Chapter 2 – Homology

2.1 – Homological Algebra

Let F_n be an abelian group for each $n \geq 0$, and let $f_n : F_n \to F_{n-1}$ be homomorphisms such that $f_n \circ f_{n+1} = 0$ (the trivial homomorphism) for all $n \geq 0$, with $f_0 = 0$. Then the sequence F given as

$$\cdots \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} 0 \tag{1}$$

is called a **chain complex**. In this general algebraic setting nothing compels the index n to consist strictly of the whole numbers, but unless otherwise specified we will always assume that a chain complex ends at index value (or "dimension") 0, with $f_0 = 0$. Thus, to say something holds "for all n" in this context is intended to mean for all $n \geq 0$. There will be times, in certain topological settings, when it will be convenient to extend a chain complex out to dimension $n = -1$. From $f_n \circ f_{n+1} = 0$ we obtain $\operatorname{Im} f_{n+1} \subset \operatorname{Ker} f_n \subset F_n$, so $\operatorname{Im} f_{n+1}$ is a normal subgroup of $\operatorname{Ker} f_n$ and we can meaningfully construct a quotient group

$$H_n(F) = \operatorname{Ker} f_n / \operatorname{Im} f_{n+1},$$

called the $\boldsymbol{n^{th}}$ **homology group of $\boldsymbol{F}$**, for each $n \geq 0$. The elements of $H_n(F)$ are cosets of the form $x + \operatorname{Im} f_{n+1}$, usually called **homology classes** and denoted by $[x]$ when it does not lead to ambiguity. If $x, y \in \operatorname{Ker} f_n$ are such that $[x] = [y]$, then x and y are said to be **homologous** and it follows that $x - y \in \operatorname{Im} f_{n+1}$.

Suppose that the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_{n+1} & \xrightarrow{f_{n+1}} & F_n & \xrightarrow{f_n} & F_{n-1} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \varphi_{n+1}} & & \big\downarrow{\scriptstyle \varphi_n} & & \big\downarrow{\scriptstyle \varphi_{n-1}} & & \\
\cdots & \longrightarrow & G_{n+1} & \xrightarrow{g_{n+1}} & G_n & \xrightarrow{g_n} & G_{n-1} & \longrightarrow & \cdots
\end{array}$$

has sequences F and G that are chain complexes, and suppose also that the diagram is commutative given the maps $\varphi_n : F_n \to G_n$. Then the maps φ_n taken together define a **chain map** $F \to G$, and it's convenient to denote the chain map by either $\varphi_n : F \to G$ or $\{\varphi_n\}$. Now, if $\psi_n : F \to G$ is another chain map, and if there also exist maps $\lambda_n : F_n \to G_{n+1}$ as in the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_{n+1} & \xrightarrow{f_{n+1}} & F_n & \xrightarrow{f_n} & F_{n-1} & \longrightarrow & \cdots \\
 & & {\scriptstyle \psi_{n+1}}\big\downarrow{\scriptstyle \varphi_{n+1}} & \overset{\lambda_n}{\swarrow} & {\scriptstyle \psi_n}\big\downarrow{\scriptstyle \varphi_n} & \overset{\lambda_{n-1}}{\swarrow} & {\scriptstyle \psi_{n-1}}\big\downarrow{\scriptstyle \varphi_{n-1}} & & \\
\cdots & \longrightarrow & G_{n+1} & \xrightarrow{g_{n+1}} & G_n & \xrightarrow{g_n} & G_{n-1} & \longrightarrow & \cdots
\end{array}$$

such that

$$\varphi_n - \psi_n = g_{n+1} \circ \lambda_n + \lambda_{n-1} \circ f_n$$

for all n, then we call the collection of maps $\{\lambda_n\}$ a **chain homotopy** between the chain maps $\{\varphi_n\}$ and $\{\psi_n\}$, and say that $\{\varphi_n\}$ and $\{\psi_n\}$ are **chain homotopic**.

Proposition 2.1. *If $\{\varphi_n\}$ is a chain map $F \to G$, then each map φ_n induces a well-defined homomorphism $\varphi_{n*} : H_n(F) \to H_n(G)$ given by*

$$\varphi_{n*}(x + \operatorname{Im} f_{n+1}) = \varphi_n(x) + \operatorname{Im} g_{n+1}$$

for each $x \in \operatorname{Ker} f_n$.

Proposition 2.2. *If $\{\varphi_n\}$ is a chain map $F \to G$ such that each map φ_n is an isomorphism, then each map φ_{n*} is an isomorphism.*

Proof. Suppose that each $\varphi_n : F_n \to G_n$ of the chain map $F \to G$ is an isomorphism. Fix $n \geq 0$. Suppose that $\varphi_{n*}(x + \operatorname{Im} f_{n+1}) = \operatorname{Im} g_{n+1}$. Then $\varphi_n(x) \in \operatorname{Im} g_{n+1}$, so there exists some $y \in G_{n+1}$ such that $g_{n+1}(y) = \varphi_n(x)$. Since φ_{n+1} is onto, there exists some $z \in F_{n+1}$ such that $\varphi_{n+1}(z) = y$. Now, $\varphi_n \circ f_{n+1} = g_{n+1} \circ \varphi_{n+1}$, so

$$\varphi_n(f_{n+1}(z)) = g_{n+1}(\varphi_{n+1}(z)) = \varphi_n(x)$$

and the one-to-oneness of φ_n implies that $f_{n+1}(z) = x$. Hence $x \in \operatorname{Im} f_{n+1}$, from which it follows that $x + \operatorname{Im} f_{n+1} = \operatorname{Im} f_{n+1}$ and therefore φ_{n*} is one-to-one.

Next, let $y + \operatorname{Im} g_{n+1} \in H_n(G)$, so $y \in \operatorname{Ker} g_n$. Since $y \in G_n$ and φ_n is onto, there exists some $x \in F_n$ such that $\varphi_n(x) = y$. If $n = 0$ then $x \in \operatorname{Ker} f_0$ also, so suppose that $n > 0$. We obtain

$$\varphi_{n-1}(f_n(x)) = g_n(\varphi_n(x)) = g_n(y) = 0,$$

whence the one-to-oneness of φ_{n-1} gives $f_n(x) = 0$ so that $x \in \operatorname{Ker} f_n$. Thus, $x + \operatorname{Im} f_{n+1} \in H_n(F)$, and

$$\varphi_{n*}(x + \operatorname{Im} f_{n+1}) = \varphi_n(x) + \operatorname{Im} g_{n+1} = y + \operatorname{Im} g_{n+1}$$

shows that φ_{n*} is onto.

Therefore φ_{n*} is an isomorphism. ■

One useful result that derives easily from a chain homotopy is the following proposition, which will be used in later developments.

Proposition 2.3. *If $\{\varphi_n\}$ and $\{\psi_n\}$ are chain-homotopic chain maps $F \to G$, then $\varphi_{n*} = \psi_{n*}$ for all n.*

Proof. Suppose that $\{\varphi_n\}$ and $\{\psi_n\}$ are chain-homotopic maps. Fix n and $x \in \operatorname{Ker} f_n$. For simplicity denote $\varphi_n, \psi_n : F_n \to G_n$ by φ and ψ, so $\varphi_*, \psi_* : H_n(F) \to H_n(G)$. It must be demonstrated that

$$\varphi_*(x + \operatorname{Im} f_{n+1}) = \varphi(x) + \operatorname{Im} g_{n+1} = \psi(x) + \operatorname{Im} g_{n+1} = \psi_*(x + \operatorname{Im} f_{n+1}),$$

or equivalently

$$(\varphi(x) - \psi(x)) + \operatorname{Im} g_{n+1} = \operatorname{Im} g_{n+1}.$$

But by hypothesis there exist maps λ_{n-1} and λ_n such that

$$\varphi - \psi = g_{n+1} \circ \lambda_n + \lambda_{n-1} \circ f_n,$$

so

$$\begin{aligned}(\varphi - \psi)(x) &= g_{n+1}(\lambda_n(x)) + \lambda_{n-1}(f_n(x)) \\ &= g_{n+1}(\lambda_n(x)) + \lambda_{n-1}(0) = g_{n+1}(\lambda_n(x)).\end{aligned}$$

Hence $(\varphi - \psi)(x) \in \operatorname{Im} g_{n+1}$, which implies that $(\varphi - \psi)(x) + \operatorname{Im} g_{n+1} = \operatorname{Im} g_{n+1}$ and therefore

$$\varphi_*(x + \operatorname{Im} f_{n+1}) = \psi_*(x + \operatorname{Im} f_{n+1})$$

as desired. ∎

In what follows, for any chain complex F (see the top row of the diagram above) let $\mathbb{1}_{F_n} : F_n \to F_n$ be identity maps, which taken together form the chain map $\mathbb{1}_{F_n} : F \to F$. Also let $\mathbb{1}_{H_n(F)}$ be the identity map on $H_n(F)$ for all n.

Proposition 2.4. *Let $\theta_n : E \to F$ and $\varphi_n : F \to G$ be chain maps. Then, for all n,*

(1) $\mathbb{1}_{F_n *} = \mathbb{1}_{H_n(F)}$
(2) $(\varphi_n \circ \theta_n)_* = \varphi_{n*} \circ \theta_{n*}$

Referring to the diagram above, if there exists a chain map $\theta_n : G \to F$ such that $\{\theta_n \circ \varphi_n\}$ is chain homotopic to $\{\mathbb{1}_{F_n}\}$ and $\{\varphi_n \circ \theta_n\}$ is chain homotopic to $\{\mathbb{1}_{G_n}\}$, then $\{\varphi_n\}$ is called a **chain-homotopy equivalence**.

Proposition 2.5. *If $\{\varphi_n\}$ is a chain-homotopy equivalence, then φ_{n*} is an isomorphism for all n.*

Proof. Suppose that the chain map $\varphi_n : F \to G$ is a chain-homotopy equivalence. Then there exists a chain map $\theta_n : G \to F$ such that $\{\theta_n \circ \varphi_n\}$ is chain homotopic to $\{\mathbb{1}_{F_n}\}$ and $\{\varphi_n \circ \theta_n\}$ is chain homotopic to $\{\mathbb{1}_{G_n}\}$. By Proposition 2.3, $(\theta_n \circ \varphi_n)_* = \mathbb{1}_{F_n *}$ and $(\varphi_n \circ \theta_n)_* = \mathbb{1}_{G_n *}$, and so by Proposition 2.4 we obtain $\theta_{n*} \circ \varphi_{n*} = \mathbb{1}_{H_n(F)}$ and $\varphi_{n*} \circ \theta_{n*} = \mathbb{1}_{H_n(G)}$. Therefore, by Proposition 1.1, φ_{n*} is an isomorphism. ∎

The chain complex (1) is said to be an **exact sequence** if $\operatorname{Im} f_{n+1} = \operatorname{Ker} f_n$ for all n, in which case $H_n(F) = 0$. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 0$$

is called a **short exact sequence**. Let E, F, and G be chain complexes in the mold of (1), and let $i_n : E \to F$ and $j_n : F \to G$ be chain maps such that

$$0 \longrightarrow E_n \xrightarrow{\ i_n\ } F_n \xrightarrow{\ j_n\ } G_n \longrightarrow 0$$

is a short exact sequence for every n. Then the diagram in Figure 1 is commutative and is called a **short exact sequence of chain complexes**. By Proposition 2.1 there are well-defined homomorphisms $i_{n*} : H_n(E) \to H_n(F)$ and $j_{n*} : H_n(F) \to H_n(G)$, and what we're interested

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & E_{n+1} & \xrightarrow{e_{n+1}} & E_n & \xrightarrow{e_n} & E_{n-1} & \longrightarrow & \cdots \qquad (E) \\
 & & \downarrow{\scriptstyle i_{n+1}} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n-1}} & & \\
\cdots & \longrightarrow & F_{n+1} & \xrightarrow{f_{n+1}} & F_n & \xrightarrow{f_n} & F_{n-1} & \longrightarrow & \cdots \qquad (F) \\
 & & \downarrow{\scriptstyle j_{n+1}} & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n-1}} & & \\
\cdots & \longrightarrow & G_{n+1} & \xrightarrow{g_{n+1}} & G_n & \xrightarrow{g_n} & G_{n-1} & \longrightarrow & \cdots \qquad (G) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

FIGURE 1

in doing is constructing homomorphisms $\rho_n : H_n(G) \to H_{n-1}(E)$ such that we obtain a **long exact sequence of homology groups**

$$\cdots \longrightarrow H_n(E) \xrightarrow{i_{n*}} H_n(F) \xrightarrow{j_{n*}} H_n(G) \xrightarrow{\rho_n} H_{n-1}(E) \xrightarrow{i_{n-1*}} H_{n-1}(F) \longrightarrow \cdots$$

Let $[z] \in H_n(G)$, so $z \in \operatorname{Ker} g_n$. Since j_n is onto, $z = j_n(y)$ for some $y \in F_n$. Now, $f_n(y) \in F_{n-1}$, and since

$$j_{n-1}(f_n(y)) = g_n(j_n(y)) = g_n(z) = 0$$

it follows that $f_n(y) \in \operatorname{Ker} j_{n-1} = \operatorname{Im} i_{n-1}$ and so there exists some $x \in E_{n-1}$ such that $i_{n-1}(x) = f_n(y)$. Since

$$i_{n-2}(e_{n-1}(x)) = f_{n-1}(i_{n-1}(x)) = f_{n-1}(f_n(y)) = 0$$

(F is a chain complex so $f_{n-1} \circ f_n = 0$) and i_{n-2} is one-to-one, we conclude that $e_{n-1}(x) = 0$ and hence x represents a homology class $[x] \in H_{n-1}(E)$. We define $\rho_n([z]) = [x]$.

Theorem 2.6. *The sequence*

$$\cdots \longrightarrow H_n(E) \xrightarrow{i_{n*}} H_n(F) \xrightarrow{j_{n*}} H_n(G) \xrightarrow{\rho_n} H_{n-1}(E) \xrightarrow{i_{n-1*}} H_{n-1}(F) \longrightarrow \cdots$$

is exact.

We round out the section with two last results that are purely a matter of homological algebra but will have wide utility when dealing with topological matters.

Lemma 2.7 (**The Five-Lemma**)**.** *In a commutative diagram of abelian groups as given below, if the two rows are exact and the maps $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is also an isomorphism.*

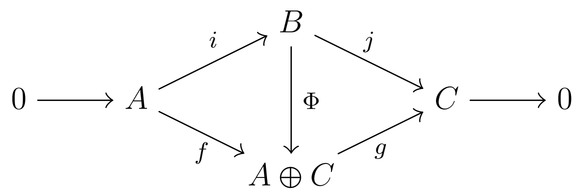
$$
\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E'
\end{array}
$$

Lemma 2.8 (The Splitting Lemma). *For a short exact sequence*

$$0 \longrightarrow A \xrightarrow{\;i\;} B \xrightarrow{\;j\;} C \longrightarrow 0$$

of abelian groups the following statements are equivalent:

(1) *There exists a homomorphism* $p : B \to A$ *such that* $p \circ i = \mathbb{1}_A$.
(2) *There exists a homomorphism* $s : C \to B$ *such that* $j \circ s = \mathbb{1}_C$.
(3) *If* $f : A \to A \oplus C$ *is given by* $f(a) = (a, 0)$ *and* $g : A \oplus C \to C$ *is given by* $g(a, c) = c$, *then there is an isomorphism* $\Phi : B \to A \oplus C$ *such that the following diagram is commutative:*



2.2 – Simplicial Homology

Recall that a set $A \subset \mathbb{R}^m$ is **convex** if $tv + (1-t)w \in A$ whenever $v, w \in A$ and $0 < t < 1$, which is to say that the line segment connecting any two points in A must also lie in A. The following result will prove useful later on.

Proposition 2.9. *Let $A \subset \mathbb{R}^m$ be a convex set. If $v_1, \ldots, v_n \in A$ and $t_1, \ldots, t_n \geq 0$ such that $\sum_k t_k = 1$, then $\sum_k t_k v_k \in A$.*

Proof. The statement is clearly true in the case $n = 1$. Suppose that it is true for some $n \in \mathbb{N}$. Let $v_1, \ldots, v_{n+1} \in A$ and $t_1, \ldots, t_{n+1} \geq 0$ such that $t_1 + \cdots + t_{n+1} = 1$.

If $t_{n+1} = 1$ then we must have $t_k = 0$ for all $1 \leq k \leq n$, whence $\sum_k t_k v_k = v_{n+1} \in A$ and we're done.

Assuming that $t_{n+1} \neq 1$, observe that from $\sum_{k=1}^n t_k = 1 - t_{n+1}$ we have

$$\sum_{k=1}^{n} \frac{t_k}{1 - t_{n+1}} = 1.$$

Now,

$$\sum_{k=1}^{n+1} t_k v_k = \sum_{k=1}^{n} t_k v_k + t_{n+1} v_{n+1} = (1 - t_{n+1}) \sum_{k=1}^{n} \frac{t_k}{1 - t_{n+1}} v_k + t_{n+1} v_{n+1},$$

where by the inductive hypothesis

$$\sum_{k=1}^{n} \frac{t_k}{1 - t_{n+1}} v_k$$

is an element of A. Thus, since $v_{n+1} \in A$ and A is convex, we conclude that

$$\sum_{k=1}^{n+1} t_k v_k$$

is also in A. ∎

The **convex hull** of a set $A \subset \mathbb{R}^m$, denoted by $\mathcal{C}(A)$, is the intersection of all convex sets that contain A; that is,

$$\mathcal{C}(A) = \bigcap \{C : A \subset C \text{ and } C \text{ is convex}\}$$

Proposition 2.10. *If $A = \{v_0, \ldots, v_n\}$, then*

$$\mathcal{C}(A) = \left\{ \sum_{k=0}^{n} t_k v_k \ : \ \forall k \, (t_k \geq 0) \ \text{ and } \ \sum_{k=0}^{n} t_k = 1 \right\}.$$

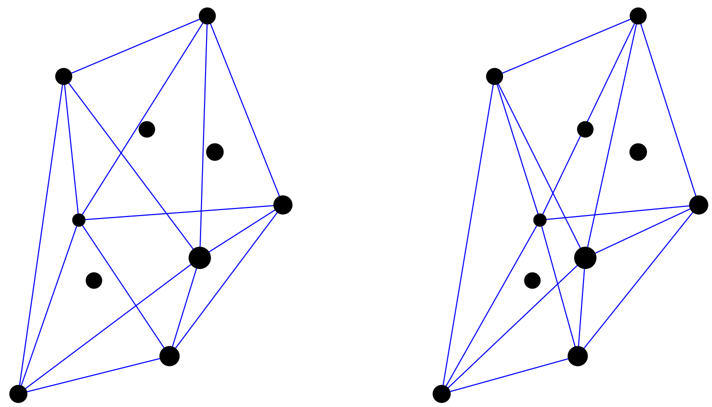


FIGURE 2. The convex hull of points in space.

Proof. Suppose that $A = \{v_0, \ldots, v_n\}$, and let

$$S = \left\{ \sum\nolimits_k t_k v_k \ : \ \forall k \, (t_k \geq 0) \ \text{ and } \ \sum\nolimits_k t_k = 1 \right\}.$$

It is clear that $A \subset S$.

Let $p, q \in S$. Then $p = \sum_k s_k v_k$ and $q = \sum_k t_k v_k$ for some nonnegative reals s_k, t_k such that $\sum_k s_k = \sum_k t_k = 1$. Now, for any $0 < r < 1$,

$$rp + (1-r)q = r \sum\nolimits_k s_k v_k + (1-r) \sum\nolimits_k t_k v_k = \sum\nolimits_k (rs_k + (1-r)t_k) v_k,$$

where $rs_k + (1-r)t_k \geq 0$ for each k, and

$$\sum\nolimits_k (rs_k + (1-r)t_k) = r \sum\nolimits_k s_k + (1-r) \sum\nolimits_k t_k = r + (1-r) = 1.$$

Hence $rp + (1-r)q \in S$ and S is a convex set containing A. From this conclusion it follows that $\mathcal{C}(A) \subset S$.

Next, let C be any convex set such that $A \subset C$. For any $q \in S$ there exist scalars $t_k \geq 0$ such that $\sum_k t_k v_k = 1$ and $q = \sum_k t_k v_k$. Now, since $v_k \in C$ for each k and C is convex, it follows from Proposition 2.9 that $q \in C$. Hence $S \subset C$, and since C is an arbitrary convex set that contains A, we obtain $S \subset \mathcal{C}(A)$. ■

The stereoscopic figure pair in Figure 2 illustrates the convex hull for a set A of ten points in $\mathbb{R}^3$. Of course, if the three points that appear in the interior of $\mathcal{C}(A)$ were removed from A, the same convex hull would result. We will be particularly interested in finite sets of points for which no one point can be removed without altering the convex hull.

If $A = \{v_0, \ldots, v_n\} \subset \mathbb{R}^m$ for some $m \geq n+1$ and the vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent, then $\mathcal{C}(A)$ is called an ***n*-simplex**, the points v_k are called the **vertices** of the simplex, and $\mathcal{C}(A)$ is denoted by $[v_0, \ldots, v_n]$. The ordering of the vertices in the symbol $[v_0, \ldots, v_n]$ further specifies an **orientation** on the n-simplex that is considered an essential part of its definition: namely, any **edge** $[v_i, v_j]$ of $[v_0, \ldots, v_n]$ is oriented in the direction of the vector $v_j - v_i$ if $i < j$, or $v_i - v_j$ if $i > j$. Proposition 2.10 along with the definition of an n-simplex make clear that any point in $[v_0, \ldots, v_n]$ is uniquely expressible in the form $\sum_k t_k v_k$, with the **barycentric coordinates** t_k of the point being nonnegative scalars that sum to 1.

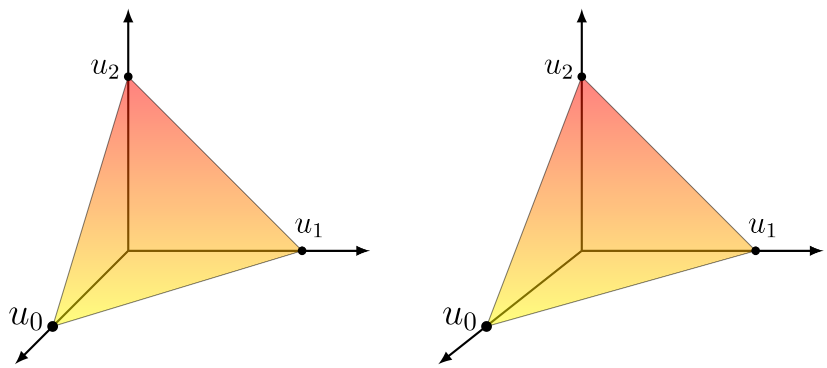


FIGURE 3. The standard 2-simplex Δ^2.

Defining $u_{i-1} = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^{n+1}$ with the 1 in the ith position for $1 \leq i \leq n+1$, the **standard $\boldsymbol{n}$-simplex** Δ^n is the n-simplex $[u_0, \ldots, u_n] \subset \mathbb{R}^{n+1}$, and so since $\sum_k t_k u_k = (t_0, \ldots, t_n)$,

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \ : \ \forall k \, (t_k \geq 0) \ \text{ and } \ \sum_{k=0}^{n} t_k = 1 \right\}.$$

See Figure 3 for an illustration of the standard 2-simplex.

The **canonical linear homeomorphism** from Δ^n to any n-simplex $[v_0, \ldots, v_n]$ is the linear transformation $\sum_k t_k u_k \mapsto \sum_k t_k v_k$ that preserves orientation.

2.3 – Singular Homology

Given a topological space X, a **singular n-simplex** is a continuous map $\sigma : \Delta^n \to X$. For each $n \geq 0$ define $C_n(X)$ to be the free abelian group with basis the set of all singular n-simplices associated with X. The elements of $C_n(X)$ are called **n-chains**, and are written as finite formal sums $\sum_{i=1}^k n_i\sigma_i$, where $k, n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. The **singular boundary maps** $\partial_n : C_n(X) \to C_{n-1}(X)$ are homomorphisms defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i \sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} \circ \delta_i^{n-1} \tag{2}$$

for each basis element $\sigma \in C_n(X)$, where $\delta_i^{n-1} : \Delta^{n-1} \to [u_0, \ldots, \hat{u}_i, \ldots, u_n]$ is the canonical linear homeomorphism as discussed in the previous section (it is often suppressed in the interests of brevity). As with the simplicial boundary maps it can be shown that $\partial_n \circ \partial_{n+1} = 0$, so we obtain a chain complex

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots,$$

called the **singular chain complex of X** and denoted by $C(X)$, which gives rise to homology groups

$$H_n(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$$

called the **singular homology groups of X**.

Now, suppose Y is another topological space, and let $\varphi : X \to Y$ be continuous. For each n the map φ induces a homomorphism[1] $\varphi_n : C_n(X) \to C_n(Y)$ defined by $\varphi_n(\sigma) = \varphi \circ \sigma$ for each basis element $\sigma : \Delta^n \to X$ of $C_n(X)$. Thus, for any n-chain $\sum_i n_i\sigma_i$ we have

$$\varphi_n\left(\sum\nolimits_i n_i\sigma_i\right) = \sum\nolimits_i n_i\varphi_n(\sigma_i) = \sum\nolimits_i n_i(\varphi_n \circ \sigma_i).$$

Denoting the singular boundary maps $C_n(X) \to C_{n-1}(X)$ and $C_n(Y) \to C_{n-1}(Y)$ by ∂_n^X and ∂_n^Y, respectively, we find that

$$\begin{aligned}(\varphi_{n-1} \circ \partial_n^X)(\sigma) &= \varphi_{n-1}\left(\sum\nolimits_i (-1)^i \sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} \circ \delta_i^{n-1}\right) \\ &= \sum\nolimits_i (-1)^i((\varphi \circ \sigma)|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} \circ \delta_i^{n-1}) \\ &= \partial_n^Y(\varphi \circ \sigma) = \partial_n^Y(\varphi_n(\sigma)) = (\partial_n^Y \circ \varphi_n)(\sigma),\end{aligned}$$

and therefore the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}^X} & C_n(X) & \xrightarrow{\partial_n^X} & C_{n-1}(X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\ \cdots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial_{n+1}^Y} & C_n(Y) & \xrightarrow{\partial_n^Y} & C_{n-1}(Y) & \longrightarrow & \cdots \end{array}$$

[1] Other books would represent φ_n by $\varphi_\sharp$, and I once was tempted to use $\varphi_{n\sharp}$; but $\varphi_\sharp$ is uninformative for obvious reasons, and $\varphi_{n\sharp}$ conveys no more information than φ_n already does.

is commutative and it follows that $\{\varphi_n\}$ forms a chain map $C(X) \to C(Y)$. By Proposition 2.1 the maps φ_n induce homomorphisms $\varphi_{n*} : H_n(X) \to H_n(Y)$ between homology groups.

The first significant general result we are in a position to obtain in singular homology is the following.

Proposition 2.11. *Let $\{X_\alpha\}_{\alpha\in A}$ be the path-components of a topological space X. Then $H_n(X) \cong \bigoplus_{\alpha\in A} H_n(X_\alpha)$.*

Proof. For $[\sigma] := \sigma + \partial_{n+1}(C_{n+1}(X)) \in H_n(X)$ we have $\sigma = \sum_{i=1}^{a} n_i\sigma_i$ for singular n-simplices $\sigma_i : \Delta^n \to X$. Each image $\sigma_i(\Delta^n)$ in X is path-connected, so for each $1 \le i \le a$ there exists some $\alpha_i \in A$ such that $\sigma_i : \Delta^n \to X_{\alpha_i}$.

Let $i_1 = 1$, $i_2 = \min\{1 < i \le a : \alpha_i \neq \alpha_1\}$, and in general $i_k = \min\{i : \alpha_i \neq \alpha_{i_1}, \dots, \alpha_{i_{k-1}}\}$ for $1 \le k \le b$, where $b \le a$. Then $X_{\alpha_{i_1}}, \dots, X_{\alpha_{i_b}}$ are the *distinct* path components of X that contain the images of the σ_i, and for convenience we can designate each $X_{\alpha_{i_k}}$ by X_{i_k}. Let $B_k = \{i \mid \sigma_i : \Delta^n \to X_{i_k}\}$ for each k, and define $\varphi_k = \sum_{i\in B_k} n_i\sigma_i$ so that $\varphi_k \in C_n(X_{i_k})$. Now $\sigma = \sum_k \varphi_k$, and $\sigma \in \operatorname{Ker}\partial_n$ implies $\sum_k \partial_n\varphi_k = 0$ and therefore $\partial_n\varphi_k = 0$ for each k since $C_n(X_{i_r}) \cap C_n(X_{i_s}) = \varnothing$ whenever $r \neq s$. It follows that each φ_k is in the kernel of ∂_n restricted to $C_n(X_{i_k})$, so that

$$[\varphi_k]_{i_k} := \varphi_k + \partial_{n+1}(C_{n+1}(X_{i_k}))$$

is in $H_n(X_{i_k})$ and we can define a map $\Omega : H_n(X) \to \bigoplus_{\alpha\in A} H_n(X_\alpha)$ by

$$\Omega([\sigma]) = \sum_{k=1}^{b} [\varphi_k]_{i_k}.$$

Suppose that $[\sigma] = [\tau]$, so $\sigma - \tau \in \partial(C_{n+1}(X))$ and there's some $\xi \in C_{n+1}(X)$ such that $\partial\xi = \sigma - \tau$. As before, we can write $\sigma = \sum_{k=1}^{a} \varphi_{\alpha_k}$ such that $\varphi_{\alpha_k} \in C_n(X_{\alpha_k})$ and $\alpha_i \neq \alpha_j$ whenever $i \neq j$. Similarly, $\tau = \sum_{k=1}^{b} \psi_{\beta_k}$ such that $\psi_{\beta_k} \in C_n(X_{\beta_k})$ and $\beta_i \neq \beta_j$ whenever $i \neq j$. By definition $\Omega([\sigma]) = ([\varphi_\alpha])_{\alpha\in A}$, where $[\varphi_\alpha]$ is a class in $H_n(X_\alpha)$ with $\varphi_\alpha = 0$ if $\alpha \neq \alpha_k$ for all $1 \le k \le a$. In similar fashion $\Omega([\tau]) = ([\psi_\alpha])_\alpha$ with $\psi_\alpha = 0$ if $\alpha \neq \beta_k$ for all $1 \le k \le b$. With this kind of arrangement we can write

$$\sigma = \sum_{\alpha\in A} \varphi_\alpha \ \text{ and } \ \tau = \sum_{\alpha\in A} \psi_\alpha,$$

so $\partial\xi = \sum_\alpha(\varphi_\alpha - \psi_\alpha)$. However, ξ itself is expressible as $\sum_{\alpha\in A} \xi_\alpha$ with $\xi_\alpha \in C_{n+1}(X_\alpha)$ for each α, and clearly we must have

$$\partial\xi_\alpha = \varphi_\alpha - \psi_\alpha \in C_n(X_\alpha)$$

since the X_α are disjoint. Since

$$\varphi_\alpha - \psi_\alpha \in \partial(C_{n+1}(X_\alpha)),$$

we find that $[\varphi_\alpha] = [\psi_\alpha]$ as classes in $H_n(X_\alpha)$, whence $\Omega([\sigma]) = \Omega([\tau])$ and Ω is well-defined.

