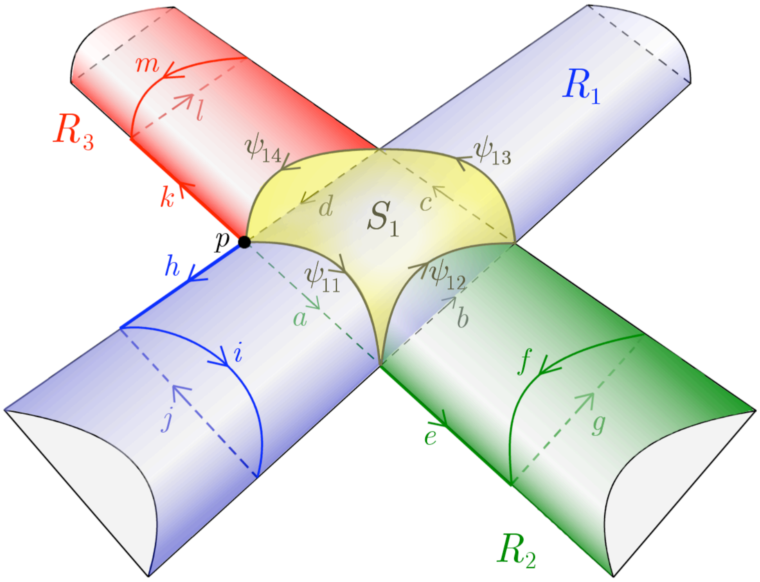

FIGURE 4

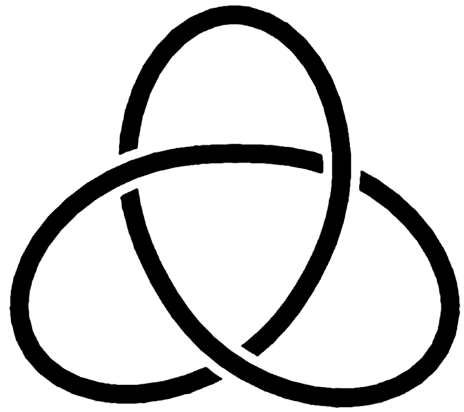

FIGURE 5

We move on now to the simplest knot in $\mathbb{R}^3$, the trefoil knot K shown in Figure 5. As before, we begin by constructing a particular subspace of $\mathbb{R}^3 - K$ which starts with the space Z' illustrated in Figure 6. We view Z' as being an extension of the space Z in Figure 2 (note the point p in both figures). It's easily seen that $\pi_1(Z', p) \cong \pi_1(Z, p) \cong \langle x, y, z, x', y', z' \rangle$.

Now, as shown in Figure 7, we attach three 2-cells R_1, R_2 and R_3 to obtain a new space Y' with fundamental group $\pi_1(Y', p) \cong \pi_1(Y, p) \cong \langle x, y, z \rangle$. A space X' is then constructed from Y' by attaching the 2-cells S_1, S_2 and S_3. The cell S_1 is the same one already shown in Figure 4, and it's attached as before by the path $x\bar{y}\bar{x}z$. In an entirely analogous fashion, S_2 is attached via $y\bar{z}\bar{y}x$ and S_3 is attached via $z\bar{x}\bar{z}y$ (note how the roles of x, y and z interchange at each knot crossing with respect to the orientation of the knot as indicated by the arrows in Figure 6).

Applying Propositon 1.26 again, and once again letting $x_1 = x$, $x_2 = y$ and $x_3 = z$, we obtain

$$\begin{aligned}
\pi_1(X', p) &\cong \langle x, y, z \mid x\bar{y}\bar{x}z, y\bar{z}\bar{y}x, z\bar{x}\bar{z}y \rangle \\
&\cong \langle x, y, z \mid \bar{z}xy\bar{x}, \bar{x}yz\bar{y}, \bar{y}zx\bar{z} \rangle \\
&\cong \langle x_1, x_2, x_3 \mid x_1x_2x_1^{-1} = x_3,\ x_2x_3x_2^{-1} = x_1,\ x_3x_1x_3^{-1} = x_2 \rangle.
\end{aligned}$$

It can be seen that X' is a deformation retraction of $\mathbb{R}^3 - K$, and therefore, dropping the reference to p owing to the path-connectedness of the involved spaces, $\pi_1(X') \cong \pi_1(\mathbb{R}^3 - K)$.