Now assume simply that $[\sigma], [\tau] \in H_n(X)$. Once again rearrange to write $\sigma = \sum_{k=1}^{a} \varphi_{\alpha_k}$ and $\tau = \sum_{k=1}^{b} \psi_{\beta_k}$, only such that $\alpha_k = \beta_k$ for all $1 \le k \le c \le a$ (assuming $a \le b$ for definiteness) and

$$\{\alpha_k : c+1 \le k \le a\} \cap \{\beta_k : c+1 \le k \le b\} = \varnothing.$$

Redesignate indices as follows: $\beta_{c+1} = \alpha_{a+1}$, $\beta_{c+2} = \alpha_{a+2}, \ldots, \beta_b = \alpha_{a+r}$, where $b = c + r$. Now, since Ω is well-defined,

$$\begin{aligned}\Omega([\sigma] + [\tau]) &= \Omega\left(\left[\sum_{k=1}^{a} \varphi_{\alpha_k} + \sum_{k=1}^{b} \psi_{\beta_k}\right]\right) \\ &= \Omega\left(\left[\sum_{k=1}^{c}(\varphi_{\alpha_k} + \psi_{\alpha_k}) + \sum_{k=c+1}^{a} \varphi_{\alpha_k} + \sum_{k=a+1}^{a+r} \psi_{\alpha_k}\right]\right)\end{aligned}$$

We can let $\xi_k = \varphi_{\alpha_k} + \psi_{\alpha_k}$ for $1 \le k \le c$, $\xi_k = \varphi_{\alpha_k}$ for $c + 1 \le k \le a$, and $\xi_k = \psi_{\alpha_k}$ for $a + 1 \le k \le a + r$, so that

$$\begin{aligned}\Omega([\sigma] + [\tau]) &= \Omega\left(\left[\sum_{k=1}^{a+r} \xi_k\right]\right) = \sum_{k=1}^{a+r} [\xi_k]_{\alpha_k} \\ &= \sum_{k=1}^{c}[\varphi_{\alpha_k} + \psi_{\alpha_k}]_{\alpha_k} + \sum_{k=c+1}^{a} [\varphi_{\alpha_k}]_{\alpha_k} + \sum_{k=a+1}^{a+r} [\psi_{\alpha_k}]_{\alpha_k}.\end{aligned}$$

Since

$$[\varphi_{\alpha_k} + \psi_{\alpha_k}]_{\alpha_k} = [\varphi_{\alpha_k}]_{\alpha_k} + [\psi_{\alpha_k}]_{\alpha_k},$$

$\alpha_k = \beta_k$ for $1 \le k \le c$, and $\psi_{\alpha_k} = \psi_{\beta_{k-a+c}}$ for $a + 1 \le k \le a + r$,

$$\begin{aligned}\Omega([\sigma] + [\tau]) &= \sum_{k=1}^{a}[\varphi_{\alpha_k}]_{\alpha_k} + \sum_{k=1}^{c}[\psi_{\alpha_k}]_{\alpha_k} + \sum_{k=a+1}^{a+r} [\psi_{\beta_{k-a+c}}]_{\beta_{k-a+c}} \\ &= \sum_{k=1}^{a}[\varphi_{\alpha_k}]_{\alpha_k} + \sum_{k=1}^{c}[\psi_{\beta_k}]_{\beta_k} + \sum_{k=c+1}^{c+r} [\psi_{\beta_k}]_{\beta_k}.\end{aligned}$$

Recalling that $b = c + r$, we obtain

$$\Omega([\sigma] + [\tau]) = \sum_{k=1}^{a}[\varphi_{\alpha_k}]_{\alpha_k} + \sum_{k=1}^{b}[\psi_{\beta_k}]_{\beta_k} = \Omega([\sigma]) + \Omega([\tau])$$

and see that Ω is a homomorphism.

Next, suppose that $\Omega([\sigma]) = 0$. Proceeding as with the well-definedness argument, we obtain $([\varphi_\alpha])_{\alpha\in A} = 0$ and hence $\varphi_\alpha \in \partial(C_{n+1}(X_\alpha))$ for all $\alpha \in A$. But then $\varphi_\alpha \in \partial(C_{n+1}(X))$ for all α and

$$[\sigma] = \left[\sum_{\alpha\in A} \varphi_\alpha\right] = \left[\sum_{k=1}^{a} \varphi_{\alpha_k}\right] = \sum_{k=1}^{a} [\varphi_{\alpha_k}] = \sum_{k=1}^{a} (\varphi_{\alpha_k} + \partial(C_{n+1}(X))) = 0,$$

where the third equality holds since each φ_{α_k} is, by construction, in the kernel of ∂_n and so represents a class in $H_n(X)$. Therefore Ω is injective.

Now suppose that $([\varphi_\alpha])_{\alpha\in A} \in \bigoplus_{\alpha\in A} H_n(X_\alpha)$, so the set $S = \{\alpha \in A : [\varphi_\alpha] \neq 0\}$ must be finite. For each $\alpha \in S$ we have

$$\varphi_\alpha \in \mathrm{Ker}[\partial_n : C_n(X_\alpha) \to C_{n-1}(X_\alpha)],$$

and thus

$$\varphi := \sum_{\alpha\in S} \varphi_\alpha \in \operatorname{Ker}[\partial_n : C_n(X) \to C_{n-1}(X)]$$

so that $[\varphi] := \varphi + \partial(C_{n+1}(X)) \in H_n(X)$. Now,

$$\Omega([\varphi]) = \sum_{\alpha\in S}[\varphi_\alpha] = \sum_{\alpha\in A}[\varphi_\alpha] = ([\varphi_\alpha])_{\alpha\in A}$$

since $[\varphi_\alpha] = 0$ for $\alpha \in A - S$. Therefore Ω is surjective. ∎

In what follows recall the notational convention whereby $[u_0]$ and $[u_0, u_1]$ denote the standard simplices Δ^0 and Δ^1, respectively.

Proposition 2.12. *If $X \neq \varnothing$ is path-connected, then $H_0(X) \cong \mathbb{Z}$.*

Proof. Define the homomorphism $\epsilon : C_0(X) \to \mathbb{Z}$ by $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$, so that $\epsilon(\sigma) = 1$ for each singular 1-simplex σ. If $\sigma : \Delta^0 \to X$ is given by $\sigma(u_0) = x_0$ for some $x_0 \in X$, then for any $k \in \mathbb{Z}$ we have $\epsilon(k\sigma) = k$ for $k\sigma \in C_0(X)$, showing that ϵ is surjective and thus $C_0(X)/\operatorname{Ker}\epsilon \cong \mathbb{Z}$.

Fix $\tau : \Delta^1 \to X$ in $C_1(X)$. Then

$$\epsilon(\partial_1\tau) = \epsilon(\tau|_{[u_1]} - \tau|_{[u_0]}) = \epsilon(\tau|_{[u_1]}) - \epsilon(\tau|_{[u_0]}) = 1 - 1 = 0.$$

Thus, if $\sigma \in \operatorname{Im}\partial_1$, then there exists some $\sum_i n_i\tau_i \in C_1(X)$ such that $\partial_1(\sum_i n_i\tau_i) = \sigma$, whence

$$\epsilon(\sigma) = \epsilon\left(\sum\nolimits_i n_i\,\partial_1\tau_i\right) = \sum\nolimits_i n_i\,\epsilon(\partial_1\tau_i) = 0$$

and we conclude that $\operatorname{Im}\partial_1 \subset \operatorname{Ker}\epsilon$.

Now suppose that $\varphi = \sum_i n_i\sigma_i \in \operatorname{Ker}\epsilon$, so $\sum_i n_i = 0$. Fix $x_0 \in X$. Define $\sigma_0 \in C_0(X)$ by $\sigma_0(u_0) = x_0$. For each i, since $\sigma_i(u_0), x_0 \in X$ and X is path-connected, there exists a path $\tau_i : \Delta^1 \to X$ such that $\tau_i(u_0) = x_0$ and $\tau_i(u_1) = \sigma_i(u_0)$. Then

$$\begin{aligned}\partial_1\left(\sum\nolimits_i n_i\tau_i\right) &= \sum\nolimits_i n_i\,\partial_1\tau_i = \sum\nolimits_i n_i(\tau_i|_{[u_1]} - \tau_i|_{[u_0]}) = \sum\nolimits_i n_i\tau_i|_{[u_1]} - \sum\nolimits_i n_i\tau_i|_{[u_0]} \\ &= \sum\nolimits_i n_i\sigma_i - \sum\nolimits_i n_i\sigma_0 = \sum\nolimits_i n_i\sigma_i - \sigma_0\sum\nolimits_i n_i = \sum\nolimits_i n_i\sigma_i = \varphi,\end{aligned}$$

which shows that $\varphi \in \operatorname{Im}\partial_1$ and hence $\operatorname{Ker}\epsilon \subset \operatorname{Im}\partial_1$.

Therefore $H_0(X) = C_0(X)/\operatorname{Im}\partial_1 = C_0(X)/\operatorname{Ker}\epsilon \cong \mathbb{Z}$. ∎

Combining the two propositions above it follows that for any space X, if $\{X_\alpha\}_{\alpha\in A}$ are the path-components of X, then $H_0(X) \cong \bigoplus_{\alpha\in A}\mathbb{Z}$.

Proposition 2.13. *If $X = \{p\}$, then $H_n(X) = 0$ for all $n \geq 1$.*

Proof. For X a single point p we find that $C_n(X) \cong \mathbb{Z}$ with generator p_n, where $p_n : \Delta^n \to X$ is the singular n-simplex given by $p_n(x) = p$ for all $x \in \Delta^n$. Fix $\sigma \in \operatorname{Ker}\partial_n \subset C_n(X)$, so $\sigma = mp_n$ for some $m \in \mathbb{Z}$ such that $\partial_n(mp_n) = 0$

Suppose that n is odd. Now, $mp_{n+1} \in C_{n+1}(X)$, and

$$\partial_{n+1}(mp_{n+1}) = m\sum_{i=0}^{n+1}(-1)^i p_{n+1}|_{[u_0,\ldots,\hat{u}_i,\ldots,u_{n+1}]} = m\sum_{i=0}^{n+1}(-1)^i p_n = mp_n = \sigma$$

shows that $\sigma \in \operatorname{Im}\partial_{n+1}$. Therefore $[\sigma] = 0$.

Suppose that n is even. $\partial_n\sigma = 0$ implies that $m\partial_n p_n = 0$; that is,

$$m\sum_{i=0}^{n}(-1)^i p_n|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} = m\sum_{i=0}^{n}(-1)^i p_{n-1} = mp_{n-1} = 0,$$

which implies that $m = 0$. Thus $\sigma = 0$ and once again we obtain $[\sigma] = 0$.

It's concluded, then, that $H_n(X) = 0$ for any $n \geq 1$. ■

So for $X = \{p\}$ Proposition 2.12 implies that $H_0(X) \cong \mathbb{Z}$, and thus there's an exception to the pattern in Proposition 2.13. To fix this (if indeed a "fix" is desired), we can modify the definition for the 0th homology class to make it also trivial in the case when X is a single point. The way to do this is to replace the map $\partial_0 : C_0(X) \to 0$ with $\epsilon : C_0(X) \to \mathbb{Z}$ to form the augmented chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

then define $\widetilde{H}_0(X) = \operatorname{Ker}\epsilon/\operatorname{Im}\partial_1$. Since it was found in the proof of Proposition 2.12 that $\operatorname{Ker}\epsilon = \operatorname{Im}\partial_1$ when X is nonempty and path-connected, we find in particular $\widetilde{H}_0(\{p\}) = 0$. Setting $\widetilde{H}_n(X) = H_n(X)$ for $n \geq 1$, we obtain what are known as **reduced homology groups** which have the virtue of being trivial in all nonnegative dimensions for one-point spaces.

If X is nonempty but not path-connected we still have $\operatorname{Im}\partial_1 \subset \operatorname{Ker}\epsilon$. Define $\bar{\epsilon} : H_0(X) \to \mathbb{Z}$ by $\bar{\epsilon}([\sigma]) = \epsilon(\sigma)$, and note that $\bar{\epsilon}$ is surjective and $\operatorname{Ker}\bar{\epsilon} = \widetilde{H}_0(X)$. Hence $H_0(X)/\widetilde{H}_0(X) \cong \mathbb{Z}$, or equivalently $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

Theorem 2.14. *If the maps $f, g : X \to Y$ are homotopic, then $f_{n*} = g_{n*} : H_n(X) \to H_n(Y)$ for all n.*

Proof. Suppose that $f, g : X \to Y$ are homotopic. In light of Proposition 2.3 it will suffice to show that the chain maps $f_n, g_n : C_n(X) \to C_n(Y)$ are chain-homotopic. That is, it must be shown that there exists, for all $n \geq 0$, maps $\lambda_n : C_n(X) \to C_{n+1}(Y)$ such that

$$g_n - f_n = \partial_{n+1}^Y \circ \lambda_n + \lambda_{n-1} \circ \partial_n^X.$$

Let $F : X \times I \to Y$ be a homotopy from f to g, so $F(\cdot, 0) = f(\cdot)$ and $F(\cdot, 1) = g(\cdot)$. As usual Δ^n will be designated by $[u_0, \ldots, u_n]$, and for the product space $\Delta^n \times I$ define $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$. Finally, define the homomorphism $\lambda_n : C_n(X) \to C_{n+1}(Y)$ by

$$\lambda_n(\sigma) = \sum_{i=0}^{n}(-1)^i F \circ (\sigma \times \mathbb{1})|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}$$

for each $\sigma : \Delta^n \to X$, where $F \circ (\sigma \times \mathbb{1}) : \Delta^n \times I \to X \times I \to Y$ and each term in the sum we take to be precomposed by the canonical linear homeomorphism

$$\Delta^{n+1} \mapsto [v_0, \ldots, v_i, w_i, \ldots, w_n] \subset \Delta^n \times I.$$

It's instructive to examine the $n = 0$ case and show that $g_0 - f_0 = \partial_1^Y \circ \lambda_0 + \lambda_{-1} \circ \partial_0^X$. Designating $\lambda_{-1} \equiv 0$, this entails showing $\partial_1^Y \circ \lambda_0 = g_0 - f_0$. Letting δ^n denote appropriate canonical linear homeomorphisms $\Delta^n \mapsto [z_0, \ldots, z_n]$ for any n-simplex $[z_0, \ldots, z_n]$, we have for any $\sigma : \Delta^0 \to X$,

$$(\partial_1^Y \circ \lambda_0)(\sigma) = \partial_1^Y (F \circ (\sigma \times \mathbb{1})|_{[v_0,w_0]} \circ \delta^1) = \partial_1^Y (F \circ (\sigma \times \mathbb{1}) \circ \delta^1),$$

where the second equality holds since $[v_0, w_0] = \Delta^0 \times I$. Pressing on,

$$(\partial_1^Y \circ \lambda_0)(\sigma) = F \circ (\sigma \times \mathbb{1}) \circ \delta^1|_{[u_1]} \circ \delta^0 - F \circ (\sigma \times \mathbb{1}) \circ \delta^1|_{[u_0]} \circ \delta^0.$$

Now,

$$\begin{aligned}(F \circ (\sigma \times \mathbb{1}) \circ \delta^1|_{[u_1]} \circ \delta^0)(u_0) &= (F \circ (\sigma \times \mathbb{1}) \circ \delta^1)(u_1) \\ &= (F \circ (\sigma \times \mathbb{1}))(u_0, 1) \\ &= F(\sigma(u_0), 1),\end{aligned}$$

where $F(\sigma(u_0), 1) = g(\sigma(u_0)) = (g_0(\sigma))(u_0)$ and thus $F \circ (\sigma \times \mathbb{1}) \circ \delta^1|_{[u_1]} \circ \delta^0 = g_0(\sigma)$. In similar fashion we find $F \circ (\sigma \times \mathbb{1}) \circ \delta^1|_{[u_0]} \circ \delta^0 = f_0(\sigma)$, and so $(\partial_1^Y \circ \lambda_0)(\sigma) = g_0(\sigma) - f_0(\sigma)$ and we're done.

The $n = 2$ case is the highest dimensioned case that can be readily visualized, so let's consider it next. Let $\sigma : \Delta^2 \to X$. For brevity let $H = F \circ (\sigma \times \mathbb{1})$ and $[j] = [u_0, \ldots, \hat{u}_j, \ldots, u_3]$. Also it will be convenient to define $[z_0, \ldots, z_n]_j = [z_0, \ldots, \hat{z}_j, \ldots, z_n]$. Now,

$$\begin{aligned}\partial_3^Y(\lambda_2(\sigma)) = & \sum_{j=0}^{3} (-1)^j H|_{[v_0,w_0,w_1,w_2]} \circ \delta^3|_{[j]} \circ \delta^2 - \sum_{j=0}^{3} (-1)^j H|_{[v_0,v_1,w_1,w_2]} \circ \delta^3|_{[j]} \circ \delta^2 \\ & + \sum_{j=0}^{3} (-1)^j H|_{[v_0,v_1,v_2,w_2]} \circ \delta^3|_{[j]} \circ \delta^2 \\ = & \sum_{j=0}^{3} (-1)^j \left[H|_{[v_0,w_0,w_1,w_2]_j} \circ \delta^2 - H|_{[v_0,v_1,w_1,w_2]_j} \circ \delta^2 + H|_{[v_0,v_1,v_2,w_2]_j} \circ \delta^2 \right],\end{aligned}$$

and from $\lambda_1(\partial_2^X(\sigma)) = \sum_{j=0}^{2} (-1)^j \lambda_1(\sigma|_{[u_0,\ldots,\hat{u}_j,\ldots,u_2]} \circ \delta^1)$ we obtain

$$\begin{aligned}\lambda_1(\partial_2^X(\sigma)) = & \sum_{j=0}^{2} (-1)^j \big[F \circ (\sigma|_{[u_0,\ldots,\hat{u}_j,\ldots,u_2]} \circ \delta^1 \times \mathbb{1})|_{[v_0,w_0,w_1]} \circ \delta^2 \\ & - F \circ (\sigma|_{[u_0,\ldots,\hat{u}_j,\ldots,u_2]} \circ \delta^1 \times \mathbb{1})|_{[v_0,v_1,w_1]} \circ \delta^2 \big] \end{aligned} \tag{3}$$

Consider the workings of

$$(\sigma|_{[u_1,u_2]} \circ \delta^1 \times \mathbb{1})|_{[v_0,w_0,w_1]} \circ \delta^2 : \Delta^2 \to \Delta^1 \times I \to X \times I.$$

The 2-simplex $[v_0, w_0, w_1]$ is regarded as a subspace of $\Delta^1 \times I$, so for a point $p \in \Delta^2$ we have $\delta^2(p) = (q, t)$ for $q \in \Delta^1$ and $0 \leq t \leq 1$. Now,

$$(\sigma|_{[u_1,u_2]} \circ \delta^1 \times \mathbb{1})(q, t) = (\sigma|_{[u_1,u_2]} \times \mathbb{1})(\delta^1(q), t) = (\sigma \times \mathbb{1})(\delta^1(q), t),$$

where $\delta^1(q) \in [u_1, u_2]$ with $[u_1, u_2]$ regarded as a subspace of Δ^2 so that

$$(\delta^1(q), t) \in [v_1, w_1, w_2] \subset \Delta^2 \times I.$$

Thus $(\sigma|_{[u_1,u_2]} \circ \delta^1 \times \mathbb{1})|_{[v_0,w_0,w_1]} \circ \delta^2$ is equal to $(\sigma \times \mathbb{1})|_{[v_1,w_1,w_2]} \circ \delta^2$ since

$$((\sigma \times \mathbb{1})|_{[v_1,w_1,w_2]} \circ \delta^2)(p) = (\sigma \times \mathbb{1})|_{[v_1,w_1,w_2]}(\delta^1(q), t) = (\sigma \times \mathbb{1})(\delta^1(q), t)$$

(note that $\delta^2 : \Delta^2 \to [v_1, w_1, w_2] \subset \Delta^2 \times I$ will canonically carry p directly to $(\delta^1(q), t)$). Expanding (3) (and suppressing the canonical homeomorphisms), we obtain

$$\begin{aligned}\lambda_1(\partial_2^X(\sigma)) = &\big[F \circ (\sigma|_{[u_1,u_2]} \times \mathbb{1})|_{[v_0,w_0,w_1]} - F \circ (\sigma|_{[u_1,u_2]} \times \mathbb{1})|_{[v_0,v_1,w_1]}\big] \\ &- \big[F \circ (\sigma|_{[u_0,u_2]} \times \mathbb{1})|_{[v_0,w_0,w_1]} - F \circ (\sigma|_{[u_0,u_2]} \times \mathbb{1})|_{[v_0,v_1,w_1]}\big] \\ &+ \big[F \circ (\sigma|_{[u_0,u_1]} \times \mathbb{1})|_{[v_0,w_0,w_1]} - F \circ (\sigma|_{[u_0,u_1]} \times \mathbb{1})|_{[v_0,v_1,w_1]}\big]\end{aligned}$$

and thus

$$\lambda_1(\partial_2^X(\sigma)) = H|_{[v_1,w_1,w_2]} - H|_{[v_1,v_2,w_2]} - H|_{[v_0,w_0,w_2]} + H|_{[v_0,v_2,w_2]} + H|_{[v_0,w_0,w_1]} - H|_{[v_0,v_1,w_1]}.$$

Now at last we add:

$$\partial_3^Y(\lambda_2(\sigma)) + \lambda_1(\partial_2^X(\sigma)) = F \circ (\sigma \times \mathbb{1})|_{[w_0,w_1,w_2]} \circ \delta^2 - F \circ (\sigma \times \mathbb{1})|_{[v_0,v_1,v_2]} \circ \delta^2.$$

For $p \in \Delta^2$,

$$\begin{aligned}(F \circ (\sigma \times \mathbb{1})|_{[w_0,w_1,w_2]} \circ \delta^2)(p) &= F \circ (\sigma \times \mathbb{1})|_{[w_0,w_1,w_2]}(p, 1) \\ &= F(\sigma(p), 1) = g(\sigma(p)) \\ &= (g \circ \sigma)(p) = (g_2(\sigma))(p),\end{aligned}$$

so $F \circ (\sigma \times \mathbb{1})|_{[w_0,w_1,w_2]} \circ \delta^2 = g_2(\sigma)$, and similarly $F \circ (\sigma \times \mathbb{1})|_{[v_0,v_1,v_2]} \circ \delta^2 = f_2(\sigma)$. Hence

$$(\partial_3^Y \circ \lambda_2 + \lambda_1 \circ \partial_2^X)(\sigma) = (g_2 - f_2)(\sigma)$$

for the basis element σ of $C_2(X)$.

The general case for arbitrary n requires careful bookkeeping and will be addressed at a later time. ■

Lemma 2.15. *Let X, Y, Z be topological spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then, for all n,*

(1) $(\mathbb{1}_X)_{n*} = \mathbb{1}_{H_n(X)}$
(2) $(g \circ f)_{n*} = g_{n*} \circ f_{n*}$

Proof. Fix n. For $\sigma : \Delta^n \to X$ we have $(\mathbb{1}_X)_n(\sigma) = \mathbb{1}_X \circ \sigma = \sigma$ so that $(\mathbb{1}_X)_n = \mathbb{1}_{C_n(X)}$. Now, by Proposition 2.4, $(\mathbb{1}_X)_{n*} = \mathbb{1}_{C_n(X)*} = \mathbb{1}_{H_n(X)}$, which proves part (1).

Next, from $g \circ f : X \to Z$ we obtain the map $(g \circ f)_n : C_n(X) \to C_n(Z)$, where

$$(g \circ f)_n(\sigma) = (g \circ f) \circ \sigma = g \circ (f \circ \sigma) = g_n(f \circ \sigma) = g_n(f_n(\sigma)) = (g_n \circ f_n)(\sigma),$$

and so we obtain

$$(g \circ f)_{n*} = (g_n \circ f_n)_* = g_{n*} \circ f_{n*}$$

by once again appealing to Proposition 2.4. This proves part (2). ∎

Corollary 2.16. *If $f : X \to Y$ is a homotopy equivalence, then $f_{n*} : H_n(X) \to H_n(Y)$ is an isomorphism for all n.*

Proof. Suppose that $f : X \to Y$ is a homotopy equivalence. Then there is a map $g : Y \to X$ such that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$. By Theorem 2.14, $(g \circ f)_{n*} = (\mathbb{1}_X)_{n*}$ and $(f \circ g)_{n*} = (\mathbb{1}_Y)_{n*}$, and thus by Lemma 2.15 we obtain $g_{n*} \circ f_{n*} = \mathbb{1}_{H_n(X)}$ and $f_{n*} \circ g_{n*} = \mathbb{1}_{H_n(Y)}$. Therefore f_{n*} is an isomorphism by Proposition 1.1. ∎

If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood of X, then the pair (X, A) is known as a **good pair**. The following theorem will be proven over course of the next two sections, with the maps $\hat{\partial}_n$ to be determined along the way.

Theorem 2.17. *If (X, A) is a good pair, then there is an exact sequence*

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_{n*}} \widetilde{H}_n(X) \xrightarrow{q_{n*}} \widetilde{H}_n(X/A) \xrightarrow{\hat{\partial}_{n*}} \widetilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0,$$

where i_{n*} is induced by the inclusion map $i : A \hookrightarrow X$ and q_{n*} is induced by the quotient map $q : X \to X/A$.

2.4 – Relative Homology

Given a topological space X, let $A \subset X$ be a subspace. From the free abelian groups $C_n(X)$ and $C_n(A)$ we form the quotient group $C_n(X, A) = C_n(X)/C_n(A)$, and define $\bar{\partial}_n$ to be the homomorphism induced by

$$\partial_n : (C_n(X), C_n(A)) \to (C_{n-1}(X), C_{n-1}(A))$$

so that

$$\bar{\partial}_n(\sigma + C_n(A)) = \partial_n(\sigma) + C_{n-1}(A)$$

for each $\sigma \in C_n(X)$. It's easy to check that $\bar{\partial}_n \circ \bar{\partial}_{n+1} = 0$ holds for all n, and so a chain complex

$$\cdots \longrightarrow C_{n+1}(X, A) \xrightarrow{\bar{\partial}_{n+1}} C_n(X, A) \xrightarrow{\bar{\partial}_n} C_{n-1}(X, A) \longrightarrow \cdots$$

results. The homology groups associated with this chain complex, given as

$$H_n(X, A) = \frac{\operatorname{Ker}[\bar{\partial}_n : C_n(X, A) \to C_{n-1}(X, A)]}{\operatorname{Im}[\bar{\partial}_{n+1} : C_{n+1}(X, A) \to C_n(X, A)]},$$

are called **relative homology groups**.

If $A \subset X$ and $B \subset Y$, a continuous map $f : (X, A) \to (Y, B)$ induces homomorphisms

$$f_n : (C_n(X), C_n(A)) \to (C_n(Y), C_n(B))$$

in the usual fashion given in section 2.1, and these in turn induce homomorphisms $\bar{f}_n : C_n(X, A) \to C_n(Y, B)$ given by

$$\bar{f}_n(\sigma + C_n(A)) = f_n(\sigma) + C_n(B) = f \circ \sigma + C_n(B)$$

for each basis element $\sigma \in C_n(X)$. The maps $\bar{f}_n$ constitute a chain map $C(X, A) \to C(Y, B)$, and so by Proposition 2.1 they induce well-defined homomorphisms $\bar{f}_{n*} : H_n(X, A) \to H_n(Y, B)$ given by

$$\bar{f}_{n*}\big((\sigma + C_n(A)) + \bar{\partial}_{n+1}(C_{n+1}(X, A))\big) = \bar{f}_n(\sigma + C_n(A)) + \bar{\partial}_{n+1}(C_{n+1}(Y, B)).$$

Slightly more compactly we can write

$$\bar{f}_{n*}\big((\sigma + C_n(A)) + \operatorname{Im} \bar{\partial}^X_{n+1}\big) = (f \circ \sigma + C_n(B)) + \operatorname{Im} \bar{\partial}^Y_{n+1}.$$

Let $i : A \hookrightarrow X$ be the inclusion map. This map induces homomorphisms $i_n : C_n(A) \to C_n(X)$ given by $i_n(\sigma) = i \circ \sigma$ for each map $\sigma : \Delta^n \to A$. Also we introduce the homomorphism $j_n : C_n(X) \to C_n(X, A)$ given by $j_n(\sigma) = \sigma + C_n(A)$ for each $\sigma : \Delta^n \to X$. Clearly each i_n is injective and each j_n is surjective. If $\sigma \in \operatorname{Im} i_n$, then $\sigma \in C_n(A)$ and so $j_n(\sigma) = \sigma + C_n(A) = C_n(A)$ (the zero element of $C_n(X, A)$), which shows that $\sigma \in \operatorname{Ker} j_n$. If $\sigma \in \operatorname{Ker} j_n$, then $\sigma \in C_n(A)$ and it follows that $i_n(\sigma) = i \circ \sigma = \sigma$, which shows that $\sigma \in \operatorname{Im} i_n$. Thus, $\operatorname{Im} i_n = \operatorname{Ker} j_n$, and we conclude that

$$0 \longrightarrow C_n(A) \xrightarrow{i_n} C_n(X) \xrightarrow{j_n} C_n(X, A) \longrightarrow 0$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{n+1}(A) & \xrightarrow{\partial^A_{n+1}} & C_n(A) & \xrightarrow{\partial^A_n} & C_{n-1}(A) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle i_{n+1}} & & \Big\downarrow{\scriptstyle i_n} & & \Big\downarrow{\scriptstyle i_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial^X_{n+1}} & C_n(X) & \xrightarrow{\partial^X_n} & C_{n-1}(X) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle j_{n+1}} & & \Big\downarrow{\scriptstyle j_n} & & \Big\downarrow{\scriptstyle j_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1}(X,A) & \xrightarrow{\bar{\partial}_{n+1}} & C_n(X,A) & \xrightarrow{\bar{\partial}_n} & C_{n-1}(X,A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 4

is a short exact sequence. The claim is that $i_n : C(A) \to C(X)$ and $j_n : C(X) \to C(X,A)$ are chain maps, which is to say the diagram in Figure 4 is commutative and therefore a short exact sequence of chain complexes. We have, for any $\sigma : \Delta^n \to A$,

$$
\begin{aligned}
(i_{n-1} \circ \partial^A_n)(\sigma) &= i_{n-1}\left(\sum\nolimits_i (-1)^i \sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]}\right) = \sum\nolimits_i (-1)^i i_{n-1}\big(\sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]}\big) \\
&= \sum\nolimits_i (-1)^i \sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} = \partial^X_n(\sigma) = \partial^X_n(i_n(\sigma)) = (\partial^X_n \circ i_n)(\sigma);
\end{aligned}
$$

and for any $\sigma : \Delta^n \to X$,

$$
\begin{aligned}
(j_{n-1} \circ \partial^X_n)(\sigma) &= j_{n-1}\left(\sum\nolimits_i (-1)^i \sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]}\right) \\
&= \sum\nolimits_i (-1)^i \left(\sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} + C_{n-1}(A)\right) \\
&= \left(\sum\nolimits_i (-1)^i \sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]}\right) + C_{n-1}(A) = \partial_n(\sigma) + C_{n-1}(A) \\
&= \bar{\partial}_n(\sigma + C_n(A)) = \bar{\partial}_n(j_n(\sigma)) = (\bar{\partial}_n \circ j_n)(\sigma).
\end{aligned}
$$

We now define homomorphisms $\rho_n : H_n(X,A) \to H_{n-1}(A)$ in the same manner as the maps ρ_n in section 2.1, where $H_{n-1}(A) = \operatorname{Ker} \partial^A_{n-1} / \operatorname{Im} \partial^A_n$. Let

$$(\sigma + C_n(A)) + \operatorname{Im} \bar{\partial}_{n+1} \in H_n(X,A),$$

so $\sigma + C_n(A) \in \operatorname{Ker} \bar{\partial}_n \subset C_n(X,A)$. Since $\sigma \in C_n(X)$ we have $j_n(\sigma) = \sigma + C_n(A)$, and from $j_{n-1} \circ \partial^X_n = \bar{\partial}_n \circ j_n$ comes $\partial^X_n \sigma \in \operatorname{Ker} j_{n-1} = \operatorname{Im} i_{n-1}$. So there is some $\tau \in C_{n-1}(A)$ such that $i_{n-1}(\tau) = \partial^X_n \sigma$, which immediately implies $\tau = \partial^X_n \sigma$. Define

$$\rho_n\left((\sigma + C_n(A)) + \operatorname{Im} \bar{\partial}_{n+1}\right) = \partial^X_n \sigma + \operatorname{Im} \partial^A_n.$$

By Theorem 2.6 we obtain a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_{n*}} H_n(X) \xrightarrow{j_{n*}} H_n(X,A) \xrightarrow{\rho_n} H_{n-1}(A) \xrightarrow{i_{n-1*}} H_{n-1}(X) \longrightarrow \cdots$$

Proposition 2.18. *For $A \subset X$ and $B \subset Y$, let $f : (X, A) \to (Y, B)$ be a map such that $f : X \to Y$ and $f|_A : A \to B$ are each homotopy equivalences. Then for all n the maps $\bar{f}_{n*} : H_n(X, A) \to H_n(Y, B)$ are isomorphisms.*

Proof. By Corollary 2.16 the maps $f_{n*} : H_n(X) \to H_n(Y)$ and $f^A_{n*} := (f|_A)_{n*} : H_n(A) \to H_n(B)$ are isomorphisms for all n. Fix $n \geq 0$. The diagram

$$\begin{array}{ccccccccc}
H_n(A) & \xrightarrow{i^X_{n*}} & H_n(X) & \xrightarrow{j^X_{n*}} & H_n(X,A) & \xrightarrow{\rho^X_n} & H_{n-1}(A) & \xrightarrow{i^X_{n-1*}} & H_{n-1}(X) \\
\downarrow{\scriptstyle f^A_{n*}} & & \downarrow{\scriptstyle f_{n*}} & & \downarrow{\scriptstyle \bar{f}_{n*}} & & \downarrow{\scriptstyle f^A_{n-1*}} & & \downarrow{\scriptstyle f_{n-1*}} \\
H_n(B) & \xrightarrow{i^Y_{n*}} & H_n(Y) & \xrightarrow{j^Y_{n*}} & H_n(Y,B) & \xrightarrow{\rho^Y_n} & H_{n-1}(B) & \xrightarrow{i^Y_{n-1*}} & H_{n-1}(Y)
\end{array}$$

has exact rows, and the claim is it's also commutative. It's easily verified that the diagram

$$\begin{array}{ccccc}
C_n(A) & \xrightarrow{i^X_n} & C_n(X) & \xrightarrow{j^X_n} & C_n(X,A) \\
\downarrow{\scriptstyle f^A_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \bar{f}_n} \\
C_n(B) & \xrightarrow{i^Y_n} & C_n(Y) & \xrightarrow{j^Y_n} & C_n(Y,B)
\end{array}$$

is commutative. For instance

$$(\bar{f}_n \circ j^X_n)(\sigma) = \bar{f}_n(\sigma + C_n(A)) = f_n(\sigma) + C_n(B) = j^Y_n(f_n(\sigma)) = (j^Y_n \circ f_n)(\sigma)$$

clinches commutativity of the second square, and a similar routine will show commutativity of the first square. Now, $\{f_n\}$, $\{\bar{f}_n\}$, $\{j^X_n\}$ and $\{j^Y_n\}$ are chain maps that induce the well-defined homomorphisms found in the first diagram, and since the functor $*$ preserves commutativity, we conclude that the second and first squares in the first diagram are likewise commutative.

Next, for

$$(\sigma + C_n(A)) + \operatorname{Im} \bar{\partial}_{n+1} \in H_n(X, A)$$

we obtain

$$\begin{aligned}
(f^A_{n-1*} \circ \rho^X_n)((\sigma + C_n(A)) + \operatorname{Im} \bar{\partial^X}_{n+1}) &= f^A_{n-1*}(\partial^X_n \sigma + \operatorname{Im} \partial^A_n) \\
&= f^A_{n-1}(\partial^X_n(\sigma)) + \operatorname{Im} \partial^B_n = f_{n-1}(\partial^X_n(\sigma)) + \operatorname{Im} \partial^B_n \\
&= \partial^Y_n(f_n(\sigma)) + \operatorname{Im} \partial^B_n = \rho^Y_n((f_n(\sigma) + C_n(B)) + \operatorname{Im} \bar{\partial}^Y_n) \\
&= (\rho^Y_n \circ \bar{f}_{n*})((\sigma + C_n(A)) + \operatorname{Im} \bar{\partial^X}_{n+1}),
\end{aligned}$$

where the third equality holds since $\partial^X_n \sigma \in C_{n-1}(A) \subset C_{n-1}(X)$ and f^A_{n-1} is the restriction of f_{n-1} to $C_{n-1}(A)$; and the fourth equality holds since $\{f_n\}$ is a chain map $C(X) \to C(Y)$. This shows commutativity of the third square.

Therefore, by the Five-Lemma, the maps $\bar{f}_{n*}$ are isomorphisms. ∎

Example 2.19. Show a map $f : (\mathbb{D}^n, \mathbb{S}^{n-1}) \hookrightarrow (\mathbb{D}^n, \mathbb{D}^n - \{0\})$ cannot be a homotopy equivalence of pairs; that is, there's no map $g : (\mathbb{D}^n, \mathbb{D}^n - \{0\}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ such that $f \circ g$ and $g \circ f$ are homotopic to $\mathbb{1}_{\mathbb{D}^n}$ through maps $(\mathbb{D}^n, \mathbb{D}^n - \{0\}) \to (\mathbb{D}^n, \mathbb{D}^n - \{0\})$ and $(\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$, respectively.

Solution. Suppose there is such a map g. Let

$$\{\varphi_t : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})\}_{t \in I}$$

be a homotopy such that $\varphi_0 = \mathbb{1}_{\mathbb{D}^n}$ and $\varphi_1 = g \circ f$. Since $f(x) = x$ for all $x \in \mathbb{D}^n$, we have $\varphi_1(x) = g(f(x)) = g(x)$ so that $\varphi_1 = g$ and hence $g \simeq \mathbb{1}_{\mathbb{D}^n}$.

Let $i : \mathbb{S}^{n-1} \hookrightarrow \mathbb{D}^n$ be the inclusion map. Since $g : \mathbb{D}^n - \{0\} \to \mathbb{S}^{n-1}$ is continuous and 0 is in the closure of $\mathbb{D}^n - \{0\}$, we must have $g(0) \in \mathbb{S}^{n-1}$ so that $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ and hence $g \circ i = g|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$.

Now, for each $n \geq 1$ we have

$$\widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \xrightarrow{i_*} \widetilde{H}_{n-1}(\mathbb{D}^n) \xrightarrow{g_*} \widetilde{H}_{n-1}(\mathbb{S}^{n-1}),$$

where $\widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ and $\widetilde{H}_{n-1}(\mathbb{D}^n) = 0$ so that $g_* \circ i_* = 0$. On the other hand the maps $\{\varphi_t|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}\}_{t \in I}$ constitute a homotopy $g \circ i \simeq \mathbb{1}_{\mathbb{S}^{n-1}}$, so by Theorem 2.14 and Lemma 2.15 we obtain

$$g_* \circ i_* = (g \circ i)_* = \mathbb{1}_{\mathbb{S}^{n-1}*} = \mathbb{1}_{\widetilde{H}_{n-1}(\mathbb{S}^{n-1})},$$

and hence for $1 \in \widetilde{H}_{n-1}(\mathbb{S}^{n-1})$ we obtain $(g_* \circ i_*)(1) = 1$; that is, $g_* \circ i_* \neq 0$, which is a contradiction. ■

2.5 – The Excision Theorem

Let A and B be subspaces of X. The inclusion map $j : (B, A \cap B) \hookrightarrow (X, A)$ induces homomorphisms $j_n : C_n(B) \to C_n(X)$ given by $j_n(\beta) = j \circ \beta$ for a basis element $\beta : \Delta^n \to B$ of $C_n(B)$. Each j_n, in turn, induces a homomorphism of quotient groups

$$\bar{\jmath}_n : C_n(B, A \cap B) \to C_n(X, A)$$

given by

$$\bar{\jmath}_n(\beta + C_n(A \cap B)) = j_n(\beta) + C_n(A) = j \circ \beta + C_n(A) = \beta + C_n(A). \tag{4}$$

The Excision Theorem states circumstances when the maps $\bar{\jmath}_n$ will induce isomorphisms of relative homology groups.

Theorem 2.20 (Excision Theorem). *If $A, B \subset X$ such that $X = A^\circ \cup B^\circ$, then the inclusion $j : (B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $\bar{\jmath}_{n*} : H_n(B, A \cap B) \to H_n(X, A)$ for all n.*

The reason for the use of the term "excision" is perhaps made clearer by the following result.

Corollary 2.21. *Given subspaces $Z \subset A \subset X$ such that $\bar{Z} \subset A^\circ$, then the inclusion $j : (X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $\bar{\jmath}_{n*} : H_n(X - Z, A - Z) \to H_n(X, A)$ for all n.*

Proof. Let $B = X - Z$, and note that $A \cap B = A - Z$. It remains to show that $A^\circ \cup B^\circ = X$. Let $x \in X$, and suppose that $x \notin A^\circ$. Since $\bar{Z} \subset A^\circ$, we have $x \notin \bar{Z}$ and thus $x \in X - \bar{Z}$. Now, $X - \bar{Z}$ is open, so there exists some open set $\mathcal{O}$ such that

$$x \in \mathcal{O} \subset X - \bar{Z} \subset X - Z = B.$$

Hence $x \in B^\circ$ and we conclude that $X = A^\circ \cup B^\circ$.

Therefore, by the Excision Theorem, $j : (X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \cong H_n(X, A)$. ■

To prove the Excision Theorem there is a technical result that first needs to be established. Let $\mathcal{U} = \{U_k\}$ be a collection of subspaces of X such that $X = \bigcup_k U_k^\circ$, and let $C_n^{\mathcal{U}}(X)$ be the subgroup of $C_n(X)$ consisting of n-chains $\sum_i n_i \sigma_i$ such that for each i there is some k for which $\sigma_i(\Delta^n) \subset U_k$. It's easy to see that if $\alpha \in C_n^{\mathcal{U}}(X)$, then $\partial_n \alpha \in C_{n-1}^{\mathcal{U}}(X)$; thus, if we let $\partial_n^{\mathcal{U}}$ denote the restriction of ∂_n to $C_n^{\mathcal{U}}(X)$, and define $\iota_n : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ to be the inclusion map, we obtain a commutative diagram of chain complexes:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{n+1}^{\mathcal{U}}(X) & \xrightarrow{\partial_{n+1}^{\mathcal{U}}} & C_n^{\mathcal{U}}(X) & \xrightarrow{\partial_n^{\mathcal{U}}} & C_{n-1}^{\mathcal{U}}(X) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \iota_{n+1}} & & \downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \iota_{n-1}} & \\
\cdots \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow \cdots
\end{array}$$

As usual $H_n(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$, and now we also define $H_n^{\mathcal{U}}(X) = \operatorname{Ker}\partial_n^{\mathcal{U}} / \operatorname{Im}\partial_{n+1}^{\mathcal{U}}$. In accordance with the developments in section 2.1, the induced map $\iota_{n*} : H_n^{\mathcal{U}}(X) \to H_n(X)$ is given by

$$\iota_{n*}(\alpha + \operatorname{Im}\partial_{n+1}^{\mathcal{U}}) = \iota_n(\alpha) + \operatorname{Im}\partial_{n+1}$$

for each $\alpha \in \operatorname{Ker}\partial_n^{\mathcal{U}}$.

Lemma 2.22. *Let $\mathcal{U} = \{U_k\}$ be a collection of subspaces of X such that $X = \bigcup_k U_k^\circ$. If $\iota_n : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is the inclusion map, then the map $\iota_{n*} : H_n^{\mathcal{U}}(X) \to H_n(X)$ is an isomorphism.*

To prove the lemma it will be shown that the chain map $\iota_n : C^{\mathcal{U}}(X) \to C(X)$ is a chain homotopy equivalence, which means there is a chain map $\varphi_n : C(X) \to C^{\mathcal{U}}(X)$ such that $\{\iota_n \circ \varphi_n\}$ is chain homotopic to $\{\mathbb{1}_n\}$ (the identity maps on $C_n(X)$) and $\{\varphi_n \circ \iota_n\}$ is chain homotopic to $\{\mathbb{1}_n^{\mathcal{U}}\}$ (the identity maps on $C_n^{\mathcal{U}}(X)$). The result follows from Proposition 2.5.

Proof. We start with barycentric subdivision of simplices.[2] By definition of an n-simplex,

$$[v_0, \dots, v_n] = \left\{ \sum_{i=0}^n t_i v_i \ : \ \forall i\,(t_i \geq 0) \text{ and } \sum_{i=0}^n t_i = 1 \right\} \subset \mathbb{R}^m$$

for some $m \geq n+1$, where the set of vectors $\{v_1 - v_0, \dots, v_n - v_0\}$ is linearly independent. The **barycenter** of $[v_0, \dots, v_n]$ is the point $b = \sum_i \frac{1}{n+1} v_i$, and the **barycentric subdivision** of $[v_0, \dots, v_n]$ is the subdivision of $[v_0, \dots, v_n]$ into smaller n-simplices of the form $[b, w_0, \dots, w_{n-1}]$ which we now specify inductively. When $n = 0$, the barycentric subdivision of $[v_0]$ is defined to be $[v_0]$ itself. For $n = 1$ we decompose $[v_0, v_1]$ with barycenter $b = \frac{1}{2}v_0 + \frac{1}{2}v_1$ into $[b, v_0]$ and $[b, v_1]$. For $n = 2$, let b_{ij} be the barycenter of the face $[v_i, v_j]$ of $[v_0, v_1, v_2]$, and decompose $[v_0, v_1, v_2]$ into $[b, b_{01}, v_0]$, $[b, b_{01}, v_1]$, $[b, b_{02}, v_0]$, $[b, b_{02}, v_2]$, $[b, b_{12}, v_1]$, and $[b, b_{12}, v_2]$. For $n = 3$, if b_{ijk} denotes the barycenter of the face $[v_i, v_j, v_k]$ of $[v_0, v_1, v_2, v_3]$, then a couple of the 24 members of the decomposition are $[b, b_{012}, b_{01}, v_0]$ and $[b, b_{012}, b_{01}, v_1]$, which we could write as $[b_{0123}, b_{012}, b_{01}, b_0]$ and $[b_{0123}, b_{012}, b_{01}, b_1]$ if we wished to employ our notation to its fullest extent (the barycenter b_i of $[v_i]$ being, of course, v_i itself). See Figure 5. In general the barycentric subdivision of $[v_0, \dots, v_n]$ is the collection $\mathcal{B}[v_0, \dots, v_n]$ of n-simplices

$$\left\{ \left[b, b_{\ell_0^{n-1} \cdots \ell_{n-1}^{n-1}}, b_{\ell_0^{n-2} \cdots \ell_{n-2}^{n-2}}, \dots, b_{\ell_0^1 \ell_1^1}, v_{\ell_0^0}\right] \ : \ \ell_{i-1}^k < \ell_i^k \ \ \& \ \ \{\ell_0^{k-1}, \dots, \ell_{k-1}^{k-1}\} \subset \{\ell_0^k, \dots, \ell_k^k\} \right\},$$

where of course $\ell_i^k \in \{0, \dots, n\}$, and

$$b_{\ell_0^k \cdots \ell_k^k} = \sum_{i=0}^k \frac{1}{k+1} v_{\ell_i^k}.$$

is the barycenter of the k-dimensional face $[v_{\ell_0^k}, \dots, v_{\ell_k^k}]$ of $[v_0, \dots, v_n]$ for $1 \leq k \leq n$. Simply put, a member of $\mathcal{B}[v_0, \dots, v_n]$ has as its vertices the barycenter of $[v_0, \dots, v_n]$, the barycenter of an $(n-1)$-dimensional face F of $[v_0, \dots, v_n]$, the barycenter of an $(n-2)$-dimensional face of F, and so on down the dimensions to conclude with a point that is a vertex of F.

[2] As with many results hereabouts, this proof is modeled along the lines of the one found in Allen Hatcher's "Algebraic Topology."

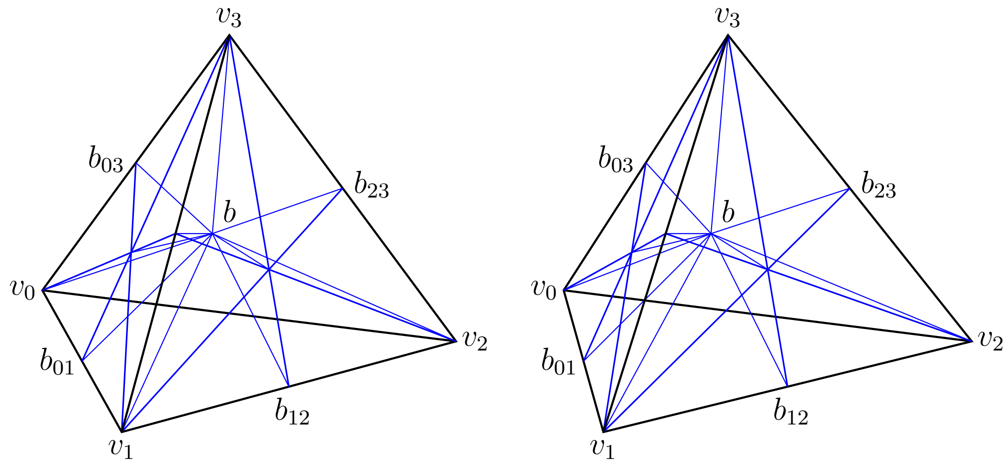


FIGURE 5. Part of the barycentric subdivision of a 3-simplex $[v_0, v_1, v_2, v_3]$.

Letting $|p-q|$ denote the Euclidean distance between points p and q in $\mathbb{R}^m$, we define the **diameter** of $[v_0, \ldots, v_n]$ to be

$$\operatorname{diam}[v_0, \ldots, v_n] = \max\{|p-q| \ : \ p, q \in [v_0, \ldots, v_n]\},$$

which is a real value that is attained for some $\hat{p}, \hat{q} \in [v_0, \ldots, v_n]$ since the set is compact. If p and $\sum_i t_i v_i$ are two points in $[v_0, \ldots, v_n]$, then

$$\begin{aligned} \left|p - \sum\nolimits_i t_i v_i\right| &= \left|\sum\nolimits_i t_i p - \sum\nolimits_i t_i v_i\right| = \left|\sum\nolimits_i t_i (p - v_i)\right| \\ &\leq \sum\nolimits_i t_i |p - v_i| = \sum\nolimits_i t_i \max_{0 \leq i \leq n} |p - v_i| \\ &= \max_{0 \leq i \leq n} |p - v_i| = |p - v_{i_0}| \end{aligned}$$

for some $i_0 \in \{0, \ldots, n\}$. Letting $p = \sum_i s_i v_i$ and repeating the process, we find that

$$|p - v_{i_0}| = \left|v_{i_0} - \sum\nolimits_i s_i v_i\right| \leq |v_{i_0} - v_{i_1}|$$

for some i_1. Hence for any $p, q \in [v_0, \ldots, v_n]$ we find that $|p - q| \leq |v_i - v_j|$ for some $i, j \in \{0, \ldots, n\}$, and therefore $\operatorname{diam}[v_0, \ldots, v_n] = \max_{0 \leq i,j \leq n} |v_i - v_j|$.

What we will want to show is that

$$\max\{\operatorname{diam}(W) \ : \ W \in \mathcal{B}[v_0, \ldots, v_n]\} \leq \frac{n}{n+1} \operatorname{diam}[v_0, \ldots, v_n]. \tag{5}$$

This is trivially true when $n = 0$ since the only member of $\mathcal{B}[v_0]$ is $[v_0]$ itself, and $\operatorname{diam}[v_0] = 0$. For $n = 1$ we have $\mathcal{B}[v_0, v_1] = \{[b, v_0], [b, v_1]\}$, with

$$\operatorname{diam}[b, v_0] = |b - v_0| = \left|\tfrac{1}{2} v_0 + \tfrac{1}{2} v_1 - v_0\right| = \tfrac{1}{2}|v_1 - v_0| = \tfrac{1}{2} \operatorname{diam}[v_0, v_1]$$

and similarly $\operatorname{diam}[b, v_1] = \frac{1}{2} \operatorname{diam}[v_0, v_1]$.

Employing induction, let n be arbitrary and suppose (5) holds for every n-simplex $[v_0, \ldots, v_n]$. Let $V = [v_0, \ldots, v_{n+1}]$ be any $(n+1)$-simplex, and let $\mathcal{B}V$ be its barycentric subdivision. Let

$$W = [b, w_0, \ldots, w_n] \in \mathcal{B}V,$$

b being the barycenter of V. First suppose that w_i and w_j are vertices of W such that $w_i, w_j \neq b$. Then the points w_i and w_j must lie on the n-dimensional face F of V that has w_0 as its barycenter, and since $[w_0, \ldots, w_n]$ is a member of $\mathcal{B}F$ and F is an n-simplex, by (5) we obtain

$$|w_i - w_j| \leq \operatorname{diam}([w_0, \ldots, w_n]) \leq \max\{\operatorname{diam}(U) \,:\, U \in \mathcal{B}F\} \leq \frac{n}{n+1} \operatorname{diam}(F).$$

Of course, $F \subset V$ implies that $\operatorname{diam}(F) \leq \operatorname{diam}(V)$, so finally

$$|w_i - w_j| \leq \frac{n}{n+1} \operatorname{diam}(V) \leq \frac{n+1}{n+2} \operatorname{diam}(V).$$

Next, take vertices w_j and b of W. Since $w_j, b \in V$ we have

$$|b - w_j| \leq \max_{0 \leq i \leq n+1} |b - v_i| = |b - v_k|$$

for some k. Let b_k be the barycenter of $[v_0, \ldots, \hat{v}_k, \ldots, v_{n+1}]$, so

$$b_k = \sum_{i=0, i\neq k}^{n+1} \frac{1}{n+1} v_i.$$

Now, since

$$\frac{1}{n+2} v_k + \frac{n+1}{n+2} b_k = \frac{1}{n+2} v_k + \sum_{i=0, i\neq k}^{n+1} \frac{1}{n+2} v_i = b,$$

we obtain

$$\begin{aligned} |v_k - b| &= \left| v_k - \left(\frac{1}{n+2} v_k + \frac{n+1}{n+2} b_k \right) \right| = \left| \frac{n+1}{n+2} v_k - \frac{n+1}{n+2} b_k \right| \\ &= \frac{n+1}{n+2} |v_k - b_k| \leq \frac{n+1}{n+2} \operatorname{diam}(V) \end{aligned}$$

and therefore

$$|b - w_j| \leq \frac{n+1}{n+2} \operatorname{diam}(V).$$

Combining the two cases analyzed above leads to the general result

$$\operatorname{diam}(W) = \max_{p,q \in \{b, w_0, \ldots, w_n\}} |p - q| \leq \frac{n+1}{n+2} \operatorname{diam}(V),$$

which finally implies

$$\max\{\operatorname{diam}(W) \,:\, W \in \mathcal{B}V\} \leq \frac{n+1}{n+2} \operatorname{diam}(V).$$

We move on now to the next stage of the proof. Let $Y \subset \mathbb{R}^m$ be a convex set, and let $C(Y)$ be the singular chain complex

$$\cdots \longrightarrow C_{n+1}(Y) \xrightarrow{\partial_{n+1}} C_n(Y) \xrightarrow{\partial_n} C_{n-1}(Y) \xrightarrow{\partial_{n-1}} \cdots$$

A linear transformation $\ell : \Delta^n \to Y$ can be uniquely determined by defining, for each $0 \leq i \leq n$, some $w_i \in Y$ for which $\ell(u_i) = w_i$; indeed, since for each $q \in \Delta^n$ there exist nonnegative scalars t_i such that $\sum_i t_i = 1$ and $q = \sum_i t_i u_i$, we obtain $\ell(q) = \ell(\sum_i t_i u_i) =$

$\sum_i t_i \ell(u_i) = \sum_i t_i w_i \in Y$ by Proposition 2.9, and therefore $\mathrm{Im}(\ell) \subset Y$ as required. Now, if $\mathcal{L}(\Delta^n, Y)$ is the collection of all linear transformations $\ell : \Delta^n \to Y$, then we can define $L_n(Y)$ to be the subgroup of $C_n(Y)$ generated by $\mathcal{L}(\Delta^n, Y)$. The boundary maps $\partial_n : C_n(Y) \to C_{n-1}(Y)$ then give rise to a chain complex $L(Y)$

$$\cdots \longrightarrow L_{n+1}(Y) \xrightarrow{\partial_{n+1}} L_n(Y) \xrightarrow{\partial_n} L_{n-1}(Y) \longrightarrow \cdots$$

which is called a **subcomplex** of $C(Y)$. In what follows it will be convenient to denote a map $\ell \in \mathcal{L}(\Delta^n, Y)$ given by $\ell(u_i) = w_i$ by the symbol $\lfloor w_0, \ldots, w_n \rfloor$.[3] It will also be convenient to extend $L(Y)$ to dimension -1 by letting $L_{-1}(Y)$ be the free group generated by the unique map $[\varnothing] \mapsto Y$, where $[\varnothing]$ is taken to be the "empty simplex" that has no vertices; we can denote this map by $\lfloor \varnothing \rfloor$, so $L_{-1}(Y) \cong \langle \lfloor \varnothing \rfloor \rangle \cong \mathbb{Z}$.

For any point $y \in Y$ define a homomorphism $\mathbb{y}_n : L_n(Y) \to L_{n+1}(Y)$ by

$$\mathbb{y}_n(\lfloor w_0, \ldots, w_n \rfloor) = \lfloor y, w_0, \ldots, w_n \rfloor,$$

where $\lfloor y, w_0, \ldots, w_n \rfloor : \Delta^{n+1} \to Y$ is given by $\lfloor y, w_0, \ldots, w_n \rfloor(u_i) = w_{i-1}$ for $1 \leq i \leq n+1$, and $\lfloor y, w_0, \ldots, w_n \rfloor(u_0) = y$. Now,

$$\begin{aligned}
\partial_{n+1}(\mathbb{y}_n(\lfloor w_0, \ldots, w_n \rfloor)) &= \sum_{i=0}^{n+1} (-1)^i \lfloor y, w_0, \ldots, w_n \rfloor|_{[u_0, \ldots, \hat{u}_i, \ldots, u_{n+1}]} \circ \delta^n \\
&= \lfloor y, w_0, \ldots, w_n \rfloor|_{[u_1, \ldots, u_{n+1}]} \circ \delta^n + \sum_{i=1}^{n+1} (-1)^i \lfloor y, w_0, \ldots, w_n \rfloor|_{[u_0, \ldots, \hat{u}_i, \ldots, u_{n+1}]} \circ \delta^n \\
&= \lfloor w_0, \ldots, w_n \rfloor + \sum_{i=1}^{n+1} (-1)^i \lfloor y, w_0, \ldots, \hat{w}_{i-1}, \ldots, w_n \rfloor \\
&= \lfloor w_0, \ldots, w_n \rfloor + \sum_{i=0}^{n} (-1)^{i+1} \lfloor y, w_0, \ldots, \hat{w}_i, \ldots, w_n \rfloor \\
&= \lfloor w_0, \ldots, w_n \rfloor - \sum_{i=0}^{n} (-1)^i \mathbb{y}_{n-1}(\lfloor w_0, \ldots, \hat{w}_i, \ldots, w_n \rfloor) \\
&= \lfloor w_0, \ldots, w_n \rfloor - \sum_{i=0}^{n} (-1)^i \mathbb{y}_{n-1}(\lfloor w_0, \ldots, w_n \rfloor|_{[u_0, \ldots, \hat{u}_i, \ldots, u_n]} \circ \delta^{n-1}) \\
&= \lfloor w_0, \ldots, w_n \rfloor - \mathbb{y}_{n-1}(\partial_n(\lfloor w_0, \ldots, w_n \rfloor)),
\end{aligned}$$

where, for instance, the sixth equality is justified as follows: $\lfloor w_0, \ldots, \hat{w}_i, \ldots, w_n \rfloor$ takes $u_k \in \Delta^{n-1}$ and returns w_k for $k < i$, and w_{k+1} for $k \geq i$, while $\lfloor w_0, \ldots, w_n \rfloor|_{[u_0, \ldots, \hat{u}_i, \ldots, u_n]} \circ \delta^{n-1}$ maps as

$$u_k \in \Delta^{n-1} \mapsto u_k \in \Delta^n \mapsto w_k$$

for $k < i$, and

$$u_k \in \Delta^{n-1} \mapsto u_{k+1} \in \Delta^n \mapsto w_{k+1}$$

for $k \geq i$.

[3] It seems to me highly inadvisable to denote ℓ by $[w_0, \ldots, w_n]$, since this symbol is already "taken" and might lead one to wrongly believe that $\mathrm{Im}(\ell)$ must necessarily be an n-simplex).

Since $\lfloor w_0, \dots, w_n \rfloor$ is a basis element for $L_n(Y)$, the preceding shows that

$$(\partial_{n+1} \circ \mathrm{y}_n)(\sigma) = \sigma - (\mathrm{y}_{n-1} \circ \partial_n)(\sigma) = (\mathbb{1}_n - \mathrm{y}_{n-1} \circ \partial_n)(\sigma). \tag{6}$$

for any $\sigma \in L_n(Y)$, with $\mathbb{1}_n : L_n(Y) \to L_n(Y)$ being the identity map. Therefore we have

$$\partial_{n+1} \circ \mathrm{y}_n + \mathrm{y}_{n-1} \circ \partial_n = \mathbb{1}_n.$$

We now define a family of homomorphisms $S_n : L_n(Y) \to L_n(Y)$ inductively. To start, we have $S_{-1} : L_{-1}(Y) \to L_{-1}(Y)$ given by $S_{-1}(\lfloor \varnothing \rfloor) = \lfloor \varnothing \rfloor$. For any k, given

$$\ell = \lfloor w_0, \dots, w_k \rfloor \in \mathcal{L}(\Delta^k, Y),$$

let b_k be the barycenter of Δ^k, set $\ell(b_k) = p$, and define the homomorphism $\mathrm{p}_k : L_k(Y) \to L_{k+1}(Y)$ by

$$\mathrm{p}_k(\ell) = \lfloor p, w_0, \dots, w_k \rfloor.$$

Then we define S_n for $n \geq 0$ to be given by

$$S_n(\ell) = (\mathrm{p}_{n-1} \circ S_{n-1} \circ \partial_n)(\ell).$$

In particular

$$S_0(\lfloor p \rfloor) = \mathrm{p}_{-1}(S_{-1}(\partial_0(\lfloor p \rfloor))) = \mathrm{p}_{-1}(S_{-1}(\lfloor \varnothing \rfloor)) = \mathrm{p}_{-1}(\lfloor \varnothing \rfloor) = \lfloor p \rfloor$$

shows that S_0 is the identity on $L_0(Y)$.

If $\ell = \lfloor y_0, y_1 \rfloor : \Delta^1 \to Y$ is in $L_1(Y)$ and has image equalling the simplicial 1-simplex $[y_0, y_1]$, so that $y_0 \neq y_1$, then $\ell(b_1) = p$ with $p \neq y_0, y_1$, and

$$\begin{aligned} S_1(\ell) &= (\mathrm{p}_0 \circ S_0 \circ \partial_1)(\ell) = \mathrm{p}_0(S_0(\ell|_{[u_1]} - \ell|_{[u_0]})) \\ &= \mathrm{p}_0(\ell|_{[u_1]} - \ell|_{[u_0]}) = \mathrm{p}_0(\lfloor y_1 \rfloor - \lfloor y_0 \rfloor) \\ &= \lfloor p, y_1 \rfloor - \lfloor p, y_0 \rfloor \end{aligned}$$

shows that $S_1(\ell)$ equals a linear combination of singular 1-simplices with images $[p, y_0]$ and $[p, y_1]$, which are elements of $\mathcal{B}[y_0, y_1]$.