We examine now the abelianization of $\pi_1(\mathbb{R}^3 - K)$, the "knot group" of K. Identifying elements of the abelianization of the group with elements of the group itself in the obvious way, we find that the relation $x_1x_2x_1^{-1} = x_3$ implies $x_2 = x_3$, $x_2x_3x_2^{-1} = x_1 \Rightarrow x_3 = x_1$, and $x_3x_1x_3^{-1} = x_2 \Rightarrow x_1 = x_2$. Hence

$$\mathrm{Ab}(\pi_1(\mathbb{R}^3 - K)) \cong \langle x_1, x_2, x_3 \mid x_2 = x_3,\ x_3 = x_1,\ x_1 = x_2 \rangle \cong \langle x_1 \rangle \cong \mathbb{Z}.$$

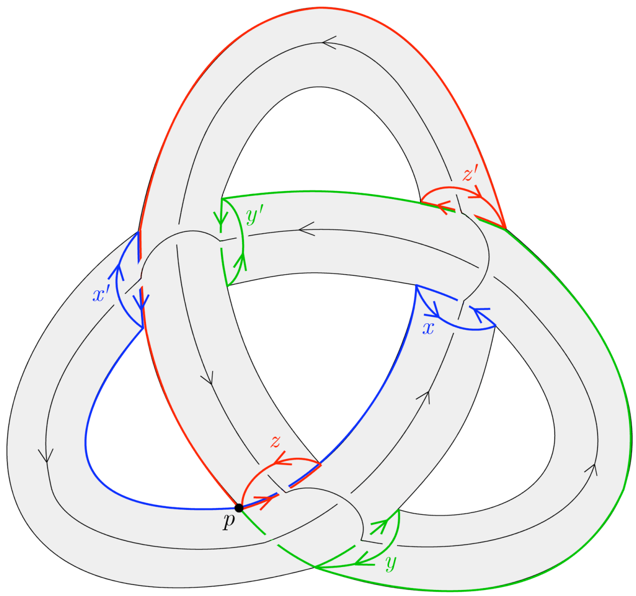

FIGURE 6

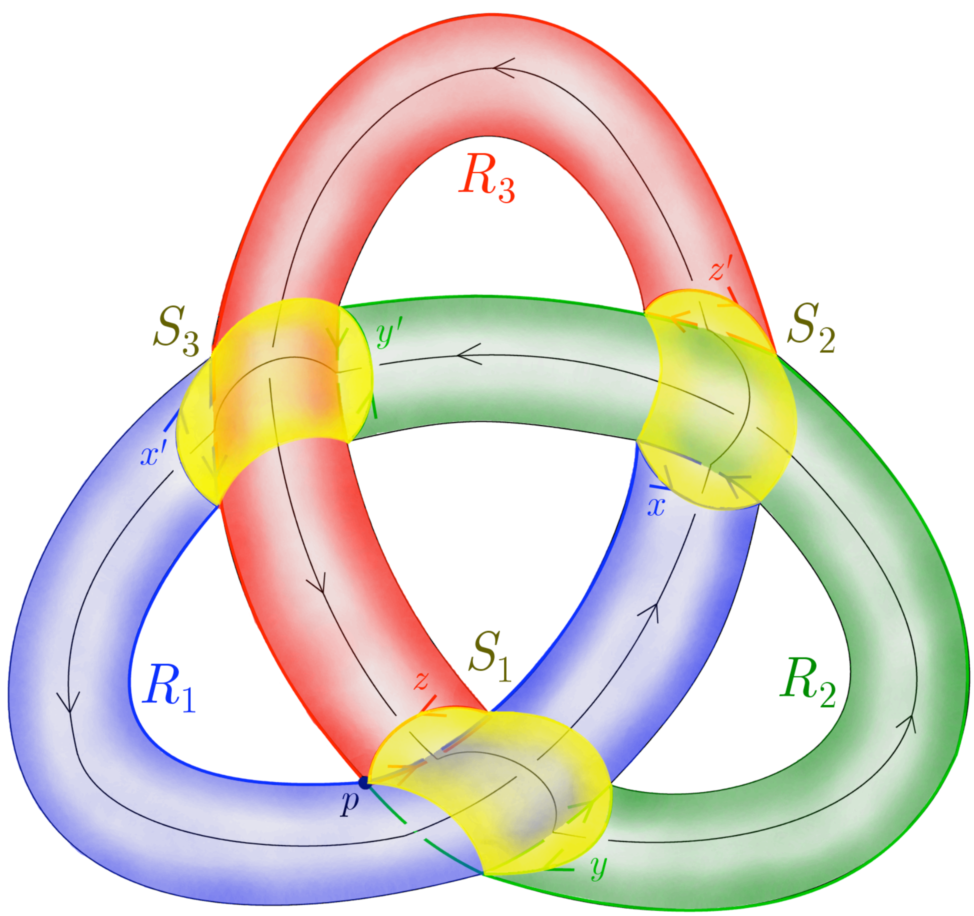

FIGURE 7