Proceeding with an induction argument, let $n \geq 1$ and suppose that for any $\ell = \lfloor y_0, \dots, y_n \rfloor \in L_n(Y)$ with $\mathrm{Im}(\ell) = [y_0, \dots, y_n]$, $S_n(\ell)$ is a linear combination of singular n-simplices with images that are elements of $\mathcal{B}[y_0, \dots, y_n]$. For any

$$\ell = \lfloor y_0, \dots, y_{n+1} \rfloor \in L_{n+1}(Y)$$

with

$$\mathrm{Im}(\ell) = [y_0, \dots, y_{n+1}] := V$$

and $\ell(b_{n+1}) = p$,

$$\begin{aligned} S_{n+1}(\ell) &= (\mathrm{p}_n \circ S_n \circ \partial_{n+1})(\lfloor y_0, \dots, y_{n+1} \rfloor) \\ &= \mathrm{p}_n(S_n(\partial_{n+1} \lfloor y_0, \dots, y_{n+1} \rfloor)) \\ &= \sum_{i=0}^{n+1} (-1)^i \mathrm{p}_n(S_n(\lfloor y_0, \dots, \hat{y}_i, \dots, y_{n+1} \rfloor)) \end{aligned}$$

For each i,

$$\mathrm{Im}(\lfloor y_0,\dots,\hat{y}_i,\dots,y_{n+1}\rfloor := \ell_i) = [y_0,\dots,\hat{y}_i,\dots,y_{n+1}] := V_i,$$

and so by hypothesis $S_n(\ell_i)$ is a linear combination of singular n-simplices with images that are elements of $\mathcal{B}V_i$. Let $W \in \mathcal{B}V_i$ be any one of these images. Then $W = [w_0,\dots,w_n]$, where w_0 is the barycenter of V_i, w_1 is the barycenter of an $(n-1)$-dimensional face F_{n-1} of V_i, w_2 is the barycenter of an $(n-2)$-dimensional face F_{n-2} of F_{n-1}, and so on until we arrive at w_n, which will be a vertex of F_{n-1}. Now

$$\mathbb{p}_n(\lfloor w_0,\dots,w_n\rfloor) = \lfloor p, w_0,\dots,w_n\rfloor,$$

which is a basis element of $L_{n+1}(Y)$ with image $[p,w_0,\dots,w_n]$, and since p is the barycenter of V, w_0 is the barycenter of the n-dimensional face $F_n = V_i$ of V, and all lower-dimensioned faces $F_{n-1},\dots,F_0$ of V_i are faces of V, it is clear that $[p,w_0,\dots,w_n] \in \mathcal{B}V$. Therefore $S_{n+1}(\ell)$ is a linear combination of singular $(n+1)$-simplices with images that are elements of $\mathcal{B}V$.

To show that the maps S_n define a chain map $L(Y) \to L(Y)$, the commutativity property $S_{n-1}\circ\partial_n = \partial_n \circ S_n$ must be verified. The base case (when $n=-1$) is clear:

$$(S_{-1}\circ\partial_0)(\lfloor p\rfloor) = S_{-1}(\lfloor\varnothing\rfloor) = \lfloor\varnothing\rfloor = \partial_0(\lfloor p\rfloor) = (\partial_0\circ S_0)(\lfloor p\rfloor).$$

For arbitary n suppose that $\partial_n \circ S_n = S_{n-1}\circ\partial_n$ is true. Let $\ell \in \mathcal{L}(\Delta^{n+1},Y)$ with $\ell(b)=p$. Noting that

$$\partial_{n+1}\circ\mathbb{p}_n = \mathbb{1}_n - \mathbb{p}_{n-1}\circ\partial_n,$$

we obtain

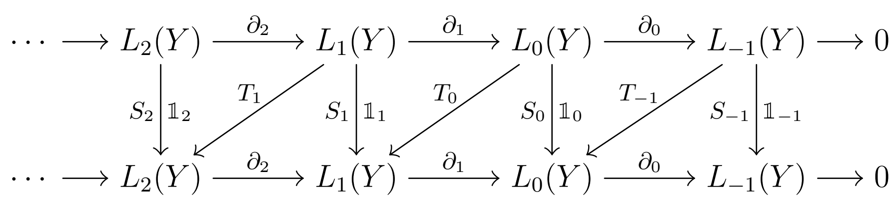
$$\begin{aligned}(\partial_{n+1}\circ S_{n+1})(\ell) &= (\partial_{n+1}\circ\mathbb{p}_n\circ S_n\circ\partial_{n+1})(\ell)\\ &= ((\mathbb{1}_n - \mathbb{p}_{n-1}\circ\partial_n)\circ(S_n\circ\partial_{n+1}))(\ell)\\ &= (S_n\circ\partial_{n+1})(\ell) - (\mathbb{p}_{n-1}\circ\partial_n\circ S_n\circ\partial_{n+1})(\ell)\\ &= (S_n\circ\partial_{n+1})(\ell) - (\mathbb{p}_{n-1}\circ S_{n-1}\circ\partial_n\circ\partial_{n+1})(\ell)\\ &= (S_n\circ\partial_{n+1})(\ell),\end{aligned}$$

since $\partial_n\circ\partial_{n+1}\equiv 0$.

Now, define homomorphisms $T_n : L_n(Y)\to L_{n+1}(Y)$ inductively as follows. Let $T_{-1}(\lfloor\varnothing\rfloor) = 0$, and for $n \geq 0$ set

$$T_n(\ell) = \mathbb{p}_n(\ell - T_{n-1}(\partial_n(\ell)))$$

for each $\ell \in \mathcal{L}(\Delta^n, Y)$, with $\mathbb{p}_n$ defined as above. Referring to the diagram



it will be shown by induction that the collection of maps $\{T_n\}_{n=-1}^{\infty}$ is a chain homotopy between the chain maps $\{S_n\}_{n=-1}^{\infty}$ and $\{\mathbb{1}_n\}_{n=-1}^{\infty}$, which entails demonstrating that

$$\mathbb{1}_n - S_n = \partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n \tag{7}$$

for $n \geq -1$. The base case is easy to secure since $T_{-1} \equiv 0$ and $T_{-2} \equiv 0$ (by definition), and $S_{-1} = \mathbb{1}_{-1}$. For arbitrary n suppose that

$$\mathbb{1}_n - S_n = \partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n$$

holds. Then, for $\ell \in \mathcal{L}(\Delta^{n+1}, Y)$,

$$\begin{aligned}
(\partial_{n+2} \circ T_{n+1})(\ell) &= \partial_{n+2}(\mathbb{p}_{n+1}(\ell - T_n(\partial_{n+1}(\ell)))) \\
&= (\partial_{n+2} \circ \mathbb{p}_{n+1})(\ell - T_n(\partial_{n+1}(\ell))) \\
&= (\mathbb{1}_{n+1} - \mathbb{p}_n \circ \partial_{n+1})(\ell - T_n(\partial_{n+1}(\ell))) \\
&= (\mathbb{1}_{n+1}(\ell - T_n(\partial_{n+1}(\ell))) - (\mathbb{p}_n \circ \partial_{n+1})(\ell - T_n(\partial_{n+1}(\ell))) \\
&= (\mathbb{1}_{n+1} - T_n \circ \partial_{n+1} - \mathbb{p}_n \circ \partial_{n+1} + \mathbb{p}_n \circ (\partial_{n+1} \circ T_n) \circ \partial_{n+1})(\ell) \\
&= (\mathbb{1}_{n+1} - T_n \circ \partial_{n+1} - \mathbb{p}_n \circ \partial_{n+1} + \mathbb{p}_n \circ (\mathbb{1}_n - S_n - T_{n-1} \circ \partial_n) \circ \partial_{n+1})(\ell) \\
&= (\mathbb{1}_{n+1} - T_n \circ \partial_{n+1} - \mathbb{p}_n \circ \partial_{n+1} + \mathbb{p}_n \circ \partial_{n+1} - \mathbb{p}_n \circ S_n \circ \partial_{n+1})(\ell) \\
&= (\mathbb{1}_{n+1} - T_n \circ \partial_{n+1} - S_{n+1})(\ell),
\end{aligned}$$

where again we make use of $\partial_n \circ \partial_{n+1} \equiv 0$, and therefore

$$\mathbb{1}_{n+1} - S_{n+1} = \partial_{n+2} \circ T_{n+1} + T_n \circ \partial_{n+1}.$$

Since $T_{-1} \equiv 0$ on $L_{-1}(Y) \cong \mathbb{Z}$, we can replace $L_{-1}(Y)$ with 0 and obtain a truncated diagram in which $\{T_n\}_{n=0}^{\infty}$ is a chain homotopy between $\{S_n\}_{n=0}^{\infty}$ and $\{\mathbb{1}_n\}_{n=0}^{\infty}$.

Now begins the third part of the proof. Fix $n \geq 0$. For each generator $\sigma : \Delta^n \to X$ of $C_n(X)$ there are induced homomorphisms $\bar{\sigma}_k^n : C_k(\Delta^n) \to C_k(X)$ given by $\bar{\sigma}_k^n(f) = \sigma \circ f$ for each integer k and map $f : \Delta^k \to \Delta^n$. Also there are the maps $S_k^n : L_k(\Delta^n) \to L_k(\Delta^n)$ that operate in the manner discussed above and define a chain map $L(\Delta^n) \to L(\Delta^n)$. Finally there are the identity maps $\mathbb{1}_{\Delta^k} = \lfloor u_0, \ldots, u_k \rfloor : \Delta^k \to \Delta^k$. Using all these maps, we define a new homomorphism $\bar{S}_n : C_n(X) \to C_n(X)$ by $\bar{S}_n(\sigma) = \bar{\sigma}_n^n(S_n^n(\mathbb{1}_{\Delta^n}))$. To show the maps $\bar{S}_n$ define a chain map $C(X) \to C(X)$, we verify the commutativity property $\bar{S}_{n-1} \circ \partial_n = \partial_n \circ \bar{S}_n$ for each n. In doing so, we use the symbol δ_i^{n-1} to denote the canonical linear homeomorphism $\Delta^{n-1} \to [u_0, \ldots, \hat{u}_i, \ldots, u_n]$. Thus,

$$\begin{aligned}
(\partial_n \circ \bar{S}_n)(\sigma) = \partial_n(\bar{\sigma}_n^n(S_n^n(\mathbb{1}_{\Delta^n}))) &= (\partial_n \circ \bar{\sigma}_n^n)(S_n^n(\mathbb{1}_{\Delta^n})) \\
&= (\bar{\sigma}_{n-1}^n \circ \partial_n)(S_n^n(\mathbb{1}_{\Delta^n})) = (\bar{\sigma}_{n-1}^n \circ (\partial_n \circ S_n^n))(\mathbb{1}_{\Delta^n}) \\
&= (\bar{\sigma}_{n-1}^n \circ (S_{n-1}^n \circ \partial_n))(\mathbb{1}_{\Delta^n}) = (\bar{\sigma}_{n-1}^n \circ S_{n-1}^n)(\partial_n \mathbb{1}_{\Delta^n})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{n}(-1)^i(\bar{\sigma}^n_{n-1} \circ S^n_{n-1})(\mathbb{1}_{\Delta^n}|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} \circ \delta_i^{n-1}) \\
&= \sum_{i=0}^{n}(-1)^i \left(\overline{(\sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} \circ \delta_i^{n-1})}^{n-1}_{n-1} \circ S^{n-1}_{n-1}\right)(\mathbb{1}_{\Delta^{n-1}}) \\
&= \sum_{i=0}^{n}(-1)^i \bar{S}_{n-1}(\sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} \circ \delta_i^{n-1}) \\
&= \bar{S}_{n-1}\left(\sum_{i=0}^{n}(-1)^i \sigma|_{[u_0,\ldots,\hat{u}_i,\ldots,u_n]} \circ \delta_i^{n-1}\right) \\
&= \bar{S}_{n-1}(\partial_n\sigma) = (\bar{S}_{n-1} \circ \partial_n)(\sigma),
\end{aligned}$$

where it must be admitted that there is a "leap" in going from the fourth to the fifth row that would require some work to justify.

Next, define $\bar{T}_n : C_n(X) \to C_{n+1}(X)$ by

$$\bar{T}_n(\sigma) = \bar{\sigma}^n_{n+1}(T^n_n(\mathbb{1}_{\Delta^n})),$$

where for each k the map $T^n_k : L_k(\Delta^n) \to L_{k+1}(\Delta^n)$ operates as described above. To be shown is that $\{\bar{T}_n\}_{n=0}^\infty$ is a chain homotopy between the chain maps $\{\bar{S}_n\}_{n=0}^\infty$ and $\{\mathbb{1}_n : C_n(X) \to C_n(X)\}_{n=0}^\infty$ (the identity maps on $C(X)$). Thus it must be shown that

$$\mathbb{1}_n - \bar{S}_n = \partial_{n+1} \circ \bar{T}_n + \bar{T}_{n-1} \circ \partial_n \tag{8}$$

for all $n \geq 0$, which will be done inductively.

Let $n = 0$. Since $S^0_0 : L_0(\Delta^0) \to L_0(\Delta^0)$ is the identity map, we have for any generator $\sigma : \Delta^0 \to X$ of $C_0(X)$,

$$\bar{S}_0(\sigma) = \bar{\sigma}^0_0(S^0_0(\mathbb{1}_{\Delta^0})) = \bar{\sigma}^0_0(\mathbb{1}_{\Delta^0}) = \sigma \circ \mathbb{1}_{\Delta^0} = \sigma,$$

so that $\bar{S}_0 = \mathbb{1}_0 : C_0(X) \to C_0(X)$ and we obtain $\mathbb{1}_0 - \bar{S}_0 = 0$. Since $T^k_{-1} \equiv 0$ for any k, we have $\bar{T}_{-1} \equiv 0$ and so

$$\begin{aligned}
(\partial_1 \circ \bar{T}_0 + \bar{T}_{-1} \circ \partial_0)(\sigma) &= \partial_1(\bar{T}_0(\sigma)) = \partial_1(\bar{\sigma}^0_1(T^0_0(\mathbb{1}_{\Delta^0}))) \\
&= (\partial_1 \circ \bar{\sigma}^0_1)(\mathbb{p}_0(\mathbb{1}_{\Delta^0} - T^0_{-1}(\partial_0(\mathbb{1}_{\Delta^0})))) \\
&= (\partial_1 \circ \bar{\sigma}^0_1)(\mathbb{p}_0(\mathbb{1}_{\Delta^0})) = (\partial_1 \circ \bar{\sigma}^0_1)(\mathbb{p}_0(\lfloor u_0 \rfloor)) \\
&= (\partial_1 \circ \bar{\sigma}^0_1)(\lfloor u_0, u_0 \rfloor) = \partial_1(\sigma \circ \lfloor u_0, u_0 \rfloor) = 0,
\end{aligned}$$

where $\mathbb{p}(\lfloor u_0 \rfloor) = \lfloor u_0, u_0 \rfloor$ since the barycenter of Δ^0 is u_0 and $\mathbb{1}_{\Delta^0}(u_0) = u_0$, and the last equality follows from the observation that $\sigma \circ \lfloor u_0, u_0 \rfloor : \Delta^1 \to \Delta^0 \to X$ is a constant function. It has now been established that

$$\mathbb{1}_0 - \bar{S}_0 = \partial_1 \circ \bar{T}_0 + \bar{T}_{-1} \circ \partial_0.$$

For the inductive step, let $n \geq 0$ be arbitrary and suppose that equation (8) holds. Take $\mathbb{1}^{\triangle}_{n+1}$ to be the identity map on $C_{n+1}(\Delta^{n+1})$ so that $\mathbb{1}^{\triangle}_{n+1}(\mathbb{1}_{\Delta^{n+1}}) = \mathbb{1}_{\Delta^{n+1}}$. Referring to the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{n+2}(\Delta^{n+1}) & \xrightarrow{\partial^{\triangle}_{n+2}} & C_{n+1}(\Delta^{n+1}) & \xrightarrow{\partial^{\triangle}_{n+1}} & C_n(\Delta^{n+1}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \bar{\sigma}^{n+1}_{n+2}} & & \downarrow{\scriptstyle \bar{\sigma}^{n+1}_{n+1}} & & \downarrow{\scriptstyle \bar{\sigma}^{n+1}_{n}} & \\
\cdots \longrightarrow & C_{n+2}(X) & \xrightarrow{\partial_{n+2}} & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \longrightarrow \cdots
\end{array}$$

we obtain, for any basis element $\sigma : \Delta^{n+1} \to X$ of $C_{n+1}(X)$,

$$\begin{aligned}
(\partial_{n+2} \circ \bar{T}_{n+1})(\sigma) &= \partial_{n+2}(\bar{\sigma}^{n+1}_{n+2}(T^{n+1}_{n+1}(\mathbb{1}_{\Delta^{n+1}}))) = (\partial_{n+2} \circ \bar{\sigma}^{n+1}_{n+2})(T^{n+1}_{n+1}(\mathbb{1}_{\Delta^{n+1}})) \\
&= (\bar{\sigma}^{n+1}_{n+1} \circ \partial^{\triangle}_{n+2})(T^{n+1}_{n+1}(\mathbb{1}_{\Delta^{n+1}})) = (\bar{\sigma}^{n+1}_{n+1} \circ (\partial^{\triangle}_{n+2} \circ T^{n+1}_{n+1}))(\mathbb{1}_{\Delta^{n+1}}) \\
&= (\bar{\sigma}^{n+1}_{n+1} \circ (\mathbb{1}^{\triangle}_{n+1} - T^{n+1}_{n} \circ \partial^{\triangle}_{n+1} - S^{n+1}_{n+1}))(\mathbb{1}_{\Delta^{n+1}}), \quad \text{by (7)} \\
&= (\bar{\sigma}^{n+1}_{n+1} \circ \mathbb{1}^{\triangle}_{n+1} - \bar{\sigma}^{n+1}_{n+1} \circ T^{n+1}_{n} \circ \partial^{\triangle}_{n+1} - \bar{\sigma}^{n+1}_{n+1} \circ S^{n+1}_{n+1})(\mathbb{1}_{\Delta^{n+1}}) \\
&= \sigma \circ \mathbb{1}_{\Delta^{n+1}} - (\bar{\sigma}^{n+1}_{n+1} \circ T^{n+1}_{n} \circ \partial^{\triangle}_{n+1})(\mathbb{1}_{\Delta^{n+1}}) - \bar{S}_{n+1}(\sigma) \\
&= \mathbb{1}_{n+1}(\sigma) - (\bar{T}_n \circ \partial_{n+1})(\sigma) - \bar{S}_{n+1}(\sigma) \\
&= (\mathbb{1}_{n+1} - \bar{T}_n \circ \partial_{n+1} - \bar{S}_{n+1})(\sigma),
\end{aligned}$$

where $\sigma \circ \mathbb{1}_{\Delta^{n+1}} = \sigma = \mathbb{1}_{n+1}(\sigma)$, and the remaining justifications for the eighth equality are left to the reader.

Now the fourth and last stage of the lemma's proof commences. For $m \geq 0$ let $\bar{S}^{\circ m}_n : C_n(X) \to C_n(X)$ be the mth iterate of $\bar{S}_n$, with the understanding that $\bar{S}^{\circ 0}_n = \mathbb{1}_n$. (The notation $\bar{S}^{\circ m}_n$ is used here instead of $\bar{S}^{m}_n$ simply because superscripts are already being used liberally for indexing purposes in the proof.) Define a homomorphism $D^m_n : C_n(X) \to C_{n+1}(X)$ by

$$D^m_n(\sigma) = \sum_{i=0}^{m-1} (\bar{T}_n \circ \bar{S}^{\circ i}_n)(\sigma)$$

for $\sigma : \Delta^n \to X$ and $m \geq 0$ (with $D^0_n(\sigma) = 0$), resulting in the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow \cdots \\
& {\scriptstyle \bar{S}^{\circ m}_{n+1}} \Big\downarrow \Big\downarrow {\scriptstyle \mathbb{1}_{n+1}} & \swarrow{\scriptstyle D^m_n} & {\scriptstyle \bar{S}^{\circ m}_{n}} \Big\downarrow \Big\downarrow {\scriptstyle \mathbb{1}_{n}} & \swarrow{\scriptstyle D^m_{n-1}} & {\scriptstyle \bar{S}^{\circ m}_{n-1}} \Big\downarrow \Big\downarrow {\scriptstyle \mathbb{1}_{n-1}} & \\
\cdots \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow \cdots
\end{array}$$

For fixed m it will be shown that the maps D^m_n provide a chain homotopy between $\{\mathbb{1}_n\}^{\infty}_{n=0}$ and $\{\bar{S}^{\circ m}_n\}^{\infty}_{n=0}$; that is,

$$\mathbb{1}_n - \bar{S}^{\circ m}_n = \partial_{n+1} \circ D^m_n + D^m_{n-1} \circ \partial_n \tag{9}$$

for all $n \geq 0$. Since the maps $\bar{S}_n^{\circ 1} = \bar{S}_n$ have been verified to define a chain map $C(X) \to C(X)$, it easily follows that $\partial_n \circ \bar{S}_n^{\circ i} = \bar{S}_{n-1}^{\circ i} \circ \partial_n$ holds for all i. Now, for any $\sigma : \Delta^n \to X$ and $m \geq 1$,

$$\begin{aligned}
(\partial_{n+1} \circ D_n^m + D_{n-1}^m \circ \partial_n)(\sigma) &= \partial_{n+1}(D_n^m(\sigma)) + D_{n-1}^m(\partial_n(\sigma)) \\
&= \partial_{n+1}\left(\sum_{i=0}^{m-1} (\bar{T}_n \circ \bar{S}_n^{\circ i})(\sigma)\right) + \sum_{i=0}^{m-1} (\bar{T}_{n-1} \circ \bar{S}_{n-1}^{\circ i})(\partial_n(\sigma)) \\
&= \sum_{i=0}^{m-1} \left[(\partial_{n+1} \circ \bar{T}_n)(\bar{S}_n^{\circ i}(\sigma)) + (\bar{T}_{n-1} \circ \bar{S}_{n-1}^{\circ i} \circ \partial_n)(\sigma)\right] \\
&= \sum_{i=0}^{m-1} \left[(\partial_{n+1} \circ \bar{T}_n)(\bar{S}_n^{\circ i}(\sigma)) + (\bar{T}_{n-1} \circ \partial_n)(\bar{S}_n^{\circ i}(\sigma))\right] \\
&= \sum_{i=0}^{m-1} (\partial_{n+1} \circ \bar{T}_n + \bar{T}_{n-1} \circ \partial_n)(\bar{S}_n^{\circ i}(\sigma)) = \sum_{i=0}^{m-1} (\mathbb{1}_n \circ \bar{S}_n)(\bar{S}_n^{\circ i}(\sigma)) \\
&= \sum_{i=0}^{m-1} (\bar{S}_n^{\circ i} - \bar{S}_n^{\circ (i+1)})(\sigma) = (\bar{S}_n^{\circ 0} - \bar{S}_n^{\circ m})(\sigma) = (\mathbb{1}_n - \bar{S}_n^{\circ m})(\sigma)
\end{aligned}$$

which verifies (9).

Recall that for any $\lfloor y_0, \ldots, y_n \rfloor : \Delta^n \to Y$ in $L_n(Y)$ with image the n-simplex $[y_0, \ldots, y_n]$, $S_n(\lfloor y_0, \ldots, y_n \rfloor)$ is a chain of singular n-simplices with images that are elements of $\mathcal{B}[y_0, \ldots, y_n]$, and so $S_n^n(\mathbb{1}_{\Delta^n})$ is a chain of maps with images that are elements of $\mathcal{B}\Delta^n$. Thus, $\bar{S}_n(\sigma)$ is a chain of maps of the form

$$\sigma \circ \alpha_{i_1} : \Delta^n \xrightarrow[\text{onto}]{\alpha_{i_1}} W_{i_1} \in \mathcal{B}\Delta^n \xrightarrow[\text{onto}]{\sigma} \sigma(W_{i_1}) \subset X.$$

For any of the maps $\sigma \circ \alpha_{i_1}$ we find that $\bar{S}_n(\sigma \circ \alpha_{i_1})$ in turn yields a chain of maps of the form

$$\sigma \circ \alpha_{i_1} \circ \alpha_{i_2} : \Delta^n \xrightarrow[\text{onto}]{\alpha_{i_2}} W_{i_2} \in \mathcal{B}\Delta^n \xrightarrow[\text{onto}]{\alpha_{i_1}} W_{i_1} \in \mathcal{B}\Delta^n \xrightarrow[\text{onto}]{\sigma} \sigma(W_{i_1}) \subset X.$$

In general $\bar{S}_n^{\circ m}(\sigma)$ is a linear combination of maps of the form $\sigma \circ \alpha_{i_1} \circ \cdots \circ \alpha_{i_m}$:

$$\Delta^n \xrightarrow{\alpha_{i_m}} W_{i_m} \xrightarrow{\alpha_{i_{m-1}}} W_{i_{m-1}} \xrightarrow{\alpha_{i_{m-2}}} \cdots \xrightarrow{\alpha_{i_2}} W_{i_2} \xrightarrow{\alpha_{i_1}} W_{i_1} \xrightarrow{\sigma} \sigma(W_{i_1}) \subset X, \tag{10}$$

where $W_{i_j} \in \mathcal{B}\Delta^n$ and $\alpha_{i_j} : \Delta^n \to W_{i_j}$ for each j.

For any n-simplex V, define $\mathcal{B}^1 V = \mathcal{B}V$, $\mathcal{B}^2 V = \bigcup\{\mathcal{B}W_1 \ : \ W_1 \in \mathcal{B}^1 V\}$, and in general

$$\mathcal{B}^n V = \bigcup \left\{\mathcal{B}W_{n-1} \ : \ W_{n-1} \in \mathcal{B}^{n-1} V\right\}$$

for $n \geq 1$. Recalling (5), it's seen that if $W_2 \in \mathcal{B}^2 V$, then $W_2 \in \mathcal{B}W_1$ for some $W_1 \in \mathcal{B}V$, and so

$$\operatorname{diam}(W_2) \leq \frac{n}{n+1} \operatorname{diam}(W_1) \leq \left(\frac{n}{n+1}\right)^2 \operatorname{diam}(V).$$

More generally for $W_m \in \mathcal{B}^m V$,

$$\operatorname{diam}(W_m) \leq \left(\frac{n}{n+1}\right)^m \operatorname{diam}(V). \tag{11}$$

Now, for any $\ell = \lfloor y_0, \ldots, y_n \rfloor : \Delta^n \to [y_0, \ldots, y_n]$, if $W \in \mathcal{B}\Delta^n$ then $\ell(W) \in \mathcal{B}[y_0, \ldots, y_n]$. This is easily seen by noticing that if b is a barycenter for some k-dimensional face of Δ^n, where $0 \le k \le n$, then $\ell(b)$ will be the barycenter of the corresponding face of $[y_0, \ldots, y_n]$. Referring to (10), it follows that $\alpha_{i_{m-1}}$ maps $W_{i_m} \in \mathcal{B}\Delta^n$ to some

$$W'_{i_{m-1}} \in \mathcal{B}W_{i_{m-1}} \subset \mathcal{B}^2\Delta^n,$$

$\alpha_{i_{m-2}}$ maps $W'_{i_{m-1}} \in \mathcal{B}^2\Delta^n$ to some

$$W'_{i_{m-2}} \in \mathcal{B}^2 W_{i_{m-2}} \subset \mathcal{B}^3\Delta^n,$$

and so on until we arrive at $\alpha_{i_1} : \Delta^n \to W_{i_1}$, which maps $W'_{i_2} \in \mathcal{B}^{m-1}\Delta^n$ to some

$$W'_{i_1} \in \mathcal{B}^{m-1} W_{i_1} \subset \mathcal{B}^m\Delta^n.$$

It is seen, then, that $\bar{S}_n^{\circ m}(\sigma)$ is a linear combination of maps $\hat{\sigma}$ of the form $\sigma \circ \alpha$, where each α maps from Δ^n onto some $W_m \in \mathcal{B}^m\Delta^n$. Thus each $\hat{\sigma}$ is effectively a restriction of σ to some $W_m \in \mathcal{B}^m\Delta^n$, where

$$\operatorname{diam}(W_m) \le \left(\frac{n}{n+1}\right)^m \operatorname{diam}(\Delta^n)$$

by (11).

Since $X = \bigcup_k U_k^\circ$ and $\sigma : \Delta^n \to X$ is continuous, the collection $\{\sigma^{-1}(U_k^\circ)\}$ forms an open cover for Δ^n. Since Δ^n is compact there exists some $\epsilon_\sigma > 0$ (a Lebesgue number for the cover) such that, for any set $W \subset \Delta^n$ with $\operatorname{diam}(W) < \epsilon_\sigma$, there exists some k for which $W \subset \sigma^{-1}(U_k^\circ)$. Let m be sufficiently large so that

$$\left(\frac{n}{n+1}\right)^m \operatorname{diam}(\Delta^n) < \epsilon_\sigma.$$

Then $\bar{S}_n^{\circ m}(\sigma)$ is a chain of maps $\hat{\sigma}$, each having image $\hat{\sigma}(W)$ in X for some set $W \subset \Delta^n$ with $\operatorname{diam}(W) < \epsilon_\sigma$, so that $W \subset \sigma^{-1}(U_k^\circ)$ for some k. Hence, each singular n-simplex $\hat{\sigma}$ in the chain $\bar{S}_n^{\circ m}(\sigma)$ maps into some $U_k \subset X$, and therefore $\bar{S}_n^{\circ m}(\sigma) \in C_n^{\mathcal{U}}(X)$.

For each singular n-simplex σ let

$$m_\sigma = \min\{m \in \mathbb{Z} \ : \ \bar{S}_n^{\circ m}(\sigma) \in C_n^{\mathcal{U}}(X)\},$$

and define $\bar{D}_n : C_n(X) \to C_{n+1}(X)$ by

$$\bar{D}_n(\sigma) = D_n^{m_\sigma}(\sigma).$$

From (9) we have

$$\sigma - \bar{S}_n^{\circ m_\sigma}(\sigma) = (\partial_{n+1} \circ D_n^{m_\sigma})(\sigma) + (D_{n-1}^{m_\sigma} \circ \partial_n)(\sigma)$$

$$(\partial_{n+1} \circ D_n^{m_\sigma})(\sigma) + \bar{D}_{n-1}(\partial_n\sigma) = \sigma - \left[\bar{S}_n^{\circ m_\sigma}(\sigma) + (D_{n-1}^{m_\sigma} \circ \partial_n)(\sigma) - \bar{D}_{n-1}(\partial_n\sigma)\right]$$

$$(\partial_{n+1} \circ \bar{D}_n)(\sigma) + (\bar{D}_{n-1} \circ \partial_n)(\sigma) = \mathbb{1}_n(\sigma) - \varphi_n(\sigma),$$

where $\varphi_n(\sigma)$ is defined to be the expression in the brackets, and so

$$\partial_{n+1} \circ \bar{D}_n + \bar{D}_{n-1} \circ \partial_n = \mathbb{1}_n - \varphi_n. \tag{12}$$

It will be shown that $\varphi_n : C_n(X) \to C_n^{\mathcal{U}}(X)$. Clearly $\bar{S}_n^{\circ m_\sigma}(\sigma) \in C_n^{\mathcal{U}}(X)$, so attention turns to $(D_{n-1}^{m_\sigma} \circ \partial_n)(\sigma) - \bar{D}_{n-1}(\partial_n \sigma)$. Let $\sigma_i = \sigma|_{[u_0,\dots,\hat{u}_i,\dots,u_n]} \circ \delta_i^{n-1}$ and

$$m_{\sigma i} = \min\{m \in \mathbb{Z} \ : \ \bar{S}_{n-1}^{\circ m}(\sigma_i) \in C_n^{\mathcal{U}}(X)\},$$

and observe that $m_{\sigma i} \leq m_\sigma$ for each $0 \leq i \leq n$. Now,

$$\begin{aligned}
(D_{n-1}^{m_\sigma} \circ \partial_n)(\sigma) - \bar{D}_{n-1}(\partial_n \sigma) &= \sum_{i=0}^{n} (-1)^i D_{n-1}^{m_\sigma}(\sigma_i) - \sum_{i=0}^{n} (-1)^i D_{n-1}^{m_{\sigma i}}(\sigma_i) \\
&= \sum_{i=0}^{n} (-1)^i \left(\sum_{j=0}^{m_\sigma - 1} (\bar{T}_{n-1} \circ \bar{S}_{n-1}^{\circ j})(\sigma_i) - \sum_{j=0}^{m_{\sigma i} - 1} (\bar{T}_{n-1} \circ \bar{S}_{n-1}^{\circ j})(\sigma_i) \right) \\
&= \sum_{i=0}^{n} \sum_{j=m_{\sigma i}}^{m_\sigma - 1} (-1)^i (\bar{T}_{n-1} \circ \bar{S}_{n-1}^{\circ j})(\sigma_i). \qquad (13)
\end{aligned}$$

Since

$$\bar{S}_{n-1}^{\circ j}(\sigma_i) \in \mathcal{C}_{n-1}^{\mathcal{U}}(X)$$

for $j \geq m_{\sigma i}$ and

$$\bar{T}_{n-1} : C_{n-1}^{\mathcal{U}}(X) \to C_n^{\mathcal{U}}(X),$$

it's readily seen from (13) that

$$(D_{n-1}^{m_\sigma} \circ \partial_n)(\sigma) - \bar{D}_{n-1}(\partial_n \sigma) \in \mathcal{C}_n^{\mathcal{U}}(X),$$

and therefore $\varphi_n(\sigma) \in \mathcal{C}_n^{\mathcal{U}}(X)$.

To show that the maps φ_n constitute a chain map $C(X) \to C^{\mathcal{U}}(X)$ as illustrated in the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1}^{\mathcal{U}}(X) & \xrightarrow{\partial_{n+1}} & C_n^{\mathcal{U}}(X) & \xrightarrow{\partial_n} & C_{n-1}^{\mathcal{U}}(X) & \longrightarrow & \cdots
\end{array}$$

we show that $\varphi_{n-1} \circ \partial_n = \partial_n \circ \varphi_n$. Using (12), we obtain

$$\partial_n \circ \varphi_n = \partial_n \circ \mathbb{1}_n - \partial_n \circ \partial_{n+1} \circ \bar{D}_n - \partial_n \circ \bar{D}_{n-1} \circ \partial_n = \partial_n - \partial_n \circ \bar{D}_{n-1} \circ \partial_n$$

and

$$\varphi_{n-1} \circ \partial_n = \mathbb{1}_{n-1} \circ \partial_n - \partial_n \circ \bar{D}_{n-1} \circ \partial_n - \bar{D}_{n-2} \circ \partial_{n-1} \circ \partial_n = \partial_n - \partial_n \circ \bar{D}_{n-1} \circ \partial_n,$$

which verifies commutativity.

Now, the inclusion maps ι_n constitute a chain map $C^{\mathcal{U}}(X) \to C(X)$, and since (12) implies

$$\partial_{n+1} \circ \bar{D}_n + \bar{D}_{n-1} \circ \partial_n = \mathbb{1}_n - \iota_n \circ \varphi_n \tag{14}$$

for all n, it follows that $\{\iota_n \circ \varphi_n\}$ is chain homotopic to $\{\mathbb{1}_n\}$.

Next, we show that $\varphi_n \circ \iota_n = \mathbb{1}_n^{\mathcal{U}}$, where $\mathbb{1}_n^{\mathcal{U}} : C_n^{\mathcal{U}}(X) \to C_n^{\mathcal{U}}(X)$ is the identity map. Let $\sigma : \Delta^n \to U_k$ for some k, so that σ is a basis element for $C_n^{\mathcal{U}}(X)$. Since $\sigma_i : \Delta^{n-1} \to U_k$ for each i, we have $m_{\sigma i} = 0$ as well as $m_\sigma = 0$. Hence,

$$\begin{aligned}(\varphi_n \circ \iota_n)(\sigma) = \varphi_n(\sigma) &= (\mathbb{1}_n - \partial_{n+1} \circ \bar{D}_n - \bar{D}_{n-1} \circ \partial_n)(\sigma) \\ &= \sigma - \partial_{n+1}(D_n^0(\sigma)) - \sum_{i=0}^{n} (-1)^i D_{n-1}^0(\sigma_i) \\ &= \sigma - \partial_{n+1}(0) - \sum_{i=0}^{n} (-1)^i (0) = \sigma,\end{aligned}$$

which completes the argument and so $\mathbb{1}_n^{\mathcal{U}} - \varphi_n \circ \iota_n = 0$. Defining $\bar{0}_n^{\mathcal{U}} : C_n^{\mathcal{U}}(X) \to C_{n+1}^{\mathcal{U}}(X)$ to be the trivial homomorphism, what we have shown is

$$\partial_{n+1} \circ \bar{0}_n^{\mathcal{U}} + \bar{0}_{n-1}^{\mathcal{U}} \circ \partial_n = 0 = \mathbb{1}_n^{\mathcal{U}} - \varphi_n \circ \iota_n,$$

and therefore $\{\varphi_n \circ \iota_n\}$ is chain homotopic to $\{\mathbb{1}_n^{\mathcal{U}}\}$.

At last we see that $\{\iota_n\}$ is a chain-homotopy equivalence, and so by Proposition 2.5 each $\iota_{n*} : H_n^{\mathcal{U}}(X) \to H_n(X)$ is an isomorphism. ■

Proof of the Excision Theorem. Let $\mathcal{U} = \{A, B\}$, where A and B are subspaces of X such that $A^\circ \cup B^\circ = X$. Define $C_n^{\mathcal{U}}(X, A) = C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$.

The maps $\iota_n : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ induce homomorphisms on quotient groups $\bar{\iota}_n : C_n^{\mathcal{U}}(X, A) \to C_n(X, A)$ given by

$$\bar{\iota}_n(\alpha + C_n^{\mathcal{U}}(A)) = \iota_n(\alpha) + C_n(A) = \alpha + C_n(A)$$

for each $\alpha \in C_n^{\mathcal{U}}(X)$. It's easy to verify that the maps $\bar{\iota}_n$ form a chain map $C^{\mathcal{U}}(X, A) \to C(X, A)$,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & C_{n+1}^{\mathcal{U}}(X,A) & \xrightarrow{\bar{\partial}_{n+1}^{\mathcal{U}}} & C_n^{\mathcal{U}}(X,A) & \xrightarrow{\bar{\partial}_n^{\mathcal{U}}} & C_{n-1}^{\mathcal{U}}(X,A) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \bar{\iota}_{n+1}} & & \downarrow{\scriptstyle \bar{\iota}_n} & & \downarrow{\scriptstyle \bar{\iota}_{n-1}} & \\ \cdots \longrightarrow & C_{n+1}(X,A) & \xrightarrow{\bar{\partial}_{n+1}} & C_n(X,A) & \xrightarrow{\bar{\partial}_n} & C_{n-1}(X,A) & \longrightarrow \cdots \end{array}$$

with the maps $\bar{\partial}_n$ in the diagram being defined as in section 2.4, and the maps $\bar{\partial}_n^{\mathcal{U}}$ being the obvious restrictions. Thus each $\bar{\iota}_n$ in turn induces a homomorphism on homology groups

$$\bar{\iota}_{n*} : \operatorname{Ker} \bar{\partial}_n^{\mathcal{U}} / \operatorname{Im} \bar{\partial}_{n+1}^{\mathcal{U}} := H_n^{\mathcal{U}}(X, A) \to \operatorname{Ker} \bar{\partial}_n / \operatorname{Im} \bar{\partial}_{n+1} := H_n(X, A)$$

defined according to the general algebraic formula given in section 2.1.

Next, the maps $\bar{\varphi}_n : C_n(X, A) \to C_n^{\mathcal{U}}(X, A)$ defined by

$$\bar{\varphi}_n(\alpha + C_n(X)) = \varphi_n(\alpha) + C_n^{\mathcal{U}}(A)$$

for each $\alpha \in C_n(X)$ give rise to a chain map $C(X, A) \to C^{\mathcal{U}}(X, A)$; and so, defining maps $\bar{\bar{D}}_n$ and $\bar{\mathbb{1}}_n$ on $C_n(X, A)$ in the canonical fashion from the maps $\bar{D}_n$ and $\mathbb{1}_n$ in the proof of Lemma 2.22, the diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow \cdots \\
& {\scriptstyle \iota_{n+1}\circ\varphi_{n+1}} \Big\downarrow {\scriptstyle \mathbb{1}_{n+1}} & \swarrow {\scriptstyle \bar{D}_n} & {\scriptstyle \iota_n\circ\varphi_n} \Big\downarrow {\scriptstyle \mathbb{1}_n} & \swarrow {\scriptstyle \bar{D}_{n-1}} & {\scriptstyle \mathbb{1}_{n-1}} \Big\downarrow {\scriptstyle \iota_{n-1}\circ\varphi_{n-1}} & \\
\cdots \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow \cdots
\end{array}$$

induces a diagram on quotient groups,

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_{n+1}(X,A) & \xrightarrow{\bar{\partial}_{n+1}} & C_n(X,A) & \xrightarrow{\bar{\partial}_n} & C_{n-1}(X,A) & \longrightarrow \cdots \\
& {\scriptstyle \bar{\iota}_{n+1}\circ\bar{\varphi}_{n+1}} \Big\downarrow {\scriptstyle \bar{\mathbb{1}}_{n+1}} & \swarrow {\scriptstyle \bar{\bar{D}}_n} & {\scriptstyle \bar{\iota}_n\circ\bar{\varphi}_n} \Big\downarrow {\scriptstyle \bar{\mathbb{1}}_n} & \swarrow {\scriptstyle \bar{\bar{D}}_{n-1}} & {\scriptstyle \bar{\mathbb{1}}_{n-1}} \Big\downarrow {\scriptstyle \bar{\iota}_{n-1}\circ\bar{\varphi}_{n-1}} & \\
\cdots \longrightarrow & C_{n+1}(X,A) & \xrightarrow{\bar{\partial}_{n+1}} & C_n(X,A) & \xrightarrow{\bar{\partial}_n} & C_{n-1}(X,A) & \longrightarrow \cdots
\end{array}$$

Again referring to the proof of Lemma 2.22, from $\varphi_n \circ \iota_n = \mathbb{1}_n^{\mathcal{U}}$ and (14) we readily obtain $\bar{\varphi}_n \circ \bar{\iota}_n = \bar{\mathbb{1}}_n^{\mathcal{U}}$ and

$$\bar{\partial}_{n+1} \circ \bar{\bar{D}}_n + \bar{\bar{D}}_{n-1} \circ \bar{\partial}_n = \bar{\mathbb{1}}_n - \bar{\iota}_n \circ \bar{\varphi}_n$$

for all n, where $\bar{\mathbb{1}}_n^{\mathcal{U}} = \mathbb{1}_{C_n^{\mathcal{U}}(X,A)}$ and $\bar{\mathbb{1}}_n = \mathbb{1}_{C_n(X,A)}$. Thus $\{\bar{\varphi}_n \circ \bar{\iota}_n\}$ is chain-homotopic to $\{\mathbb{1}_{C_n^{\mathcal{U}}(X,A)}\}$, and $\{\bar{\iota}_n \circ \bar{\varphi}_n\}$ is chain-homotopic to $\{\mathbb{1}_{C_n(X,A)}\}$, which implies that $\{\bar{\iota}_n\}$ is a chain-homotopy equivalence and therefore the maps $\bar{\iota}_{n*} : H_n^{\mathcal{U}}(X,A) \to H_n(X,A)$ are isomorphisms.

Now, define $\kappa_n : C_n(B) \hookrightarrow C_n^{\mathcal{U}}(X)$ to be inclusion maps, which induce homomorphisms $\bar{\kappa}_n : C_n(B, A\cap B) \to C_n^{\mathcal{U}}(X,A)$ given by

$$\bar{\kappa}_n(\beta + C_n(A\cap B)) = \kappa_n(\beta) + C_n^{\mathcal{U}}(A) = \beta + C_n^{\mathcal{U}}(A)$$

for each $\beta \in C_n(B)$, noting that $C_n(B) = C_n^{\mathcal{U}}(B) \subset C_n^{\mathcal{U}}(X)$. Since the maps κ_n form a chain map, the maps $\bar{\kappa}_n$ constitute a chain map at the quotient group level,

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_{n+1}(B, A\cap B) & \xrightarrow{\bar{\partial}_{n+1}^{B}} & C_n(B, A\cap B) & \xrightarrow{\bar{\partial}_n^{B}} & C_{n-1}(B, A\cap B) & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle \bar{\kappa}_{n+1}} & & \Big\downarrow {\scriptstyle \bar{\kappa}_n} & & \Big\downarrow {\scriptstyle \bar{\kappa}_{n-1}} & \\
\cdots \longrightarrow & C_{n+1}^{\mathcal{U}}(X,A) & \xrightarrow{\bar{\partial}_{n+1}^{\mathcal{U}}} & C_n^{\mathcal{U}}(X,A) & \xrightarrow{\bar{\partial}_n^{\mathcal{U}}} & C_{n-1}^{\mathcal{U}}(X,A) & \longrightarrow \cdots
\end{array}$$

and so induce homomorphisms

$$\bar{\kappa}_{n*} : \operatorname{Ker} \bar{\partial}_n^{B} / \operatorname{Im} \bar{\partial}_{n+1}^{B} := H_n(B, A\cap B) \to \operatorname{Ker} \bar{\partial}_n^{\mathcal{U}} / \operatorname{Im} \bar{\partial}_{n+1}^{\mathcal{U}} := H_n^{\mathcal{U}}(X,A).$$

Fix n, and suppose that $\bar{\kappa}_n(\beta + C_n(A\cap B)) = C_n^{\mathcal{U}}(A)$, the zero element of $C_n^{\mathcal{U}}(X,A)$. Then $\beta \in C_n^{\mathcal{U}}(A)$, but since $\beta \in C_n^{\mathcal{U}}(B)$ also, it follows that $\beta \in C_n^{\mathcal{U}}(A\cap B)$ and so

$$\beta + C_n(A\cap B) = C_n(A\cap B).$$

Thus $\bar{\kappa}_n$ is one-to-one.

Next, let $\gamma + C_n^{\mathcal{U}}(A) \in C_n^{\mathcal{U}}(X, A)$, so that $\gamma \in C_n^{\mathcal{U}}(X)$. We can assume that γ is a generator for $C_n^{\mathcal{U}}(X)$ so that $\gamma : \Delta^n \to U$ for some $U \in \mathcal{U}$. If $U = A$, then

$$\bar{\kappa}_n(C_n(A \cap B)) = C_n^{\mathcal{U}}(A) = \gamma + C_n^{\mathcal{U}}(A);$$

and if $U = B$, then $\gamma + C_n(A \cap B) \in C_n(B, A \cap B)$ with

$$\bar{\kappa}_n(\gamma + C_n(A \cap B)) = \kappa_n(\gamma) + C_n^{\mathcal{U}}(A) = \gamma + C_n^{\mathcal{U}}(A).$$

Thus $\bar{\kappa}_n$ is onto, and Proposition 2.2 implies that $\bar{\kappa}_{n*}$ is an isomorphism.

Now we have isomorphisms

$$\bar{\iota}_{n*} \circ \bar{\kappa}_{n*} : H_n(B, A \cap B) \to H_n^{\mathcal{U}}(X, A) \to H_n(X, A)$$

for all n. Since $\bar{\iota}_{n*} \circ \bar{\kappa}_{n*} = (\bar{\iota}_n \circ \bar{\kappa}_n)_*$ by Proposition 2.4, it is a routine matter to show that $\bar{\iota}_{n*} \circ \bar{\kappa}_{n*} = \bar{\jmath}_{n*}$ by showing $\bar{\iota}_n \circ \bar{\kappa}_n = \bar{\jmath}_n$, so the proof is done. ■

One result that follows fairly easily from the Excision Theorem is the following proposition, which makes a nice connection between relative and absolute homology.

Proposition 2.23. *If (X, A) is a good pair, then the quotient map*

$$q : (X, A) \to (X/A, A/A)$$

induces isomorphisms

$$\bar{q}_{n*} : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$$

for all $n \geq 0$.

Proof. Let V be a neighborhood in X that deformation retracts to A. Define homomorphisms $l_n : C_n(X, A) \to C_n(X, V)$ as follows: for each basis element $\sigma : \Delta^n \to X$ of $C_n(X)$, let

$$l_n(\sigma + C_n(A)) = \sigma + C_n(V);$$

similarly, define $k_n : C_n(V, A) \to C_n(X, A)$ by

$$k_n(\sigma + C_n(A)) = \sigma + C_n(A)$$

for each $\sigma : \Delta^n \to V$. Note that

$$0 \longrightarrow C_n(V, A) \xrightarrow{k_n} C_n(X, A) \xrightarrow{l_n} C_n(X, V) \longrightarrow 0$$

is a short exact sequence, and thus we can construct a short exact sequence of chain complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{n+1}(V,A) & \xrightarrow{\partial_{n+1}} & C_n(V,A) & \xrightarrow{\partial_n} & C_{n-1}(V,A) & \longrightarrow & \cdots \\
 & & \Big\downarrow k_{n+1} & & \Big\downarrow k_n & & \Big\downarrow k_{n-1} & & \\
\cdots & \longrightarrow & C_{n+1}(X,A) & \xrightarrow{\partial_{n+1}} & C_n(X,A) & \xrightarrow{\partial_n} & C_{n-1}(X,A) & \longrightarrow & \cdots \\
 & & \Big\downarrow l_{n+1} & & \Big\downarrow l_n & & \Big\downarrow l_{n-1} & & \\
\cdots & \longrightarrow & C_{n+1}(X,V) & \xrightarrow{\partial_{n+1}} & C_n(X,V) & \xrightarrow{\partial_n} & C_{n-1}(X,V) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with maps ∂_n defined in the usual way (verification of commutativity is omitted here). From this we obtain a long exact sequence for the triple (X, V, A),

$$\cdots \longrightarrow H_n(V,A) \xrightarrow{k_{n*}} H_n(X,A) \xrightarrow{l_{n*}} H_n(X,V) \xrightarrow{\partial_{n*}} H_{n-1}(V,A) \longrightarrow \cdots \tag{15}$$

in the manner outlined in section 2.4.

The natural quotient map $q : (X, A) \to (X/A, A/A)$ given by $q(x) = x + A$ induces homomorphisms $q_n : C_n(X) \to C_n(X/A)$ given by $q_n(\sigma) = q \circ \sigma$, which in turn induce maps $\bar{q}_n : C_n(X, A) \to C_n(X/A, A/A)$ given by

$$\bar{q}_n(\sigma + C_n(A)) = q \circ \sigma + C_n(A/A).$$

Now we construct the diagram[4]

$$
\begin{array}{ccccc}
C_n(X,A) & \xrightarrow{\quad l_n \quad} & C_n(X,V) & \xleftarrow{\quad \bar{\jmath}_n \quad} & C_n(X-A, V-A) \\
\Big\downarrow \bar{q}_n & & \Big\downarrow \bar{q}'_n & & \Big\downarrow \bar{q}''_n \\
C_n(X/A, A/A) & \xrightarrow{i_n} & C_n(X/A, V/A) & \xleftarrow{\bar{\jmath}'_n} & C_n(X/A - A/A, V/A - A/A)
\end{array}
$$

where

$$i_n(\sigma + C_n(A/A)) = \sigma + C_n(V/A)$$

for each singular n-simplex $\sigma : \Delta^n \to X/A$, and $\bar{\jmath}_n$ and $\bar{\jmath}'_n$ are defined as in equation (4). Much like $\bar{q}_n$ we have

$$\bar{q}'_n(\sigma + C_n(V)) = q \circ \sigma + C_n(V/A),$$

and

$$\bar{q}''_n(\sigma + C_n(V - A)) = q \circ \sigma + C_n(V/A - A/A).$$

The diagram is commutative since

$$
\begin{aligned}
i_n(\bar{q}_n(\sigma + C_n(A))) &= i_n(q \circ \sigma + C_n(A/A)) = q \circ \sigma + C_n(V/A) \\
&= \bar{q}'_n(\sigma + C_n(V)) = \bar{q}'_n(l_n(\sigma + C_n(A)))
\end{aligned}
$$

[4] It is a quick matter to verify that $X/A - A/A = (X - A)/A$ and so on.

and

$$\bar{q}'_n(\bar{\jmath}_n(\sigma + C_n(V - A))) = \bar{q}'_n(\sigma + C_n(V)) = q \circ \sigma + C_n(V/A)$$
$$= \bar{\jmath}'_n(q \circ \sigma + C_n(V/A - A/A)) = \bar{\jmath}'_n(\bar{q}''_n(\sigma + C_n(V - A))),$$

and therefore the diagram

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{l_{n*}} & H_n(X,V) & \xleftarrow{\bar{\jmath}_{n*}} & H_n(X-A,V-A) \\ \downarrow{\scriptstyle \bar{q}_{n*}} & & \downarrow{\scriptstyle \bar{q}'_{n*}} & & \downarrow{\scriptstyle \bar{q}''_{n*}} \\ H_n(X/A,A/A) & \xrightarrow{i_{n*}} & H_n(X/A,V/A) & \xleftarrow{\bar{\jmath}'_{n*}} & H_n(X/A-A/A,V/A-A/A) \end{array}$$

is also commutative.

The maps $\bar{\jmath}_{n*}$ and $\bar{\jmath}'_{n*}$ are isomorphisms by the Excision Theorem. The map $\bar{q}''_n$ is an isomorphism since it is induced by the homeomorphism

$$q : (X - A, V - A) \to (X/A - A/A, V/A - A/A);$$

indeed, we find that $q : X - A \to X/A - A/A$ and the restriction $q : V - A \to V/A - A/A$ are each homeomorphisms, and hence homotopy equivalences, and so we can invoke the result of Proposition 2.18

Since V deformation retracts to A, there exists a retraction $r : (V, A) \to (A, A)$. Thus, $r : V \to A$ is a homotopy equivalence, as is the restriction $r : A \to A$ (which in fact is simply the identity map on A and so is a homeomorphism). Using Proposition 2.18 once more we conclude that $r_{n*} : H_n(V, A) \to H_n(A, A)$ is an isomorphism, and since $H_n(A, A) = 0$ for all n it follows that $H_n(V, A) \cong 0$ for all n. Now, from the exact sequence (15) it can be seen that l_{n*} is an isomorphism as well. A similar argument will show that i_{n*} is an isomorphism since $r : (V, A) \to (A, A)$ induces a retraction $(V/A, A/A) \to (A/A, A/A)$.

The commutativity of the diagram above leads to the conclusion that

$$\bar{q}_{n*} = (i_{n*})^{-1} \circ \bar{\jmath}'_{n*} \circ \bar{q}''_n \circ (\bar{\jmath}_{n*})^{-1} \circ l_{n*},$$

and therefore $\bar{q}_{n*}$ is an isomorphism. ■

In section 2.4 we obtained the long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_{n*}} \widetilde{H}_n(X) \xrightarrow{j_{n*}} \widetilde{H}_n(X,A) \xrightarrow{\bar{\partial}_{n*}} \widetilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(X,A) \longrightarrow 0,$$

and since $\widetilde{H}_n(X, A) = H_n(X, A)$ for all $n \geq 0$ if $A \neq \varnothing$, it follows from Proposition 2.23 that there is an isomorphism $\varphi_{n*} : \widetilde{H}_n(X, A) \to \widetilde{H}(X/A)$, and so we arrive at an exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_{n*}} \widetilde{H}_n(X) \xrightarrow{q_{n*}} \widetilde{H}_n(X/A) \xrightarrow{\hat{\partial}_{n*}} \widetilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0$$

with $q_{n*} = \varphi_{n*} \circ j_{n*}$ and $\hat{\partial}_{n*} = \bar{\partial}_{n*} \circ \varphi_{n*}^{-1}$. Thus Theorem 2.17 is proven once it is verified that the maps q_{n*} here are indeed the maps induced by $q : X \to X/A$.

Enough machinery has now been developed to entertain a few interesting examples.

Example 2.24. Here we will find explicit relative cycles representing generators of the infinite cyclic groups $H_n(\mathbb{D}^n, \partial\mathbb{D}^n)$.

Starting with $(\mathbb{D}^n, \partial\mathbb{D}^n)$, pass to the equivalent pair $(\Delta^n, \partial\Delta^n)$. It will be shown by induction that the identity map $i_n : \Delta^n \to \Delta^n$ is a cycle generating

$$H_n(\Delta^n, \partial\Delta^n) = \frac{\operatorname{Ker}[\partial_n : C_n(\Delta^n, \partial\Delta^n) \to C_{n-1}(\Delta^n, \partial\Delta^n)]}{\operatorname{Im}[\partial_{n+1} : C_{n+1}(\Delta^n, \partial\Delta^n) \to C_n(\Delta^n, \partial\Delta^n)]}$$

for $n \geq 0$. Note an element of $H_n(\Delta^n, \partial\Delta^n)$ has the form

$$(\varphi + C_n(\partial\Delta^n)) + \partial_{n+1}(C_{n+1}(\Delta^n, \partial\Delta^n))$$

for $\varphi \in C_n(\Delta^n)$ such that $\varphi + C_n(\partial\Delta^n) \in \operatorname{Ker}\partial_n$, so in fact it's better to say it will be shown that $H_n(\Delta^n, \partial\Delta^n) = \langle[\bar{\imath}_n]\rangle$, where

$$[\bar{\imath}_n] := (i_n + C_n(\partial\Delta^n)) + \partial_{n+1}(C_{n+1}(\Delta^n, \partial\Delta^n))$$

is the generator.

Now,

$$\partial_n(i_n) = \sum_{k=0}^{n}(-1)^k i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]},$$

where for each k we have $i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]} : \Delta^{n-1} \to \partial\Delta^n$ since $[u_0, \ldots, \hat{u}_k, \ldots, u_n] \subset \partial\Delta^n$ is the kth "face" of Δ^n, and so $\partial_n(i_n) \in C_{n-1}(\partial\Delta^n)$ and we see that i_n is a relative cycle.

Let $n = 0$. We have $i_0 : \Delta^0 \to \Delta^0$ and we must show that i_0 represents a generator for

$$H_0(\Delta^0, \partial\Delta^0) = H_0(\Delta^0, \varnothing) \cong H_0(\Delta^0) = H_0(\{u_0\}) \cong \mathbb{Z}.$$

Fix $[\bar{\varphi}] \in H_0(\Delta^0, \partial\Delta^0)$. Then

$$[\bar{\varphi}] = \bar{\varphi} + \partial_1(C_1(\Delta^0, \partial\Delta^0))$$

with $\bar{\varphi} = \varphi + C_0(\partial\Delta^0) \in \operatorname{Ker}\partial$ for some $\varphi \in C_0(\Delta^0)$. Clearly $C_0(\Delta^0) = \langle i_0\rangle$, where i_0 is the map $u_0 \mapsto u_0$, and so $\varphi = ki_0$ for some $k \in \mathbb{Z}$. Hence

$$\left[\,\overline{ki_0}\,\right] = \overline{ki_0} + \partial_1(C_1(\Delta^0, \partial\Delta^0)),$$

where

$$\overline{ki_0} = ki_0 + C_0(\partial\Delta^0) = k(i_0 + C_0(\partial\Delta^0)) = k\bar{\imath}_0$$

so that

$$[\bar{\varphi}] = \left[\,\overline{ki_0}\,\right] = [k\bar{\imath}_0] = k\,[\bar{\imath}_0]$$

Therefore $H_0(\Delta^0, \partial\Delta^0) = \langle[\bar{\imath}_0]\rangle$ and the base case is done.

For the induction step, let $n \geq 1$, and suppose i_{n-1} is a cycle generating $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. Let Λ be the union of all but one of the $(n-1)$-dimensional faces of Δ^n. The first claim is that there exists isomorphisms

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\;\cong\;} H_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{\;\cong\;} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

For each $k > 1$, since Δ^n and Λ are contractible, we obtain from the long exact sequence for the pair (Δ^n, Λ)

$$\cdots \longrightarrow \underbrace{H_k(\Delta^n)}_{0} \longrightarrow H_k(\Delta^n, \Lambda) \longrightarrow \underbrace{H_{k-1}(\Lambda)}_{0} \longrightarrow \underbrace{H_{k-1}(\Delta^n)}_{0} \longrightarrow \cdots,$$

and so $H_k(\Delta^n, \Lambda) \cong 0$. Now for the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ (note $\Lambda \subset \partial\Delta^n \subset \Delta^n$) we have a long exact sequence which gives, for $n > 2$,

$$\cdots \longrightarrow \underbrace{H_n(\Delta^n, \Lambda)}_{0} \longrightarrow H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\bar{\partial}_n} H_{n-1}(\partial\Delta^n, \Lambda) \longrightarrow \underbrace{H_{n-1}(\Delta^n, \Lambda)}_{0} \longrightarrow \cdots,$$

whence $H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda)$. When $n = 2$, since (Δ^2, Λ) is a good pair and Δ^2/Λ is contractible, it follows that $H_1(\Delta^2, \Lambda) \cong \widetilde{H}_1(\Delta^2/\Lambda) = 0$ and therefore $H_2(\Delta^2, \partial\Delta^2) \cong H_1(\partial\Delta^2, \Lambda)$. The $n = 1$ case requires a more direct approach: we have

$$H_1(\Delta^1, \partial\Delta^1) \cong \widetilde{H}_1(\Delta^1/\partial\Delta^1) \cong \widetilde{H}_1(\mathbb{S}^1) \cong \mathbb{Z},$$

and

$$H_0(\partial\Delta^1, \Lambda) \cong \widetilde{H}_0(\partial\Delta^1/\Lambda) \cong \mathbb{Z},$$

where the last isomorphism obtains from the observation that

$$\partial\Delta^1/\Lambda = \{u_0 + \Lambda,\ u_1 + \Lambda\},$$

a two-point set. Hence $H_1(\Delta^1, \partial\Delta^1) \cong H_0(\partial\Delta^1, \Lambda)$, and it's seen that $H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda)$ for all $n > 0$ as desired.

The map $\bar{\partial}_n : H_n(\Delta^n, \partial\Delta^n) \to H_{n-1}(\partial\Delta^n, \Lambda)$ has now been established to be an isomorphism for all $n \geq 1$, and it remains to determine explicitly how $\bar{\partial}_n$ actually works. In general, consider the long exact sequence for the triple (X, A, B), where of course $B \subset A \subset X$:

$$\cdots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \xrightarrow{\hat{\partial}_n} H_{n-1}(A, B) \longrightarrow H_{n-1}(X, B) \longrightarrow \cdots$$

Let $[\bar{\varphi}] \in H_n(X, A)$. Define ∂_n^X to be the usual boundary map $C_n(X) \to C_{n-1}(X)$. Then since

$$H_n(X, A) = \frac{\mathrm{Ker}[\partial_n : C_n(X, A) \to C_{n-1}(X, A)]}{\mathrm{Im}[\partial_{n+1} : C_{n+1}(X, A) \to C_n(X, A)]},$$

we find that $\bar{\varphi} = \varphi + C_n(A)$ (where $\varphi \in C_n(X)$) is such that

$$\partial_n(\varphi + C_n(A)) = \partial_n^X \varphi + C_{n-1}(A) = C_{n-1}(A),$$

and hence $\partial_n^X \varphi \in C_{n-1}(A)$; that is, φ is a "relative cycle". The homomorphism $\hat{\partial}_n$ will map the class $[\bar{\varphi}]$ represented by the relative cycle φ to the class in $H_{n-1}(A, B)$ represented by the relative cycle $\partial_n^X \varphi$, denoted here by $[\overline{\partial_n^X \varphi}]$. Recalling

$$H_{n-1}(A, B) = \frac{\mathrm{Ker}[\partial_{n-1} : C_{n-1}(A, B) \to C_{n-2}(A, B)]}{\mathrm{Im}[\partial_n : C_n(A, B) \to C_{n-1}(A, B)]},$$

we see that $[\overline{\partial_n^X \varphi}] = (\partial_n^X \varphi + C_{n-1}(B)) + \partial_n(C_n(A, B))$. Hence,

$$\hat{\partial}_n[\bar{\varphi}] = \hat{\partial}_n((\varphi + C_n(A)) + \partial_{n+1}(C_{n+1}(X, A)))$$

$$= (\partial_n^X \varphi + C_{n-1}(B)) + \partial_n(C_n(A, B)).$$

Now, the map $\bar{\partial}_n : H_n(\Delta^n, \partial\Delta^n) \to H_{n-1}(\partial\Delta^n, \Lambda)$ above operates in an analogous fashion. For $[\bar{\imath}_n] \in H_n(\Delta^n, \partial\Delta^n)$ we obtain

$$\begin{aligned}\bar{\partial}_n[\bar{\imath}_n] &= \bar{\partial}_n((i_n + C_n(\partial\Delta^n)) + \partial_{n+1}(C_{n+1}(\Delta^n, \partial\Delta^n))) \\ &= (\partial_n^{\Delta^n} i_n + C_{n-1}(\Lambda)) + \partial_n(C_n(\partial\Delta^n, \Lambda)).\end{aligned}$$

As noted earlier, $\partial_n^{\Delta^n} i_n$ (or simply ∂i_n) is the singular n-chain $\sum_{k=0}^n (-1)^k i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]}$, where for each $0 \leq k \leq n$, $i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]}$ acts as the "inclusion map" of Δ^{n-1} onto the kth face of Δ^n (i.e. the face obtained when the kth vertex of Δ^n is deleted). All but one of the maps $i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]}$ therefore maps to Λ. So if Λ is, say, the union of all but the 0th face of Δ^n (i.e. all but $[\hat{u}_0, u_1, \ldots, u_n]$), then it follows that $i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]} : \Delta^{n-1} \to \Lambda$ for all $k \neq 0$, and thus $(-1)^k i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]} \in C_{n-1}(\Lambda)$ for all $k \neq 0$ and we have

$$\begin{aligned}\bar{\partial}_n[\bar{\imath}_n] &= \left(\sum_{k=0}^n (-1)^k i_n|_{[u_0,\ldots,\hat{u}_k,\ldots,u_n]} + C_{n-1}(\Lambda)\right) + \partial_n(C_n(\partial\Delta^n, \Lambda)) \\ &= (i_n|_{[\hat{u}_0,u_1,\ldots,u_n]} + C_{n-1}(\Lambda)) + \partial_n(C_n(\partial\Delta^n, \Lambda)) \\ &= \left[\,\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\,\right]. \end{aligned} \tag{16}$$

Thus the class

$$\left[\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\right] \in H_{n-1}(\partial\Delta^n, \Lambda)$$

corresponds via an isomorphism to the class $[\bar{\imath}_n] \in H_n(\Delta^n, \partial\Delta^n)$.

The next matter to verify is that there exists an isomorphism

$$\psi : H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \to H_{n-1}(\partial\Delta^n, \Lambda).$$

The map ψ will be shown to be equal to a composition of five isomorphisms $p_*^{-1} \circ \beta_* \circ \ell_* \circ \alpha_* \circ q_*$, where

$$\begin{aligned}H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) &\xrightarrow{q_*} H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}, \partial\Delta^{n-1}/\partial\Delta^{n-1}) \xrightarrow{\alpha_*} \widetilde{H}_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \\ &\xrightarrow{\ell_*} \widetilde{H}_{n-1}(\partial\Delta^n/\Lambda) \xrightarrow{\beta_*} H_{n-1}(\partial\Delta^n/\Lambda, \Lambda/\Lambda) \xrightarrow{p_*^{-1}} H_{n-1}(\partial\Delta^n, \Lambda).\end{aligned}$$

The quotient map

$$q : (\Delta^{n-1}, \partial\Delta^{n-1}) \to (\Delta^{n-1}/\partial\Delta^{n-1}, \partial\Delta^{n-1}/\partial\Delta^{n-1})$$

given by

$$q(x) = x + \partial\Delta^{n-1}$$

for each $x \in \Delta^{n-1}$ induces the homomorphism q_* defined by

$$\begin{aligned}&q_*((\varphi + C_{n-1}(\partial\Delta^{n-1})) + \partial_n(C_n(\Delta^{n-1}, \partial\Delta^{n-1}))) \\ &\qquad = (q \circ \varphi + C_{n-1}(\partial\Delta^{n-1}/\partial\Delta^{n-1})) + \partial_n(C_n(\Delta^{n-1}/\partial\Delta^{n-1}, \partial\Delta^{n-1}/\partial\Delta^{n-1})),\end{aligned}$$

where for $\varphi = \sum_k m_k\sigma_k$ we take $q \circ \varphi = \sum_k m_k(q \circ \sigma_k)$. By Proposition 2.23 q_* is in fact an isomorphism.

The isomorphism α_* is quite natural. Given that

$$\widetilde{H}_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) = \frac{\mathrm{Ker}[\partial_{n-1} : C_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \to C_{n-2}(\Delta^{n-1}/\partial\Delta^{n-1})]}{\mathrm{Im}[\partial_n : C_n(\Delta^{n-1}/\partial\Delta^{n-1}) \to C_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1})]},$$

we have

$$\alpha_*((\varphi + C_{n-1}(\partial\Delta^{n-1}/\partial\Delta^{n-1})) + \partial_n(C_n(\Delta^{n-1}/\partial\Delta^{n-1}, \partial\Delta^{n-1}/\partial\Delta^{n-1}))) \\ = \varphi + \partial_n(C_n(\Delta^{n-1}, \partial\Delta^{n-1})).$$

The proof that α_* is an isomorphism should be straightforward and will be omitted here.

Next we develop the isomorphism ℓ_*. We start with $\iota : \Delta^{n-1} \to \partial\Delta^n$, defined to be the "inclusion map" of Δ^{n-1} onto the face of Δ^n *not* included in Λ. The map ι induces

$$\ell : \Delta^{n-1}/\partial\Delta^{n-1} \to \partial\Delta^n/\Lambda,$$

given by

$$\ell(x + \partial\Delta^{n-1}) = \iota(x) + \Lambda$$

for all $x \in \Delta^{n-1}$. Clearly ℓ is bijective and continuous, and since $\Delta^{n-1}/\partial\Delta^{n-1}$ is compact (it's homeomorphic to $\mathbb{S}^{n-1}$) it follows that ℓ is a homeomorphism. Now, ℓ induces a homomorphism ℓ_*: recalling

$$\widetilde{H}_{n-1}(\partial\Delta^n/\Lambda) = \frac{\mathrm{Ker}[\partial_{n-1} : C_{n-1}(\partial\Delta^n/\Lambda) \to C_{n-2}(\partial\Delta^n/\Lambda)]}{\mathrm{Im}[\partial_n : C_n(\partial\Delta^n/\Lambda) \to C_{n-1}(\partial\Delta^n/\Lambda)]},$$

we find that

$$\ell_*(\varphi + \partial_n(C_n(\Delta^{n-1}/\partial\Delta^{n-1}))) = \ell \circ \varphi + \partial_n(C_n(\partial\Delta^n/\Lambda)),$$

and since ℓ_* is induced by a homotopy equivalence it must be an isomorphism.

The homomorphism β_*, like α_*, is a quite natural map. For

$$\varphi \in \mathrm{Ker}[\partial_{n-1} : C_{n-1}(\partial\Delta^n/\Lambda) \to C_{n-2}(\partial\Delta^n/\Lambda)]$$

we obtain

$$\beta_*(\varphi + \partial_n(C_n(\partial\Delta^n/\Lambda))) = (\varphi + C_{n-1}(\Lambda/\Lambda)) + \partial_n(C_n(\partial\Delta^n/\Lambda, \Lambda/\Lambda)).$$

The map p_*^{-1} is the inverse of the isomorphism

$$p_* : H_{n-1}(\partial\Delta^n, \Lambda) \to H_{n-1}(\partial\Delta^n/\Lambda, \Lambda/\Lambda)$$

induced by the quotient map

$$p : (\partial\Delta^n, \Lambda) \to (\partial\Delta^n/\Lambda, \Lambda/\Lambda)$$

given by $p(x) = x + \Lambda$. Specifically p_* is given by

$$p_*((\varphi + C_{n-1}(\Lambda)) + \partial_n(C_n(\partial\Delta^n, \Lambda))) = (p \circ \varphi + C_{n-1}(\Lambda/\Lambda)) + \partial_n(C_n(\partial\Delta^n/\Lambda, \Lambda/\Lambda)).$$

By the inductive hypothesis $[\bar{\imath}_{n-1}]$ is a generator for $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. Now,

$$\psi([\bar{\imath}_{n-1}]) = p^{-1}(\beta_*(\ell_*(\alpha_*(q_*((i_{n-1} + C_{n-1}(\partial\Delta^{n-1})) + \partial_n(C_n(\Delta^{n-1}, \partial\Delta^{n-1})))))))$$

$$
\begin{aligned}
&= p^{-1}(\beta_*(\ell_*(\alpha_*((q \circ i_{n-1} + C_{n-1}(\partial\Delta^{n-1}/\partial\Delta^{n-1}))\\
&\quad + \partial_n(C_n(\Delta^{n-1}/\partial\Delta^{n-1}, \partial\Delta^{n-1}/\partial\Delta^{n-1}))))))\\
&= p^{-1}(\beta_*(\ell_*(q \circ i_{n-1} + \partial_n(C_n(\Delta^{n-1}/\partial\Delta^{n-1})))))\\
&= p^{-1}(\beta_*(\ell \circ q \circ i_{n-1} + \partial_n(C_n(\partial\Delta^n/\Lambda))))\\
&= p^{-1}((\ell \circ q \circ i_{n-1} + C_{n-1}(\Lambda/\Lambda)) + \partial_n(C_n(\partial\Delta^n/\Lambda, \Lambda/\Lambda))). \qquad (17)
\end{aligned}
$$

For the sake of definiteness let us again assume that Λ is the union of all but the 0th face of Δ^n. Given this assumption, recall from (16) the class

$$\left[\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\right] \in H_{n-1}(\partial\Delta^n, \Lambda).$$

We find that

$$p_*\left(\left[\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\right]\right) = (p \circ i_n|_{[\hat{u}_0,u_1,\ldots,u_n]} + C_{n-1}(\Lambda/\Lambda)) + \partial_n(C_n(\partial\Delta^n/\Lambda, \Lambda/\Lambda)),$$

where $p \circ i_n|_{[\hat{u}_0,u_1,\ldots,u_n]} : \Delta^{n-1} \to \partial\Delta^n/\Lambda$ such that, for $x \in \Delta^{n-1}$, we have

$$p(i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}(x)) = p(x \in \text{0th face of } \Delta^n) = (x \in \text{0th face of } \Delta^n) + \Lambda.$$

On the other hand there is also the map $\ell \circ q \circ i_{n-1} : \Delta^{n-1} \to \partial\Delta^n/\Lambda$ which, for any $x \in \Delta^{n-1}$, yields

$$\ell(q(i_{n-1}(x))) = \ell(q(x)) = \ell(x + \partial\Delta^{n-1}) = \iota(x) + \Lambda = (x \in \text{0th face of } \Delta^n) + \Lambda.$$

Hence $p \circ i_n|_{[\hat{u}_0,u_1,\ldots,u_n]} = \ell \circ q \circ i_{n-1}$, which shows that

$$p_*\left(\left[\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\right]\right) = (\ell \circ q \circ i_{n-1} + C_{n-1}(\Lambda/\Lambda)) + \partial_n(C_n(\partial\Delta^n/\Lambda, \Lambda/\Lambda)),$$

and therefore since p_* is an isomorphism

$$p_*^{-1}((\ell \circ q \circ i_{n-1} + C_{n-1}(\Lambda/\Lambda)) + \partial_n(C_n(\partial\Delta^n/\Lambda, \Lambda/\Lambda))) = \left[\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\right].$$

So, from (17) it's seen that

$$\psi([\bar{\imath}_{n-1}]) = \left[\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\right],$$

which together with (16) yields

$$\bar{\partial}_n[\bar{\imath}_n] = \left[\overline{i_n|_{[\hat{u}_0,u_1,\ldots,u_n]}}\right] = \psi([\bar{\imath}_{n-1}]).$$

Thus we find that $[\bar{\imath}_n] \in H_n(\Delta^n, \partial\Delta^n)$ corresponds via the isomorphism $\psi^{-1} \circ \bar{\partial}$ to the class $[\bar{\imath}_{n-1}] \in H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$, and since $[\bar{\imath}_{n-1}]$ is a generator for $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$, it follows that $[\bar{\imath}_n]$ is a generator for $H_n(\Delta^n, \partial\Delta^n)$.

Therefore for all $n \geq 0$ the singular n-simplex $h \circ i_n : \Delta^n \to D^n$ is a relative cycle that represents a generator for $H_n(\mathbb{D}^n, \partial\mathbb{D}^n)$, where $h : (\Delta^n, \partial\Delta^n) \to (\mathbb{D}^n, \partial\mathbb{D}^n)$ is taken to be any homeomorphism. ∎

Example 2.25. We can build on the result of the previous example by finding an explicit cycle representing a generator of the infinite cyclic group $H_n(\mathbb{S}^n)$ for each $n \geq 1$.

To start, define two singular n-simplices $\tau_1 : \Delta^n \to \mathbb{S}^n$ and $\tau_2 : \Delta^n \to \mathbb{S}^n$, where τ_1 maps to one hemisphere A of $\mathbb{S}^n$, and τ_2 maps to the opposite hemisphere B such that $\tau_2 = \rho \circ \tau_1$ with $\rho : \mathbb{S}^n \to \mathbb{S}^n$ being reflection about the plane containing $A \cap B$. In the $n = 1$ case τ_1 can be thought of as mapping $\Delta^1 = [u_0, u_1]$ to $\mathbb{S}^1 \subset \mathbb{R}^2$ in linear fashion from $p = (1, 0)$ to $q = (-1, 0)$ through $\mathbb{H}^2_+$ (the "upper" semicircle), while τ_2 maps $[u_0, u_1]$ from p to q through $\mathbb{H}^2_-$ (the "lower" semicircle). Now,

$$\begin{aligned}\partial(\tau_1 - \tau_2) &= \partial\tau_1 - \partial\tau_2 \\ &= (\tau_1|_{[\hat{u}_0,u_1]} - \tau_1|_{[u_0,\hat{u}_1]}) - (\tau_2|_{[\hat{u}_0,u_1]} - \tau_2|_{[u_0,\hat{u}_1]}) \\ &= (q - p) - (q - p) = 0,\end{aligned}$$

so we see that $\tau_1 - \tau_2 \in C_1(\mathbb{S}^1)$ is a cycle, and in general $\tau_1 - \tau_2 \in C_n(\mathbb{S}^n)$ is a cycle for each $n \geq 1$. The claim will be that $[\tau_1 - \tau_2] \in H_n(\mathbb{S}^n)$ is a generator for the group. To establish this, we will examine the isomorphisms in the diagram

$$H_n(\mathbb{S}^n) \xrightarrow{j_*} H_n(\mathbb{S}^n, \tau_2(\Delta^n)) \xleftarrow{\cong} H_n(\partial\Delta^{n+1}, \Lambda) \xleftarrow{\psi} H_n(\Delta^n, \partial\Delta^n) \tag{18}$$

for $n \geq 1$. Note that $\tau_2(\Delta^n)$, being a closed hemisphere of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, is homeomorphic to $\mathbb{D}^n$, which in turn is homeomorphic to Δ^n itself.

The first isomorphism in (18) is precisely the map $j_{n*} : H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n, \tau_2(\Delta^n))$ in the long exact sequence for the pair $(\mathbb{S}^n, \tau_2(\Delta^n))$, which by definition is induced by the quotient map

$$j : C_n(\mathbb{S}^n) \to C_n(\mathbb{S}^n, \tau_2(\Delta^n))$$

given by $j(\varphi) = \varphi + C_n(\tau_2(\Delta^n))$, and therefore for any singular n-chain $\varphi \in \operatorname{Ker}[\partial_n : C_n(\mathbb{S}^n) \to C_{n-1}(\mathbb{S}^n)]$ we have

$$\begin{aligned}j_{n*}(\varphi + \partial_{n+1}(C_{n+1}(\mathbb{S}^n))) &= j(\varphi) + \partial_{n+1}(C_{n+1}(\mathbb{S}^n, \tau_2(\Delta^n))) \\ &= (\varphi + C_n(\tau_2(\Delta^n))) + \partial_{n+1}(C_{n+1}(\mathbb{S}^n, \tau_2(\Delta^n))).\end{aligned}$$

In particular,

$$\begin{aligned}j_{n*}([\tau_1 - \tau_2]) &= j_{n*}((\tau_1 - \tau_2) + \partial_{n+1}(C_{n+1}(\mathbb{S}^n))) \\ &= ((\tau_1 - \tau_2) + C_n(\tau_2(\Delta^n))) + \partial_{n+1}(C_{n+1}(\mathbb{S}^n, \tau_2(\Delta^n))) \\ &= (\tau_1 + C_n(\tau_2(\Delta^n))) + \partial_{n+1}(C_{n+1}(\mathbb{S}^n, \tau_2(\Delta^n))),\end{aligned}$$

where the last equality holds since of course $\tau_2 \in C_n(\tau_2(\Delta^n))$.

The third isomorphism is the map ψ of previous acquaintance, only with the integer $n - 1$ replaced with n. It's known from Example 2.24 that $[\bar{\iota}_n] \in H_n(\Delta^n, \partial\Delta^n)$ is a generator for the group, and if we assume that Λ is the union of all but the 0th face of Δ^{n+1} (which is opposite the vertex u_0), then

$$\psi([\bar{\imath}_n]) = (i_{n+1}|_{[\hat{u}_0,u_1,\ldots,u_{n+1}]} + C_n(\Lambda)) + \partial_{n+1}(C_{n+1}(\partial\Delta^{n+1},\Lambda))$$

Noting that $\partial\Delta^{n+1}$ is homeomorphic to $\mathbb{S}^n$ and Λ is homeomorphic to $\tau_2(\Delta^n)$, the second isomorphism above is induced by a homeomorphism

$$h : (\partial\Delta^{n+1},\Lambda) \to (\mathbb{S}^n, \tau_2(\Delta^n))$$

that can be constructed using the Pasting Lemma. Let λ denote the 0th face of Δ^{n+1}, so $\partial\Delta^{n+1} = \Lambda\cup\lambda$. Let $T : \lambda \to \Delta^n$ be the canonical linear homeomorphism and define $h|_\lambda = \tau_1\circ T$, so $h|_\lambda$ maps the 0th face of $\partial\Delta^{n+1}$ homeomorphically onto the hemisphere $\tau_1(\Delta^n)$ of $\mathbb{S}^n$. Next, let $L : \Lambda \to \Delta^n$ be a homeomorphism such that $L|_{\Lambda\cap\lambda} = T|_{\Lambda\cap\lambda}$ (observe that $\partial\Lambda = \Lambda\cap\lambda = \partial\lambda$), and define $h|_\Lambda = \tau_2 \circ L$. Then $h|_\Lambda$ maps Λ homeomorphically onto the hemisphere $\tau_2(\Delta^n)$, and since $\tau_1|_{\partial\Delta^n} = \tau_1|_{\partial\Delta^n}$ it follows that $h|_\lambda(x) = h|_\Lambda(x)$ for all $x \in \Lambda\cap\lambda$ and thus $h : \partial\Delta^{n+1} \to \mathbb{S}^n$ is a homeomorphism. It induces isomorphisms

$$h_{n*} : H_n(\partial\Delta^{n+1},\Lambda) \to H_n(\mathbb{S}^n, \tau_2(\Delta^n)).$$

given by

$$\begin{aligned} h_{n*}((i_{n+1}|_{[\hat{u}_0,u_1,\ldots,u_{n+1}]} &+ C_n(\Lambda)) + \partial_{n+1}(C_{n+1}(\partial\Delta^{n+1},\Lambda))) \\ &= (h\circ i_{n+1}|_{[\hat{u}_0,u_1,\ldots,u_{n+1}]} + C_n(\tau_2(\Delta^n))) + \partial(C_{n+1}(\mathbb{S}^n,\tau_2(\Delta^n))) \\ &= (h\circ i_{n+1}|_\lambda + C_n(\tau_2(\Delta^n))) + \partial(C_{n+1}(\mathbb{S}^n,\tau_2(\Delta^n))). \end{aligned}$$

Here $h\circ i_{n+1}|_\lambda$ is as usual implicitly precomposed with a canonical linear homeomorphism $\Delta^n \mapsto \lambda$ which is in fact T^{-1}. Now, for $x \in \Delta^n$, we have

$$\begin{aligned} (h\circ i_{n+1}|_\lambda \circ T^{-1})(x) &= h(i_{n+1}|_\lambda(T^{-1}(x))) \\ &= h(T^{-1}(x)) && \text{(since } T^{-1}(x) \in \lambda\text{)} \\ &= h|_\lambda(T^{-1}(x)) \\ &= (\tau_1\circ T)(T^{-1}(x)) && \text{(by definition of } h|_\lambda\text{)} \\ &= \tau_1(x), \end{aligned}$$

and so $h\circ i_{n+1}|_\lambda \circ T^{-1} = \tau_1$. Suppressing T^{-1} as is customary, it's seen that

$$\begin{aligned} h_{n*}((i_{n+1}|_{[\hat{u}_0,u_1,\ldots,u_{n+1}]} &+ C_n(\Lambda)) + \partial_{n+1}(C_{n+1}(\partial\Delta^{n+1},\Lambda))) \\ &= (\tau_1 + C_n(\tau_2(\Delta^n))) + \partial_{n+1}(C_{n+1}(\mathbb{S}^n,\tau_2(\Delta^n))). \end{aligned}$$

Hence

$$h_{n*}(\psi([\bar{\imath}_n])) = (\tau_1 + C_n(\tau_2(\Delta^n))) + \partial_{n+1}(C_{n+1}(\mathbb{S}^n,\tau_2(\Delta^n))) = j_{n*}([\tau_1 - \tau_2]),$$

or simply

$$[\tau_1 - \tau_2] = ((j_{n*})^{-1} \circ h_{n*} \circ \psi)([\bar{\imath}_n]).$$

So $[\tau_1 - \tau_2] \in H_n(\mathbb{S}^n)$ corresponds via isomorphism to the generator $[\bar{\imath}_n] \in H_n(\Delta^n, \partial\Delta^n)$, which shows that $[\tau_1 - \tau_2]$ is a generator for $H_n(\mathbb{S}^n)$. ■

Example 2.26. We are now in a position to find explicit generators for $H_1(\mathbb{T}^2)$, which will prove useful later on when we turn our attention toward calculating homology groups using Mayer-Vietoris sequences.

This will be done by finding the generators for $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ (where we view $\mathbb{S}^1 \times \mathbb{S}^1$ as a subspace of $\mathbb{R}^4$) and invoking the usual homeomorphism $\mathbb{S}^1 \times \mathbb{S}^1 \mapsto \mathbb{T}^2$. Considering two copies of the 1-sphere, $\mathbb{S}^1_1$ and $\mathbb{S}^1_2$, we define $\tau_1, \tau_2 : \Delta^1 \to \mathbb{S}^1_1$ as in Example 2.25, and we define $\sigma_1, \sigma_2 : \Delta^1 \to \mathbb{S}^1_2$ the same way. Then $[\tau_1 - \tau_2]$ is an explicit generator for $H_1(\mathbb{S}^1_1)$ and $[\sigma_1 - \sigma_2]$ is an explicit generator for $H_1(\mathbb{S}^1_2)$, and it follows that $\{([\tau_1 - \tau_2], 0), (0, [\sigma_1 - \sigma_2])\}$ is a basis for the free abelian group $H_1(\mathbb{S}^1_1) \oplus H_1(\mathbb{S}^1_2)$.

Define $\iota_1 : \mathbb{S}^1_1 \to \mathbb{S}^1_1 \times \mathbb{S}^1_2$ and $\iota_2 : \mathbb{S}^1_2 \to \mathbb{S}^1_1 \times \mathbb{S}^1_2$ by

$$\iota_1(x_1, y_1) = ((x_1, y_1), (1, 0)) \quad \text{and} \quad \iota_2(x_2, y_2) = ((1, 0), (x_2, y_2))$$

(viewing each copy of the 1-sphere as a subspace of $\mathbb{R}^2$). Also define $\hat{\tau}_i = \iota_i \circ \tau_i$ and $\hat{\sigma}_i = \iota_i \circ \sigma_i$, all being maps $\Delta^1 \mapsto \mathbb{S}^1_1 \times \mathbb{S}^1_2$. Finally, define $[\![\xi]\!] := \xi + \partial_2(C_2(\mathbb{S}^1_1 \times \mathbb{S}^1_2))$ for any cycle ξ. The claim here is that $[\![\hat{\tau}_1 - \hat{\tau}_2]\!]$ and $[\![\hat{\sigma}_1 - \hat{\sigma}_2]\!]$ are generators for $H_1(\mathbb{S}^1_1 \times \mathbb{S}^1_2) \cong \mathbb{Z} \oplus \mathbb{Z}$. To verify this claim, define a homomorphism

$$\omega : H_1(\mathbb{S}^1_1) \oplus H_1(\mathbb{S}^1_2) \to H_1(\mathbb{S}^1_1 \times \mathbb{S}^1_2)$$

by

$$\begin{aligned} \omega(m[\tau_1 - \tau_2], n[\sigma_1 - \sigma_2]) &:= m\,\omega([\tau_1 - \tau_2], 0) + n\,\omega(0, [\sigma_1 - \sigma_2]) \\ &= m[\![\hat{\tau}_1 - \hat{\tau}_2]\!] + n[\![\hat{\sigma}_1 - \hat{\sigma}_2]\!] \\ &= [\![m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\sigma}_1 - \hat{\sigma}_2)]\!]. \end{aligned}$$

It remains to show that ω is an isomorphism.

Suppose $\omega\,(m[\tau_1 - \tau_2], n[\sigma_1 - \sigma_2]) = [\![0]\!]$, so

$$m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\sigma}_1 - \hat{\sigma}_2) \in \partial_2(C_2(\mathbb{S}^1_1 \times \mathbb{S}^1_2))$$

and there exists some

$$\alpha = \sum_{i=1}^{\ell} k_i \alpha_i \in C_2(\mathbb{S}^1_1 \times \mathbb{S}^1_2)$$

such that $\partial_2(\alpha) = m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\sigma}_1 - \hat{\sigma}_2)$ (here $\alpha_i : \Delta^n \to \mathbb{S}^1_1 \times \mathbb{S}^1_2$ for each i). That is,

$$\sum_{i=1}^{\ell} k_i \left(\alpha_i|_{[u_1,u_2]} - \alpha_i|_{[u_0,u_2]} + \alpha_i|_{[u_0,u_1]} \right) = m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\sigma}_1 - \hat{\sigma}_2).$$

Assume to start that $n = 0$, so $\partial_2(\alpha) = m(\hat{\tau}_1 - \hat{\tau}_2)$. Letting $x \in \Delta^2$, then for each i, $\alpha_i(x) = (f_i^1(x); f_i^2(x))$ for $f_i^j : \Delta^2 \to \mathbb{S}_j^1$. Defining $\tilde{\alpha}_i : \Delta^2 \to \mathbb{S}_1^1$ by $\tilde{\alpha}_i(x) = f_i^1(x)$, we readily find for

$$\tilde{\alpha} = \sum_{i=1}^{\ell} k_i \tilde{\alpha}_i$$

that $\partial_2(\tilde{\alpha}) = m(\tau_1 - \tau_2)$ and hence $m(\tau_1 - \tau_2) \in \partial_2(C_2(\mathbb{S}_1^1))$. Now, since $\tau_1 - \tau_2$ is a generator for $H_1(\mathbb{S}_1^1)$ we cannot have $m(\tau_1 - \tau_2) \in \partial_2(C_2(\mathbb{S}_1^1))$ unless $m = 0$, otherwise we obtain $H_1(\mathbb{S}_1^1) \cong \mathbb{Z}_k$ for some $1 \leq k \leq |m|$. Thus if $n = 0$, then $m = 0$ also. Similarly $m = 0$ must imply that $n = 0$.

We must rule out the possibility that $m, n \neq 0$. To do this, for each i we redefine α_i such that any restriction of α_i to a face of Δ^2 that equals $\hat{\sigma}_1$ or $\hat{\sigma}_2$ is replaced with $\hat{\tau}_1$ or $\hat{\tau}_2$, respectively—along with a corresponding change in the definition of α_i in the interior of Δ^2 to maintain continuity. In this way we obtain a new map $\tilde{\alpha}_i$, and by extension a new chain $\tilde{\alpha}$ such that

$$\partial_2(\tilde{\alpha}) = m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\tau}_1 - \hat{\tau}_2) = (m+n)(\hat{\tau}_1 - \hat{\tau}_2).$$

Then $m + n = 0$ must hold, or $n = -m$, and we're led to conclude that

$$\partial_2(\alpha) = m(\hat{\tau}_1 - \hat{\tau}_2) - m(\hat{\sigma}_1 - \hat{\sigma}_2).$$

From here, manipulating α_i's to replace the cycle $\hat{\sigma}_1 - \hat{\sigma}_2$ with $\hat{\sigma}_2 - \hat{\sigma}_1$ will give $2m = 0$ and hence $m, n = 0$.

So $\omega\left(m[\tau_1 - \tau_2], n[\sigma_1 - \sigma_2]\right) = [\![0]\!]$ implies that $(m[\tau_1 - \tau_2], n[\sigma_1 - \sigma_2]) = (0, 0)$, which shows that $\operatorname{Ker}\omega$ is trivial and hence ω is injective.

Now let $[\![\xi]\!] \in H_1(\mathbb{S}_1^1 \times \mathbb{S}_2^1)$ be arbitrary. Then the 1-chain $\xi = \sum_{i=1}^{\ell} k_i \xi_i$ is a cycle, so $\xi_i : \Delta^1 \to \mathbb{S}_1^1 \times \mathbb{S}_2^1$ such that

$$\sum_{i=1}^{\ell} k_i(\xi_i(u_1) - \xi_i(u_0)) = 0.$$

Such a cycle will be homotopic to the cycle

$$m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\sigma}_1 - \hat{\sigma}_2)$$

for some $m, n \in \mathbb{Z}$, which is a loop based at $((1,0),(1,0))$; thus, since homotopic cycles are homologous, we find that

$$\omega(m[\tau_1 - \tau_2], n[\sigma_1 - \sigma_2]) = [\![m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\sigma}_1 - \hat{\sigma}_2)]\!] = [\![\xi]\!]$$

for $(m[\tau_1 - \tau_2], n[\sigma_1 - \sigma_2]) \in H_1(\mathbb{S}_1^1) \oplus H_1(\mathbb{S}_2^1)$. It follows that ω is surjective.

It's now established that ω is an isomorphism, and since $[\![\hat{\tau}_1 - \hat{\tau}_2]\!]$ and $[\![\hat{\sigma}_1 - \hat{\sigma}_2]\!]$ correspond via this isomorphism to the generators $([\tau_1 - \tau_2], 0)$ and $(0, [\sigma_1 - \sigma_2])$ for $H_1(\mathbb{S}_1^1) \oplus H_1(\mathbb{S}_2^1)$, it follows that

$$[\![\hat{\tau}_1 - \hat{\tau}_2]\!],\ [\![\hat{\sigma}_1 - \hat{\sigma}_2]\!] \in H_1(\mathbb{S}_1^1 \times \mathbb{S}_2^1)$$

are explicit generators for $H_1(\mathbb{S}_1^1 \times \mathbb{S}_2^1)$. Now, if $h : \mathbb{S}_1^1 \times \mathbb{S}_2^1 \to \mathbb{T}^2$ is any homeomorphism, then $[\![h \circ \hat{\tau}_1 - h \circ \hat{\tau}_2]\!]$ and $[\![h \circ \hat{\sigma}_1 - h \circ \hat{\sigma}_2]\!]$ are generators for $H_1(\mathbb{T}^2)$. ■

Example 2.27. In keeping with the philosophy that being explicit oftentimes trumps being "elegant" at least when it comes to being pedagogically useful, what follows is a simple example of how a path in $\mathbb{S}^1_1 \times \mathbb{S}^1_2$ can be homotoped to a path of the form $m(\hat{\tau}_1 - \hat{\tau}_2) + n(\hat{\sigma}_1 - \hat{\sigma}_2)$.

Let $\gamma : [0,1] \to \mathbb{S}^1_1 \times \mathbb{S}^1_2$ be given by

$$\gamma(t) = ((1, 4\pi t), (1, 2\pi t)),$$

so γ is a path in $\mathbb{S}^1_1 \times \mathbb{S}^1_2$ that runs twice around $\mathbb{S}^1_1$ whilst simultaneously running once around $\mathbb{S}^1_2$. (In $\mathbb{T}^2$ this map can be characterized as a path that wends twice around the girth of the torus while going once around its central hole.) Set $\gamma_0 = \gamma$ and define $\gamma_1 : [0,1] \to \mathbb{S}^1_1 \times \mathbb{S}^1_2$ by

$$\gamma_1(t) = \begin{cases} ((1, 8\pi t), (1, 0)), & \text{if } 0 \leq t \leq 1/2 \\ ((1, 4\pi), (1, 4\pi t - 2\pi)), & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Note that γ_1 is a path that first runs twice around $\mathbb{S}^1_1$ (staying stationary in $\mathbb{S}^1_2$), then runs once around $\mathbb{S}^1_2$ (staying stationary in $\mathbb{S}^1_1$). Thus γ_1 is in fact (homotopic to) the singular 1-chain $2(\hat{\tau}_1 - \hat{\tau}_2) + (\hat{\sigma}_1 - \hat{\sigma}_2)$. Now, let $\{\gamma_s\}_{s \in [0,1]}$ be the family of functions given by

$$\gamma_s(t) = \begin{cases} ((1, 4\pi(t + st)), (1, 2\pi(t - st))), & \text{if } 0 \leq t \leq 1/2 \\ ((1, 4\pi(t - st + s)), (1, 2\pi(t + st - s))), & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

It is straightforward to verify that $\{\gamma_s\}_{s \in [0,1]}$ is a homotopy, and therefore γ is homotopic to $2(\hat{\tau}_1 - \hat{\tau}_2) + (\hat{\sigma}_1 - \hat{\sigma}_2)$. ■

2.9 – Mayer-Vietoris Sequences

Let X be a topological space, and let $A, B \subset X$ such that $X = A^\circ \cup B^\circ$. Define $C_n(A+B)$ to be the subgroup of $C_n(X)$ consisting of chains that are reducible to the form $\sigma + \tau$,[5] where $\sigma \in C_n(A)$ and $\tau \in C_n(B)$. The boundary map $\partial_n^X : C_n(X) \to C_{n-1}(X)$ has restriction $\partial_n^+ : C_n(A+B) \to C_{n-1}(A+B)$, and also $\partial_n^\cap : C_n(A \cap B) \to C_{n-1}(A \cap B)$, $\partial_n^A : C_n(A) \to C_{n-1}(A)$, and $\partial_n^B : C_n(B) \to C_{n-1}(B)$. All boundary maps ∂ defined here give rise to chain complexes since $\partial\partial = 0$ always holds, and so in particular there are the homology groups

$$H_n^{A+B}(X) = \frac{\operatorname{Ker}[\partial_n^+ : C_n(A+B) \to C_{n-1}(A+B)]}{\operatorname{Im}[\partial_{n+1}^+ : C_{n+1}(A+B) \to C_n(A+B)]}.$$

Lemma 2.28. *Define $\varphi_n : C_n(A+B) \to C_n(A) \oplus C_n(B)$ by $\varphi_n(\sigma) = (\sigma, -\sigma)$, and define $\psi_n : C_n(A) \oplus C_n(B) \to C_n(A+B)$ by $\psi_n(\sigma, \tau) = \sigma + \tau$. Then*

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\varphi_n} C_n(A) \oplus C_n(B) \xrightarrow{\psi_n} C_n(A+B) \longrightarrow 0 \tag{19}$$

is a short exact sequence.

Proof. Let $\sigma \in \operatorname{Ker}\varphi_n$, so $\varphi_n(\sigma) = (\sigma, -\sigma) = (0,0)$. This implies that $\sigma = 0 \in C_n(A)$, and since a subgroup must have the same identity element as the group that contains it, it follows that $\sigma = 0 \in C_n(A \cap B)$. Thus $\operatorname{Ker}\varphi_n = \{0\}$.

For any $\sigma \in C_n(A \cap B)$ we have

$$\psi_n(\varphi_n(\sigma)) = \psi_n(\sigma, -\sigma) = \sigma + (-\sigma) = 0,$$

so $\operatorname{Im}\varphi_n \subset \operatorname{Ker}\psi_n$.

Next, suppose that $(\sigma, \tau) \in \operatorname{Ker}\psi_n$. Then $\psi_n(\sigma, \tau) = \sigma + \tau = 0$, implying that $\sigma = -\tau$ and so $\sigma \in C_n(A \cap B)$ (since $\sigma \in C_n(A)$, and $\tau \in C_n(B)$ implies $-\tau \in C_n(B)$). Now, $\varphi_n(\sigma) = (\sigma, -\sigma) = (\sigma, \tau)$ shows that $(\sigma, \tau) \in \operatorname{Im}\varphi_n$. So $\operatorname{Ker}\psi_n \subset \operatorname{Im}\varphi_n$.

Finally, fix $\xi \in C_n(A+B)$. Then $\xi = \sigma + \tau$ for some $\sigma \in C_n(A)$ and $\tau \in C_n(B)$. Now, $(\sigma, \tau) \in C_n(A) \oplus C_n(B)$, and $\psi_n(\sigma, \tau) = \sigma + \tau = \xi$ shows $\xi \in \operatorname{Im}\psi_n$. Therefore $\operatorname{Im}\psi_n = C_n(A+B)$. ■

The Mayer-Vietoris sequence is the long exact sequence of homology groups

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\Phi_n} H_n(A) \oplus H_n(B) \xrightarrow{\Psi_n} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(X) \longrightarrow 0 \tag{20}$$

associated with the short exact sequence of chain complexes in Figure 6 formed by the sequences (19), where we define $\partial_n^\circ = \partial_n^A \oplus \partial_n^B$. It must be verified that the diagram is commutative, and the definitions of the homomorphisms Φ_n, Ψ_n and ∂_n in (20) should be made explicit. Once this is done, it will be confirmed that (20) is indeed exact.

For $z \in C_n(A \cap B)$ we have $\varphi_{n-1}(\partial_n^\cap z) = (\partial_n^\cap z, -\partial_n^\cap z)$ while

$$\partial_n^\circ(\varphi_n(z)) = (\partial_n^A \oplus \partial_n^B)(z, -z) = (\partial_n^A z, \partial_n^B(-z)) = (\partial_n^\cap z, -\partial_n z),$$

[5] $C_n(A+B)$ is also denoted by $C_n^{\mathcal{U}}(X)$ for $\mathcal{U} = \{A, B\}$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{n+1}(A\cap B) & \xrightarrow{\partial^{\cap}_{n+1}} & C_n(A\cap B) & \xrightarrow{\partial^{\cap}_{n}} & C_{n-1}(A\cap B) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_{n}} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1}(A)\oplus C_{n+1}(B) & \xrightarrow{\partial^{\circ}_{n+1}} & C_n(A)\oplus C_n(B) & \xrightarrow{\partial^{\circ}_{n}} & C_{n-1}(A)\oplus C_{n-1}(B) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \psi_{n+1}} & & \downarrow{\scriptstyle \psi_{n}} & & \downarrow{\scriptstyle \psi_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1}(A+B) & \xrightarrow{\partial^{+}_{n+1}} & C_n(A+B) & \xrightarrow{\partial^{+}_{n}} & C_{n-1}(A+B) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 6

since $\partial^A_n|_{C_n(A\cap B)} = \partial^{\cap}_n$ and $\partial^B_n|_{C_n(A\cap B)} = \partial^{\cap}_n$. Hence $\varphi_{n-1}\circ\partial^{\cap}_n = \partial^{\circ}_n\circ\varphi_n$.

Next, for $(\sigma,\tau)\in C_n(A)\oplus C_n(B)$,

$$\psi_{n-1}(\partial^{\circ}_n(\sigma,\tau)) = \psi_{n-1}(\partial^A_n\sigma, \partial^B_n\tau) = \partial^A_n\sigma + \partial^B_n\tau,$$

while

$$\partial^+_n(\psi_n(\sigma,\tau)) = \partial^+_n(\sigma+\tau) = \partial^+_n\sigma + \partial^+_n\tau = \partial^A_n\sigma + \partial^B_n\tau,$$

where the last equality holds since $\partial^+_n : C_n(A+B)\to C_{n-1}(A+B)$ has restrictions $\partial^+_n|_{C_n(A)} = \partial^A_n$ and $\partial^+_n|_{C_n(B)} = \partial^B_n$. Hence $\psi_{n-1}\circ\partial^{\circ}_n = \partial^+_n\circ\psi_n$. So the diagram is commutative.

Define the map

$$\Phi_n : H_n(A\cap B)\to H_n(A)\oplus H_n(B)$$

by $\Phi_n([z]_\cap) = ([z]_A, [-z]_B)$. Given that

$$H_n(A\cap B) = \frac{\operatorname{Ker}[\partial^{\cap}_n : C_n(A\cap B)\to C_{n-1}(A\cap B)]}{\operatorname{Im}[\partial^{\cap}_{n+1} : C_{n+1}(A\cap B)\to C_n(A\cap B)]},$$

we could write more explicitly

$$\Phi_n(z+\partial^{\cap}_{n+1}(C_{n+1}(A\cap B))) = (z+\partial^A_{n+1}(C_{n+1}(A)), -z+\partial^B_{n+1}(C_{n+1}(B))).$$

We take $\Psi_n : H_n(A)\oplus H_n(B)\to H_n(X)$ as being given by $\Psi_n = \iota_*\circ\psi_*$, where the map $\psi_* : H_n(A)\oplus H_n(B)\to H^{A+B}_n(X)$ is the homomorphism induced by ψ_n:

$$\psi_*([\sigma]_A, [\tau]_B) = [\psi_n(\sigma,\tau)]_+ = [\sigma+\tau]_+ := (\sigma+\tau)+\partial^+_{n+1}(C_{n+1}(A+B)),$$

for any $\sigma\in\operatorname{Ker}\partial^A_n$ and $\tau\in\operatorname{Ker}\partial^B_n$. By Lemma 2.22 the inclusion map $\iota : C_n(A+B)\hookrightarrow C_n(X)$ induces an isomorphism $\iota_* : H^{A+B}_n(X)\to H_n(X)$ given by $\iota_*([z]_+) = [z]$, or more explicitly

$$\iota_*(z+\partial^+_{n+1}(C_{n+1}(A+B))) = z+\partial^X_{n+1}(C_{n+1}(X)).$$

Thus

$$\Psi_n([\sigma]_A, [\tau]_B) = \iota_*([\sigma+\tau]_+) = [\sigma+\tau],$$

where $\partial_n^A\sigma = 0$ and $\partial_n^B\tau = 0$ imply that

$$\partial_n^X(\sigma+\tau) = \partial_n^X\sigma + \partial_n^X\tau = \partial_n^A\sigma + \partial_n^B\tau = 0$$

as required.

Let $[x] \in H_n(X)$, so $x \in \operatorname{Ker}\partial_n^X \subset C_n(X)$. Since ι_* is an isomorphism there exists some $[\xi]_+ \in H_n^{A+B}(X)$ such that $\iota_*([\xi]_+) = [x]$, or $[\xi] = [x]$. Hence x is homologous to $\xi \in \operatorname{Ker}\partial_n^X$, and since $\xi \in \operatorname{Ker}\partial_n^+ \subset C_n(A+B)$ there exist $\sigma \in C_n(A)$ and $\tau \in C_n(B)$ such that $\xi = \sigma + \tau$, and thus $[x] = [\sigma + \tau]$. But $\partial_n^X\xi = 0$ implies that $\partial_n^X(\sigma+\tau) = 0$, so $\partial_n^X\sigma = -\partial_n^X\tau$; and since $\partial_n^X\sigma \in C_{n-1}(A)$ and $\partial_n^X\tau \in C_{n-1}(B)$, it follows that $\partial_n^X\sigma \in C_{n-1}(A) \cap C_{n-1}(B)$ and therefore $\partial_n^X\sigma \in C_{n-1}(A\cap B)$. Now to define $\partial_n : H_n(X) \to H_{n-1}(A\cap B)$. We have $[x] = [\sigma+\tau]$ for $\sigma+\tau \in C_n(A+B)$, and $\psi_n(\sigma,\tau) = \sigma+\tau$ for $(\sigma,\tau) \in C_n(A)\oplus C_n(B)$. Since

$$\psi_{n-1}(\partial_n^\circ(\sigma,\tau)) = \psi_{n-1}(\partial_n^A\sigma, \partial_n^B\tau) = \psi_{n-1}(\partial_n^X\sigma, \partial_n^X\tau) = \partial_n^X\sigma + \partial_n^X\tau = \partial_n^X(\sigma+\tau) = 0$$

shows that $\partial_n^\circ(\sigma,\tau) \in \operatorname{Ker}\psi_{n-1}$ and $\operatorname{Ker}\psi_{n-1} = \operatorname{Im}\varphi_{n-1}$, there must be some $z \in C_{n-1}(A\cap B)$ for which $\varphi_{n-1}(z) = \partial_n^\circ(\sigma,\tau) = (\partial_n^X\sigma, \partial_n^X\tau)$. However since

$$\varphi_{n-1}(\partial_n^X\sigma) = (\partial_n^X\sigma, -\partial_n^X\sigma) = (\partial_n^X\sigma, \partial_n^X\tau)$$

and φ_{n-1} is injective, it must be that $z = \partial_n^X\sigma$! We define $\partial_n([x]) = [\partial_n^X\sigma]_\cap$. This manner of defining ∂_n mirrors that of the previous section and thus is assured of being a well-defined homomorphism.

It remains to demonstrate that the sequence (20) is exact.

Proof.

• $\operatorname{Im}\Phi_n \subset \operatorname{Ker}\Psi_n$. Let $[z]_\cap \in H_n(A\cap B)$. Now,

$$\Psi_n(\Phi_n([z]_\cap) = \Psi_n([z]_A, [-z]_B) = [z+(-z)] = [0],$$

and so $\Phi_n([z]_\cap) \in \operatorname{Ker}\Psi_n$.

• $\operatorname{Im}\Psi_n \subset \operatorname{Ker}\partial_n$. Let $[z] \in \operatorname{Im}\Psi_n$. Then there is some $([x]_A, [y]_B) \in H_n(A)\oplus H_n(B)$ such that $\Psi_n([x]_A, [y]_B) = [z]$, whence $[z] = [x+y]$ with $x \in \operatorname{Ker}\partial_n^A \subset C_n(A)$ and $y \in \partial_n^B \subset C_n(B)$. By the definition of ∂_n,

$$\partial_n[z] = \partial_n[x+y] = [\partial_n^X x]_\cap = [\partial_n^A x]_\cap = [0]_\cap,$$

where the third equality holds since $\partial_n^A = \partial_n^X|_{C_n(A)}$. Hence $[z] \in \operatorname{Ker}\partial_n$.

• $\operatorname{Im}\partial_n \subset \operatorname{Ker}\Phi_{n-1}$. Let $[z]_\cap \in \operatorname{Im}\partial_n$, so there exists some $[x] \in H_n(X)$ with $\partial_n[x] = [z]_\cap$. As before, we can find some $\sigma \in C_n(A)$ and $\tau \in C_n(B)$ such that $[x] = [\sigma+\tau]$, and then

$$\partial_n[x] = \partial_n[\sigma+\tau] = [\partial_n^X\sigma]_\cap.$$

Thus $[z]_\cap = [\partial_n^X\sigma]_\cap$, and

$$\Phi_{n-1}([z]_\cap) = \Phi_{n-1}([\partial_n^X\sigma]_\cap) = ([\partial_n^A\sigma]_A, [-\partial_n^A\sigma]_B) = ([\partial_n^A\sigma]_A, [\partial_n^B\tau]_B),$$

where the last equality holds since

$$\partial_n^X(\sigma+\tau) = 0 \ \Rightarrow\ \partial_n^X\tau = -\partial_n^X\sigma \ \Rightarrow\ \partial_n^B\tau = -\partial_n^A\sigma.$$

Now,

$$[\partial_n^A \sigma]_A = \partial_n^A \sigma + \partial_n^A(C_n(A)) = \partial_n^A(C_n(A)) = [0]_A$$

and similarly $[\partial_n^B \tau]_B = [0]_B$. Thus we have $\Phi_{n-1}([z]_\cap) = ([0]_A, [0]_B)$, giving $[z]_\cap \in \operatorname{Ker} \Phi_{n-1}$.

• $\operatorname{Ker} \Psi_n \subset \operatorname{Im} \Phi_n$. Let $([\sigma]_A, [\tau]_B) \in \operatorname{Ker} \Psi_n$, so $\sigma \in \operatorname{Ker} \partial_n^A \subset C_n(A)$, $\tau \in \operatorname{Ker} \partial_n^B \subset C_n(B)$, and

$$\Psi([\sigma]_A, [\tau]_B) = \iota_*(\psi_*([\sigma]_A, [\tau]_B)) = 0$$

implies that $[\sigma + \tau]_+ = \psi_*([\sigma]_A, [\tau]_B) = [0]_+$ since ι_* is injective. Hence

$$(\sigma + \tau) + \partial_{n+1}^+(C_{n+1}(A + B)) = \partial_{n+1}^+(C_{n+1}(A + B)),$$

so that there exists some $c' \in C_{n+1}(A + B)$ such that $\partial_{n+1}^+ c' = \sigma + \tau = \psi_n(\sigma, \tau)$. Since ψ_{n+1} is surjective there exists some $(\sigma', \tau') \in C_{n+1}(A) \oplus C_{n+1}(B)$ such that $\psi_{n+1}(\sigma', \tau') = \sigma' + \tau' = c'$.

Now, $(\sigma, \tau) - \partial_{n+1}^\circ(\sigma', \tau') \in C_n(A) \oplus C_n(B)$, and

$$\begin{aligned}
\psi_n((\sigma, \tau) - \partial_{n+1}^\circ(\sigma', \tau')) &= \psi_n(\sigma, \tau) - \psi_n(\partial_{n+1}^\circ(\sigma', \tau')) \\
&= \psi_n(\sigma, \tau) - \partial_{n+1}^+(\psi_{n+1}(\sigma', \tau')) \\
&= (\sigma + \tau) - \partial_{n+1}^+ c' = (\sigma + \tau) - (\sigma + \tau) = 0
\end{aligned}$$

shows that

$$(\sigma, \tau) - \partial_{n+1}^\circ(\sigma', \tau') \in \operatorname{Ker} \psi_n = \operatorname{Im} \varphi_n,$$

so there's some $a \in C_n(A \cap B)$ such that $\varphi_n(a) = (\sigma, \tau) - \partial_{n+1}^\circ(\sigma', \tau')$. Since

$$(a, -a) = \varphi_n(a) = (\sigma, \tau) - (\partial_{n+1}^A \oplus \partial_{n+1}^B)(\sigma', \tau') = (\sigma - \partial_{n+1}^A \sigma', \tau - \partial_{n+1}^B \tau')$$

we have $a = \sigma - \partial_{n+1}^A \sigma'$ and $-a = \tau - \partial_{n+1}^B \tau'$.

Forging on,

$$\begin{aligned}
\varphi_{n-1}(\partial_n^\cap(a)) = \partial_n^\circ(\varphi_n(a)) &= \partial_n^\circ(\sigma, \tau) - \partial_n^\circ \partial_{n+1}^\circ(\sigma', \tau') \\
&= \partial_n^\circ(\sigma, \tau) = (\partial_n^A \sigma, \partial_n^B \tau) = (0, 0),
\end{aligned}$$

and since φ_{n-1} is injective we get $\partial_n^\cap(a) = 0$, so that $a \in \operatorname{Ker} \partial_n^\cap$ and $[a]_\cap \in H_n(A \cap B)$. Finally,

$$\Phi_n([a]_\cap) = ([a]_A, [-a]_B) = ([\sigma - \partial_{n+1}^A \sigma']_A, [\tau - \partial_{n+1}^B \tau']_B) = ([\sigma]_A, [\tau]_B),$$

where the last equality holds since

$$\begin{aligned}
[\sigma - \partial_{n+1}^A \sigma']_A &= (\sigma - \partial_{n+1}^A \sigma') + \partial_{n+1}^A(C_{n+1}(A)) \\
&= (\sigma + \partial_{n+1}^A(-\sigma')) + \partial_{n+1}^A(C_{n+1}(A)) \\
&= \sigma + \partial_{n+1}^A(C_{n+1}(A)) = [\sigma]_A,
\end{aligned}$$

and similarly for $[\tau]_B$. So $([\sigma]_A, [\tau]_B) \in \operatorname{Im} \Phi_n$.

• $\operatorname{Ker}\partial_n \subset \operatorname{Im}\Psi_n$. Recalling that $\Psi_n = \iota_* \circ \psi_*$, it will first be shown that $\operatorname{Ker}\partial_n \circ \iota_* \subset \operatorname{Im}\psi_*$. Let

$$[c]_+ \in \operatorname{Ker}\partial_n \circ \iota_* \subset H_n^{A+B}(X),$$

so $c \in \operatorname{Ker}\partial_n^+ \subset C_n(A+B)$ and there exists $\sigma \in C_n(A)$, $\tau \in C_n(B)$ such that $c = \sigma + \tau$. Set $b = (\sigma, \tau)$, so $\psi_n(b) = \sigma + \tau = c$. Now

$$\psi_{n-1}(\partial_n^\circ b) = \partial_n^+(\psi_n(b)) = \partial_n^+ c = 0$$

implies $\partial_n^\circ b \in \operatorname{Ker}\psi_{n-1} = \operatorname{Im}\varphi_{n-1}$, so there exists an $a \in C_{n-1}(A \cap B)$ such that

$$(a, -a) = \varphi_{n-1}(a) = \partial_n^\circ b = (\partial_n^A \sigma, \partial_n^B \tau).$$

We see that $\partial_n^X \sigma = \partial_n^A \sigma = a \in C_{n-1}(A \cap B)$, and from $\partial_{n-1}^\cap a = \partial_{n-1}^\cap \partial_n^X \sigma = 0$ it's clear that $[a]_\cap = [\partial_n^X \sigma]_\cap \in H_{n-1}(A \cap B)$.

By the definition of ∂_n,

$$(\partial_n \circ \iota_*)([c]_+) = \partial_n([c]) = \partial_n([\sigma + \tau]) = [\partial_n^X \sigma]_\cap = [a]_\cap.$$

Recalling $[c]_+ \in \operatorname{Ker}\partial \circ \iota_*$, we must have

$$[a]_\cap = a + \partial_n^\cap(C_n(A \cap B)) = \partial_n^\cap(C_n(A \cap B)),$$

and thus there's some $a' \in C_n(A \cap B)$ with $\partial_n^\cap a' = a$. The chain $b - \varphi_n(a') \in C_n(A) \oplus C_n(B)$ is a cycle:

$$\partial_n^\circ(b - \varphi_n(a')) = \partial_n^\circ b - \partial_n^\circ(\varphi_n(a')) = \partial_n^\circ b - \varphi_{n-1}(\partial_n^\cap a')$$

$$= \partial_n^\circ b - \varphi_{n-1}(a) = \partial_n^\circ b - \partial_n^\circ b = (0, 0);$$

but also we have

$$\partial_n^\circ(b - \varphi_n(a')) = (\partial_n^A \sigma, \partial_n^B \tau) - \partial_n^\circ(a', -a')$$

$$= (\partial_n^A \sigma, \partial_n^B \tau) - (\partial_n^A a', -\partial_n^B a')$$

$$= (\partial_n^A(\sigma - a'), \partial_n^B(\tau + a')),$$

which shows that $([\sigma - a']_A, [\tau + a']_B) \in H_n(A) \oplus H_n(B)$. Then

$$\psi_*([\sigma - a']_A, [\tau + a']_B) = [\psi(\sigma - a', \tau + a')]_+ = [(\sigma + a') + (\tau + a')]_+ = [\sigma + \tau] = [c]_+$$

shows that $[c]_+ \in \operatorname{Im}\psi_*$.

Suppose that $\chi \in \operatorname{Ker}\partial_n$, so $\chi \in H_n(X)$ such that $\partial_n \chi = 0$. Since ι_* is an isomorphism there exists $\xi \in H_n^{A+B}(X)$ such that $\iota_*^{-1}(\chi) = \xi$. Also

$$(\partial_n \circ \iota_*)(\xi) = (\partial_n \circ \iota_*)(\iota_*^{-1}(\chi)) = \partial_n \chi = 0,$$

so $\xi \in \operatorname{Ker}\partial_n \circ \iota_* \subset \operatorname{Im}\psi_*$ and it follows that there exists some $\omega \in H_n(A) \oplus H_n(B)$ such that $\psi_*(\omega) = \xi$. Then

$$\Psi_n(\omega) = \iota_*(\psi_*(\omega)) = \iota_*(\xi) = \chi$$

shows that $\chi \in \operatorname{Im}\Psi_n$.

• $\operatorname{Ker}\Phi_{n-1} \subset \operatorname{Im}\partial_n$. Let $[a]_\cap \in \operatorname{Ker}\Phi_{n-1}$, so $\partial_{n-1}^\cap a = 0$, which implies that $\partial_{n-1}^A a = \partial_{n-1}^B a = 0$ and thus $([a]_A, [-a]_B) \in H_{n-1}(A) \oplus H_{n-1}(B)$. Now, by hypothesis,

$$\Phi_{n-1}([a]_\cap) = ([a]_A, [-a]_B) = (\partial_n^A(C_n(A)), \partial_n^B(C_n(B))),$$

implying $a \in \partial_n^A(C_n(A))$ and $-a \in \partial_n^B(C_n(B))$, and hence there exists $(\sigma, \tau) \in C_n(A) \oplus C_n(B)$ such that

$$\partial_n^\circ(\sigma, \tau) = (\partial_n^A \sigma, \partial_n^B \tau) = (a, -a) = \varphi_{n-1}(a).$$

Observe that $\psi_n(\sigma, \tau) \in C_n(A + B)$ is a cycle:

$$\partial_n^+(\psi_n(\sigma, \tau)) = \psi_{n-1}(\partial_n^\circ(\sigma, \tau)) = \psi_{n-1}(\varphi_{n-1}(a)) = 0$$

since $\operatorname{Im}\varphi_{n-1} = \operatorname{Ker}\psi_{n-1}$, so $[\psi_n(\sigma, \tau)]_+ \in H_n^{A+B}(X)$. Now,

$$(\partial_n \circ \iota_*)([\psi_n(\sigma, \tau)]_+) = \partial_n([\sigma + \tau]) := [\partial_n^X \sigma]_\cap = [\partial_n^A \sigma]_\cap = [a]_\cap,$$

and therefore $[a]_\cap \in \operatorname{Im}\partial_n$. ■

The examples that will be examined below will serve to illustrate the uses of another Mayer-Vietoris sequence that arises from $A, B \subset X$, where $X = A \cup B$ with A, B being deformation retracts of open sets $U, V \subset X$ such that $A \cap B$ is a deformation retract of $U \cap V$. Thus for retractions $r^A : U \to A$ and $r^B : V \to B$ we find that $r^A|_{U\cap V} = r^B|_{U\cap V}$. To establish this sequence we start with the following commutative diagram:

$$\begin{array}{ccccccccc}
H_n(U\cap V) & \xrightarrow{\Phi_n} & H_n(U)\oplus H_n(V) & \xrightarrow{\psi_*} & H_n^{U+V}(X) & \xrightarrow{\partial_n\circ\iota_*} & H_{n-1}(U\cap V) & \xrightarrow{\Phi_{n-1}} & H_{n-1}(U)\oplus H_{n-1}(V) \\
{\scriptstyle i_*}\uparrow & & {\scriptstyle j_*}\uparrow & & {\scriptstyle k_*}\uparrow & & {\scriptstyle i_*}\uparrow & & {\scriptstyle j_*}\uparrow \\
H_n(A\cap B) & \xrightarrow{\Phi_n'} & H_n(A)\oplus H_n(B) & \xrightarrow{\psi_*'} & H_n^{A+B}(X) & \xrightarrow{\partial_n'\circ\iota_*'} & H_{n-1}(A\cap B) & \xrightarrow{\Phi_{n-1}'} & H_{n-1}(A)\oplus H_{n-1}(B)
\end{array}$$

Here i_* is induced by $i : A\cap B \hookrightarrow U\cap V$, and $j_* = j_*^A \oplus j_*^B$ with j_*^A induced by $j^A : A \hookrightarrow U$ and j_*^B induced by $j^B : B \hookrightarrow V$. Since i, j^A and j^B are homotopy equivalences, all i_* and j_* maps are isomorphisms, and so in particular j_* has inverse $r_* = r_*^A \oplus r_*^B$ (r_*^A and r_*^B being the inverses of j_*^A and j_*^B); that is, $j_*^{-1} = r_*$. The homomorphism k_* arises from $k : C_n(A+B) \hookrightarrow C_n(U+V)$ given by $k(\sigma + \tau) = j^A \circ \sigma + j^B \circ \tau$ for $\sigma \in C_n(A)$ and $\tau \in C_n(B)$ (if, say, $\sigma = n_1\sigma_1 + n_2\sigma_2$ with $\sigma_1, \sigma_2 : \Delta^n \to A$, we take $j^A \circ \sigma = n_1(j^A \circ \sigma_1) + n_2(j^A \circ \sigma_2)$), so that $k_*([\sigma + \tau]_+) = [k(\sigma + \tau)]_+$. Also we define ψ_*' in the same way as ψ_*, Φ_n' by $\Phi_n' = j_*^{-1} \circ \Phi_n \circ i_*$, and $\partial_n' : H_n(X) \to H_{n-1}(A\cap B)$ by $\partial_n' = i_*^{-1} \circ \partial_n$. Finally, $\iota_*' : H_n^{A+B}(X) \to H_n(X)$ is induced by $\iota' : C_n(A + B) \hookrightarrow C_n(X)$ so that $\iota_*'([\xi]_+) = [\iota'(\xi)] = [\xi]$. With these definitions it is straightforward to verify that the diagram is commutative. Also the exactness of the upper row follows easily from the exactness of sequence (20). What is not wholly trivial is demonstrating that the lower row is exact, but once it is confirmed to be so, the Five-Lemma implies that k_* is an isomorphism. Then the commutative diagram

$$\begin{array}{ccc} H_n^{U+V}(X) & \xrightarrow{\iota_*} & H_n(X) \\ {\scriptstyle k_*}\big\uparrow & & {\scriptstyle \cong}\big\uparrow \\ H_n^{A+B}(X) & \xrightarrow{\iota'_*} & H_n(X) \end{array}$$

makes clear that ι'_* is an isomorphism. Knowing this is all that is required in order to construct a Mayer-Vietoris sequence for the decomposition $X = A \cup B$ with $A \subset U$ and $B \subset V$ as described above.

For the benefit of the obsessive-compulsive among us, the proof of the exactness of the rows of the diagram above follows.

Proof.

• $\operatorname{Im}\Phi_n \subset \operatorname{Ker}\psi_*$. Let $x \in \operatorname{Im}\Phi_n = \operatorname{Ker}\Psi_n$, so $\Psi_n(x) = (\iota_* \circ \psi_*)(x) = 0$. Since ι_* is an isomorphism we obtain $\psi_*(x) = 0$ and thus $x \in \operatorname{Ker}\psi_*$.

• $\operatorname{Ker}\psi_* \subset \operatorname{Im}\Phi_n$. Let $x \in \operatorname{Ker}\psi_*$, so $\psi_*(x) = 0$ implies $\Psi_n(x) = \iota_*(\psi_*(x)) = 0$, which shows that $x \in \operatorname{Ker}\Psi_n = \operatorname{Im}\Phi_n$.

• $\operatorname{Im}\psi_* \subset \operatorname{Ker}(\partial_n \circ \iota_*)$. Let $\xi \in \operatorname{Im}\psi_*$, so there exists $x \in H_n(U) \oplus H_n(V)$ such that $\psi_*(x) = \xi$, and thus $\iota_*(\psi_*(x)) = \iota_*(\xi)$ implies that

$$\iota_*(\xi) \in \operatorname{Im}(\iota_* \circ \psi_*) = \operatorname{Im}\Psi_n = \operatorname{Ker}\partial_n.$$

This leads to $\partial_n(\iota_*(\xi)) = 0$ and therefore $\xi \in \operatorname{Ker}(\partial_n \circ \iota_*)$.

• $\operatorname{Ker}(\partial_n \circ \iota_*) \subset \operatorname{Im}\psi_*$. Let $\xi \in \operatorname{Ker}(\partial_n \circ \iota_*)$, so $\iota_*(\xi) \in \operatorname{Ker}\partial_n = \operatorname{Im}\Psi_n$ and $\Psi_n(x) = \iota_*(\xi)$ for some x. That is, $\iota_*(\psi_*(x)) = \iota_*(\xi)$, and since ι_* is injective, $\psi_*(x) = \xi$ and therefore $\xi \in \operatorname{Im}\psi_*$.

• $\operatorname{Im}(\partial_n \circ \iota_*) \subset \operatorname{Ker}\Phi_{n-1}$. Let $z \in \operatorname{Im}(\partial_n \circ \iota_*)$, so $\partial_n(\iota_*(\xi)) = z$ for some ξ, from which it's clear that $z \in \operatorname{Im}\partial_n = \operatorname{Ker}\Phi_{n-1}$.

• $\operatorname{Ker}\Phi_{n-1} \subset \operatorname{Im}(\partial_n \circ \iota_*)$. Let $z \in \operatorname{Ker}\Phi_{n-1} = \operatorname{Im}\partial_n$. Then $\partial_n(x) = z$ for some $x \in H_n(X)$. Since ι_* is surjective $\exists\, \xi \in H_n^{A+B}(X)$ s.t. $\iota_*(\xi) = x$, and thus $\partial_n(\iota_*(\xi)) = \partial_n(x) = z$ shows that $z \in \operatorname{Im}(\partial_n \circ \iota_*)$. This completes the verification that the top row is exact.

• $\operatorname{Im}\Phi'_n \subset \operatorname{Ker}\psi'_*$. Let $([\sigma]_A, [\tau]_B) \in \operatorname{Im}\Phi'_n$, so $\exists [\xi]_\cap \in H_n(A \cap B)$ s.t. $\Phi'_n([\xi]_\cap) = ([\sigma]_A, [\tau]_B)$, whence

$$\begin{aligned} ([\sigma]_A, [\tau]_B) &= (r_* \circ \Phi_n \circ i_*)([\xi]_\cap) = r_*([\xi]_U, [-\xi]_V) \\ &= ([r^A \circ \xi]_A, [-r^B \circ \xi]_B) = ([\xi]_A, [\xi]_B). \end{aligned}$$

The last equality holds since $\xi \in C_n(A \cap B)$, and r^A, r^B behave as the identity when restricted to $A \cap B$ so that $r^A \circ \xi = \xi, r^B \circ \xi = \xi$. Now

$$\psi'_*([\sigma]_A, [\tau]_B) = \psi'_*([\xi]_A, [\xi]_B) = [\xi - \xi]_+ = 0$$

as desired.

• $\operatorname{Ker}\psi'_* \subset \operatorname{Im}\Phi'_n$. Let $([\sigma]_A, [\tau]_B) \in \operatorname{Ker}\psi'_*$. Then

$$\psi_*(j_*([\sigma]_A, [\tau]_B)) = k_*(\psi'_*([\sigma]_A, [\tau]_B)) = k_*(0) = 0$$

by commutativity, so

$$j_*([\sigma]_A, [\tau]_B) \in \operatorname{Ker}\psi_* = \operatorname{Im}\Phi_n$$

and

$$\Phi_n([\xi]_\cap) = j_*([\sigma]_A, [\tau]_B)$$

for some $[\xi]_\cap \in H_n(U \cap V)$. Now

$$j_*^{-1}(\Phi_n([\xi]_\cap)) = ([\sigma]_A, [\tau]_B) = \Phi'_n(i_*^{-1}([\xi]_\cap))$$

shows that $([\sigma]_A, [\tau]_B) \in \operatorname{Im}\Phi'_n$.

• $\operatorname{Im}\psi'_* \subset \operatorname{Ker}(\partial'_n \circ \iota'_*)$. Let $[\xi]_+ \in \operatorname{Im}\psi'_*$, so $\exists([\sigma]_A, [\tau]_B) \in H_n(A) \oplus H_n(B)$ such that

$$\psi'_*([\sigma]_A, [\tau]_B) = [\sigma + \tau]_+ = [\xi]_+.$$

Now,

$$\psi'_*(j_*([\sigma]_A, [\tau]_B)) = [\sigma + \tau]_+ \in \operatorname{Im}\psi_* = \operatorname{Ker}(\partial_n \circ \iota_*)$$

implies that

$$\partial_n(\iota_*([\sigma + \tau]_+)) = \partial_n([\sigma + \tau]) = 0.$$

Thus

$$\partial'_n \circ \iota'_*([\xi]_+) = \partial'_n \circ \iota'_*([\sigma + \tau]_+) = \partial'_n([\sigma + \tau]) = 0$$

and we get $[\xi]_+ \in \operatorname{Ker}(\partial'_n \circ \iota'_*)$.

• $\operatorname{Ker}(\partial'_n \circ \iota'_*) \subset \operatorname{Im}\psi'_*$. Let $[\xi]_+ \in \operatorname{Ker}(\partial'_n \circ \iota'_*)$, so

$$\partial_n \circ \iota_* \circ k_*([\xi]_+) = i_* \circ \partial'_n \circ \iota'_*([\xi]_+) = 0$$

yields $k_*([\xi]_+) \in \operatorname{Ker}\partial_n \circ \iota_* = \operatorname{Im}\psi_*$, and thus there is some

$$([\mu]_U, [\nu]_V) \in H_n(U) \oplus H_n(V)$$

such that $[\mu + \nu]_+ = [\xi]_+ \in H_n^{U+V}(X)$. What remains to show is that

$$\psi'_*(j_*^{-1}([\mu]_U, [\nu]_V)) = \psi'_*([r^A \circ \mu]_A, [r^B \circ \nu]_B) = [r^A \circ \mu + r^B \circ \nu]_+ = [\xi]_+$$

in $H_n^{A+B}(X)$, or equivalently $\exists\, \omega \in C_{n+1}(A + B)$ such that

$$\partial^+_{n+1}\omega = r^A \circ \mu + r^B \circ \nu - \xi.$$

But notice $r^A \simeq \mathbb{1}_U$ and $r^B \simeq \mathbb{1}_V$ imply that $r^A_* = \mathbb{1}_{U*}$ and $r^B_* = \mathbb{1}_{V*}$, so

$$([\mu]_U, [\nu]_V) = ([r^A \circ \mu]_U, [r^B \circ \nu]_V)$$

and we obtain $[r^A \circ \mu + r^B \circ \nu]_+ = [\xi]_+$ in $H_n^{U+V}(X)$. Hence $\exists\, \omega' \in C_{n+1}(U+V)$ such that

$$\partial_{n+1}^+ \omega' = r^A \circ \mu + r^B \circ \nu - \xi,$$

and we can write $\omega' = \sigma + \tau$ for some $\sigma \in C_{n+1}(U), \tau \in C_{n+1}(V)$. Now, if we define

$$\omega := r^A \circ \sigma + r^B \circ \tau \in C_{n+1}(A+B),$$

then $\partial_{n+1}^+ \omega = r^A \circ \mu + r^B \circ \nu - \xi$ as desired. Therefore $[\xi]_+ \in \operatorname{Im} \psi'_*$.

• $\operatorname{Im}(\partial'_n \circ \iota'_*) \subset \operatorname{Ker} \Phi'_{n-1}$. Let $[z]_\cap \in \operatorname{Im}(\partial'_n \circ \iota'_*)$, so $\exists\, [\xi]_+ \in H_n^{A+B}(X)$ such that

$$\partial'_n \circ \iota'_*([\xi]_+) = [z]_\cap.$$

That is,

$$[z]_\cap = \partial'_n([\xi]) = (i_*^{-1} \circ \partial_n)([\xi]).$$

Now,

$$(\partial_n \circ \iota_*)(k_*([\xi]_+)) = (\partial_n \circ \iota_*)([\xi]_+) = \partial_n[\xi],$$

so $\partial_n[\xi] \in \operatorname{Im}(\partial_n \circ \iota_*) = \operatorname{Ker} \Phi_{n-1}$ and we obtain $\Phi_{n-1}(\partial_n[\xi]) = 0$. From $\Phi_{n-1} \circ i_* = j_* \circ \Phi'_{n-1}$ comes

$$(\Phi_{n-1} \circ i_*)([z]_\cap) = (\Phi_{n-1} \circ i_*)((i_*^{-1} \circ \partial_n)([\xi])) = \Phi_{n-1}(\partial_n[\xi]) = 0 = (j_* \circ \Phi'_{n-1})([z]_\cap),$$

so $\Phi'_{n-1}([z]_\cap) = 0$ since j_* is an isomorphism. Hence $[z]_\cap \in \operatorname{Ker} \Phi'_{n-1}$.

• $\operatorname{Ker} \Phi'_{n-1} \subset \operatorname{Im}(\partial'_n \circ \iota'_*)$. Let $[z]_\cap \in \operatorname{Ker} \Phi'_{n-1}$, so

$$\Phi'_{n-1}([z]_\cap) = ([z]_A, [-z]_B) = (0, 0)$$

and $\exists\, x \in C_n(A), y \in C_n(B)$ s.t. $\partial_n^A x = z, \partial_n^B y = -z$. Now, $x + y \in C_n(A+B)$ with

$$\partial_n^+(x+y) = \partial_n^A x + \partial_n^B y = z - z = 0,$$

so $[x+y]_+ \in H_n^{A+B}(X)$. Then, letting $r^\cap$ denote the retraction of $U \cap V$ onto $A \cap B$ so that $r_*^\cap := (r^\cap)_* = (i_*)^{-1} := i_*^{-1}$,

$$(\partial'_n \circ \iota'_*)([x+y]_+) = i_*^{-1}(\partial_n[x+y]) = i_*^{-1}([\partial_n^X x]_\cap) = r_*^\cap([\partial_n^A x]_\cap) = r_*^\cap([z]_\cap) = [r^\cap \circ z]_\cap.$$

However, z is a chain in $C_{n-1}(A \cap B)$ and $r^\cap|_{A \cap B} = \mathbb{1}_{A \cap B}$, so in fact $r^\cap \circ z = z$ and we obtain $(\partial'_n \circ \iota'_*)([x+y]_+) = [z]_\cap$. Therefore $[z]_\cap \in \operatorname{Im}(\partial'_n \circ \iota'_*)$ and the bottom row of the diagram is exact. ∎

Example 2.29. The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region $\mathcal{R}$. Two copies of $\mathcal{R}$, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Here we will compute the homology groups of X, $H_n(X)$; also we will find the relative homology groups $H_n(\mathcal{R}, M_g)$.[6]

First assume that $g = 1$, so $M_g = \mathbb{T}^2 \subset \mathbb{R}^3$ (the embedding of $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ in $\mathbb{R}^3$). Let $\mathcal{T}$ be the region in $\mathbb{R}^3$ that $\mathbb{T}^2$ bounds, so $\mathcal{T}$ is a solid "donut" in space which is homeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1 \subset \mathbb{R}^4$, and $\partial \mathcal{T} = \mathbb{T}^2$. Let $\mathcal{T}_A$ and $\mathcal{T}_B$ be two copies of $\mathcal{T}$, and let $\iota : \partial \mathcal{T}_A \to \partial \mathcal{T}_B$ be

[6]This appears as problem 2.29 in Hatcher.

the identity map. Then the space X in this case is given as $X = \mathcal{T}_A \sqcup_\iota \mathcal{T}_B$, and indeed $\mathcal{T}_A \sqcup_\iota \mathcal{T}_B$ constitutes a natural decomposition of X for which there can be found neighborhoods U and V in X that deformation retract to $\mathcal{T}_A$ and $\mathcal{T}_B$ while $U \cap V$ deformation retracts to $\mathcal{T}_A \cap \mathcal{T}_B = \mathbb{T}^2$. It has been found earlier that

$$H_n(\mathbb{T}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 1 \\ 0, & \text{if } n \geq 3, \end{cases}$$

and so, noting that $\mathcal{T}_A, \mathcal{T}_B \simeq \mathbb{S}^1$, we obtain the Mayer-Vietoris sequence

$$\cdots \longrightarrow \underbrace{\widetilde{H}_2(\mathcal{T}_A \cap \mathcal{T}_B)}_{H_2(\mathbb{T}^2)\cong\mathbb{Z}} \xrightarrow{\Phi_2} \underbrace{\widetilde{H}_2(\mathcal{T}_A) \oplus \widetilde{H}_2(\mathcal{T}_B)}_{0} \xrightarrow{\Psi_2} \widetilde{H}_2(X) \xrightarrow{\partial_2} \underbrace{\widetilde{H}_1(\mathcal{T}_A \cap \mathcal{T}_B)}_{H_1(\mathbb{T}^2)\cong\mathbb{Z}\oplus\mathbb{Z}}$$
$$\xrightarrow{\Phi_1} \underbrace{\widetilde{H}_1(\mathcal{T}_A) \oplus \widetilde{H}_1(\mathcal{T}_B)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\Psi_1} \widetilde{H}_1(X) \xrightarrow{\partial_1} \underbrace{\widetilde{H}_0(\mathcal{T}_A \cap \mathcal{T}_B)}_{\widetilde{H}_0(\mathbb{T}^2)\cong 0} \longrightarrow \cdots \quad (21)$$

Of course $H_0(X) \cong \mathbb{Z}$ since X is path-connected. From $\operatorname{Im}\Psi_1 = \operatorname{Ker}\partial_1 = H_1(X)$ we see that Ψ_1 is surjective, so

$$H_1(X) \cong H_1(\mathcal{T}_A) \oplus H_1(\mathcal{T}_B)/\operatorname{Ker}\Psi_1 \cong H_1(\mathcal{T}_A) \oplus H_1(\mathcal{T}_B)/\operatorname{Im}\Phi_1.$$

The workings of Φ_1 must be determined.

Recall the cycles $\hat{\tau} := \hat{\tau}_1 - \hat{\tau}_2$ and $\hat{\sigma} := \hat{\sigma}_1 - \hat{\sigma}_2$ from a few pages back. It has been shown that $[\![\hat{\tau}]\!]$ and $[\![\hat{\sigma}]\!]$ are generators for $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ (it seems safe at this point to drop the subscripts that formerly distinguished the two copies of $\mathbb{S}^1$). A homeomorphism $h : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{T}^2$ can be found that maps $\hat{\tau}$ and $\hat{\sigma}$ to cycles $\check{\tau}$ and $\check{\sigma}$, respectively, on the surface $\mathbb{T}^2$ as shown in Figure 7. Specifically we have $\check{\tau} = h \circ \hat{\tau}_1 - h \circ \hat{\tau}_2$, with $h \circ \hat{\tau}_1$ going halfway around the girth of the torus and $-h \circ \hat{\tau}_2$ completing the loop; and we have $\check{\sigma} = h \circ \hat{\sigma}_1 - h \circ \hat{\sigma}_2$ making a loop around the center hole of the torus. Now $[\check{\tau}], [\check{\sigma}] \in H_1(\mathbb{T}^2)$ are explicit generators for $H_1(\mathbb{T}^2)$ since they correspond via the isomorphism h_* to the generators $[\![\hat{\tau}]\!]$ and $[\![\hat{\sigma}]\!]$, and since $H_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ we can naturally identify $[\check{\tau}]$ with $(1, 0)$ and $[\check{\sigma}]$ with (0,1).

By definition $\Phi_1 = i_*^A \oplus (-i_*^B)$, where $i_*^A : H_1(\mathcal{T}_A \cap \mathcal{T}_B) \to H_1(\mathcal{T}_A)$ is the homomorphism induced by $i^A : \mathcal{T}_A \cap \mathcal{T}_B \hookrightarrow \mathcal{T}_A$, and $i_*^B : H_1(\mathcal{T}_A \cap \mathcal{T}_B) \to H_1(\mathcal{T}_B)$ is induced by $i^B : \mathcal{T}_A \cap \mathcal{T}_B \hookrightarrow \mathcal{T}_B$. Letting $\check{\tau}_i := h \circ \hat{\tau}_i$ for $i = 1, 2$, we have

$$i_*^A([\check{\tau}]) = [i^A \circ \check{\tau}_1 - i^A \circ \check{\tau}_2] = [0]$$

since the cycle $i^A \circ \check{\tau}_1 - i^A \circ \check{\tau}_2$ is a nulhomotopic loop in $\mathcal{T}_A$. Bearing in mind that $H_1(\mathcal{T}_A) \cong H_1(\mathbb{S}^1) \cong \mathbb{Z}$, we can write $i_*^A(1, 0) = 0$. Similarly $i_*^B([\check{\tau}]) = [0]$, or $i_*^B(1, 0) = 0$. Let's examine the cycle $\check{\sigma}$ in $\mathcal{T}_A \cap \mathcal{T}_B = \mathbb{T}^2$ more closely. Letting $\check{\sigma}_i := h \circ \hat{\sigma}_i$, we have $\check{\sigma} = \check{\sigma}_1 - \check{\sigma}_2$ such that $\check{\sigma}_1$ goes halfway around the center hole of the torus, and $\check{\sigma}_2$ completes the circuit. The space $\mathcal{T}_A$ deformation retracts to $\mathbb{S}^1$, and so the resultant retraction $r : \mathcal{T}_A \to \mathbb{S}^1$ is a homotopy equivalence and therefore induces an isomorphism $r_* : H_1(\mathcal{T}_A) \to H_1(\mathbb{S}^1)$. It is straightforward to engineer the function r so that it maps the paths $i^A \circ \check{\sigma}_i$ in $\mathcal{T}_A$ onto σ_i in $\mathbb{S}^1$; that is, $r \circ i^A \circ \check{\sigma}_i : \Delta^1 \to \mathbb{S}^1$ is equal to $\sigma_i : \Delta^1 \to \mathbb{S}^1$. Thus for $[i^A \circ \check{\sigma}_1 - i^A \circ \check{\sigma}_2] \in H_1(\mathcal{T}_A)$ we have

$$r_*([i^A \circ \check{\sigma}_1 - i^A \circ \check{\sigma}_2]) = [r \circ i^A \circ \check{\sigma}_1 - r \circ i^A \circ \check{\sigma}_2] = [\sigma_1 - \sigma_2] \in H_1(\mathbb{S}^1).$$

Since $[\sigma_1 - \sigma_2]$ is known to be a generator for $H_1(\mathbb{S}^1)$ is readily follows that $[i^A \circ \check{\sigma}_1 - i^A \circ \check{\sigma}_2]$ is a generator for $H_1(\mathcal{T}_A)$. Now we note that

$$[i^A \circ \check{\sigma}_1 - i^A \circ \check{\sigma}_2] = i^A_*([\check{\sigma}_1 - \check{\sigma}_2]) = i^A_*([\check{\sigma}]),$$

and so we can naturally write $i^A_*(0,1) = 1$. In similar fashion i^B_* maps $[\check{\sigma}] \in H_1(\mathcal{T}_A \cap \mathcal{T}_B)$ to the corresponding generator for $H_1(\mathcal{T}_B)$, so that $i^*_B(0,1) = 1$ also.

We have, then,

$$\Phi_1([\check{\tau}]) := \Phi_1(1,0) = (0,0) \ \text{ and } \ \Phi_1([\check{\sigma}]) := \Phi_1(0,1) = (1,-1).$$

If we let $H_1(\mathcal{T}_A) = \langle a \rangle$ and $H_1(\mathcal{T}_B) = \langle b \rangle$, then Φ_1 is given by $\Phi_1(1,0) = 0a + 0b = 0$ and $\Phi_1(0,1) = 1a - 1b = a - b$, and in general

$$\Phi_1(m,n) = \Phi_1((m,0) + (0,n)) = m\Phi_1(1,0) + n\Phi_1(0,1) = m \cdot 0 + n(a-b) = n(a-b).$$

Hence $\operatorname{Im}\Phi_1 = \langle a - b \rangle$, and we find that

$$H_1(X) \cong \frac{\langle a \rangle \oplus \langle b \rangle}{\langle a-b \rangle} = \frac{\operatorname{Ab}\langle a, b \rangle}{\langle a-b \rangle} \cong \frac{\operatorname{Ab}\langle a-b, b \rangle}{\langle a-b \rangle} \cong \langle b \rangle \cong \mathbb{Z}.$$

Next, $\operatorname{Ker}\partial_2 = \operatorname{Im}\Psi_2 = 0$ implies that ∂_2 is injective, which in turn implies that $H_2(X) \cong \operatorname{Im}\partial_2 = \operatorname{Ker}\Phi_1$. Now,

$$\begin{aligned}\operatorname{Ker}\Phi_1 &= \{(m,n) \in \mathbb{Z} \oplus \mathbb{Z} : \Phi_1(m,n) = (0,0)\} \\ &= \{(m,n) : (n,-n) = (0,0)\} \\ &= \{(m,0) : m \in \mathbb{Z}\} \cong \mathbb{Z},\end{aligned}$$

so it is concluded that $H_2(X) \cong \mathbb{Z}$.

We extend our long exact sequence (21) a little to the left,

$$\cdots \longrightarrow H_3(\mathcal{T}_A \cap \mathcal{T}_B) \xrightarrow{\Phi_3} \underbrace{H_3(\mathcal{T}_A) \oplus H_3(\mathcal{T}_B)}_{0} \xrightarrow{\Psi_3} H_3(X) \xrightarrow{\partial_3} H_2(\mathcal{T}_A \cap \mathcal{T}_B)$$

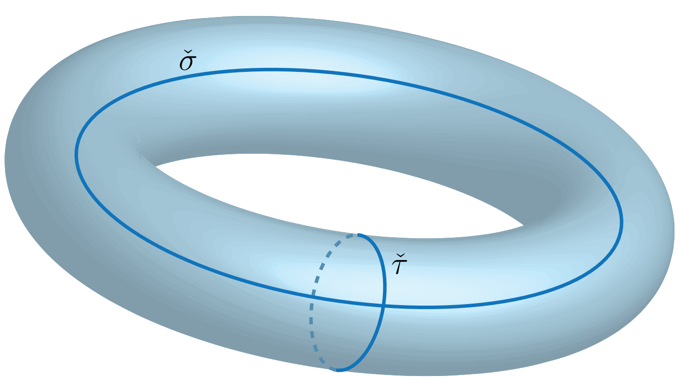


Figure 7. The cycles $\check{\tau}$ and $\check{\sigma}$ in $\mathbb{T}^2$.

$$\xrightarrow{\Phi_2} \underbrace{H_2(\mathcal{T}_A) \oplus H_2(\mathcal{T}_B)}_{0} \longrightarrow \cdots, \quad (22)$$

and find immediately that $H_3(X) \cong H_2(\mathcal{T}_A \cap \mathcal{T}_B) \cong H_2(\mathbb{T}^2) \cong \mathbb{Z}$. Therefore

$$H_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n \leq 3 \\ 0, & \text{if } n > 3. \end{cases}$$

We now turn to the task of finding the relative homology groups $H_n(\mathcal{T}, \mathbb{T}^2)$ (i.e. $H_n(\mathcal{R}, M_g)$ for $g = 1$). There is the long exact sequence for this pair

$$\cdots \longrightarrow \underbrace{\widetilde{H}_3(\mathbb{T}^2)}_{0} \xrightarrow{i_*^3} \underbrace{\widetilde{H}_3(\mathcal{T})}_{0} \xrightarrow{j_*^3} \widetilde{H}_3(\mathcal{T}, \mathbb{T}^2) \xrightarrow{\partial_*^3} \underbrace{\widetilde{H}_2(\mathbb{T}^2)}_{\mathbb{Z}} \xrightarrow{i_*^2} \underbrace{\widetilde{H}_2(\mathcal{T})}_{0} \xrightarrow{j_*^2} \widetilde{H}_2(\mathcal{T}, \mathbb{T}^2)$$

$$\xrightarrow{\partial_*^2} \underbrace{\widetilde{H}_1(\mathbb{T}^2)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{i_*^1} \underbrace{\widetilde{H}_1(\mathbb{T}^2)}_{\mathbb{Z}} \xrightarrow{j_*^1} \widetilde{H}_1(\mathcal{T}, \mathbb{T}^2) \xrightarrow{\partial_*^1} \underbrace{\widetilde{H}_0(\mathbb{T}^2)}_{0} \xrightarrow{i_*^0} \underbrace{\widetilde{H}_0(\mathcal{T})}_{0} \longrightarrow \cdots, \quad (23)$$

which straightaway gives $H_3(\mathcal{T}, \mathbb{T}^2) \cong H_2(\mathbb{T}^2) \cong \mathbb{Z}$.

Next,

$$\operatorname{Im} j_*^2 = 0 \quad \Rightarrow \quad \operatorname{Ker} \partial_*^2 = 0 \quad \Rightarrow \quad \partial_*^2 \text{ is 1-1} \quad \Rightarrow \quad H_2(\mathcal{T}, \mathbb{T}^2) \cong \operatorname{Im} \partial_*^2 = \operatorname{Ker} i_*^1.$$

The map

$$i_*^1 : H_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_1(\mathcal{T}) \cong \mathbb{Z}$$

works the same way as

$$i_*^A : H_1(\mathcal{T}_A \cap \mathcal{T}_B) \to H_1(\mathcal{T}_A),$$

so $i_*^1(1,0) = 0$ and $i_*^1(0,1) = 1$, and

$$\operatorname{Ker} i_*^1 = \{(m,n) \in \mathbb{Z} \oplus \mathbb{Z} : i_*^1(m,n) = 0\} = \{(m,n) : n = 0\} \cong \mathbb{Z}$$

implies that $H_2(\mathcal{T}, \mathbb{T}^2) \cong \mathbb{Z}$.

Moving on, $H_1(\mathcal{T}, \mathbb{T}^2) = \operatorname{Ker} \partial_*^1 = \operatorname{Im} j_*^1$ implies that j_*^1 is surjective, whence we get

$$H_1(\mathcal{T}, \mathbb{T}^2) \cong H_1(\mathcal{T}) / \operatorname{Ker} j_*^1 = H_1(\mathcal{T}) / \operatorname{Im} i_*^1.$$

But i_*^1 is surjective also: for any $n \in \mathbb{Z} \cong H_1(\mathcal{T})$ we have $(0,n) \in \mathbb{Z} \oplus \mathbb{Z} \cong H_1(\mathbb{T}^2)$ such that $i_*^1(0,n) = n i_*^1(0,1) = n$. Hence

$$H_1(\mathcal{T}, \mathbb{T}^2) \cong H_1(\mathcal{T}) / H_1(\mathcal{T}) \cong 0.$$

Finally, noting that $(\mathcal{T}, \mathbb{T}^2)$ is a good pair, Proposition 2.23 implies that $H_0(\mathcal{T}, \mathbb{T}^2) \cong \widetilde{H}_0(\mathcal{T} / \mathbb{T}^2) = 0$ (since the quotient space $\mathcal{T} / \mathbb{T}^2$ is clearly path connected). Therefore

$$H_n(\mathcal{T}, \mathbb{T}^2) \cong \begin{cases} 0, & \text{if } n = 0, 1 \text{ or } n \geq 4 \\ \mathbb{Z}, & \text{if } n = 2, 3. \end{cases}$$

Now let $g \geq 2$, and let $\mathcal{M}$ be the region in $\mathbb{R}^3$ that M_g bounds. Let $\mathcal{M}_A$ and $\mathcal{M}_B$ be two copies of $\mathcal{M}$, and let $\iota : \partial\mathcal{M}_A \to \partial\mathcal{M}_B$ be the identity map. Then the space X in this case is given as $X = \mathcal{M}_A \sqcup_\iota \mathcal{M}_B$, and indeed $\mathcal{M}_A \sqcup_\iota \mathcal{M}_B$ constitutes a natural decomposition of X for which there can be found neighborhoods U and V in X that deformation retract to $\mathcal{M}_A$ and $\mathcal{M}_B$ while $U \cap V$ deformation retracts to $\mathcal{M}_A \cap \mathcal{M}_B = M_g$. We know

$$H_n(M_g) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2 \\ \bigoplus_{i=1}^{2g} \mathbb{Z}, & \text{if } n = 1 \\ 0, & \text{if } n \geq 3, \end{cases}$$

and so, noting that $\mathcal{M}_A, \mathcal{M}_B \simeq \bigvee_{i=1}^{g} \mathbb{S}^1$, we obtain the Mayer-Vietoris sequence

$$\cdots \longrightarrow \underbrace{\widetilde{H}_2(\mathcal{M}_A \cap \mathcal{M}_B)}_{H_2(M_g) \cong \mathbb{Z}} \xrightarrow{\Phi_2} \underbrace{\widetilde{H}_2(\mathcal{M}_A) \oplus \widetilde{H}_2(\mathcal{M}_B)}_{0} \xrightarrow{\Psi_2} \widetilde{H}_2(X) \xrightarrow{\partial_2} \underbrace{\widetilde{H}_1(\mathcal{M}_A \cap \mathcal{M}_B)}_{H_1(M_g) \cong \bigoplus_{i=1}^{2g} \mathbb{Z}}$$

$$\xrightarrow{\Phi_1} \underbrace{\widetilde{H}_1(\mathcal{M}_A) \oplus \widetilde{H}_1(\mathcal{M}_B)}_{\cong (\bigoplus_{i=1}^{g} \mathbb{Z}) \oplus (\bigoplus_{i=1}^{g} \mathbb{Z})} \xrightarrow{\Psi_1} \widetilde{H}_1(X) \xrightarrow{\partial_1} \underbrace{\widetilde{H}_0(\mathcal{M}_A \cap \mathcal{M}_B)}_{\widetilde{H}_0(M_g) \cong 0} \longrightarrow \cdots \tag{24}$$

$H_0(X) \cong \mathbb{Z}$ since X is path-connected. From $\operatorname{Im} \Psi_1 = \operatorname{Ker} \partial_1 = H_1(X)$ we find Ψ_1 is surjective, so

$$H_1(X) \cong H_1(\mathcal{M}_A) \oplus H_1(\mathcal{M}_B) / \operatorname{Ker} \Psi_1 \cong H_1(\mathcal{M}_A) \oplus H_1(\mathcal{M}_B) / \operatorname{Im} \Phi_1.$$

The group $H_1(M_g)$ is generated by the homology classes $[\check{\tau}_k]$ and $[\check{\sigma}_k]$ for $1 \leq k \leq g$, where the cycles $\check{\tau}_k$ and $\check{\sigma}_k$ are shown in Figure 8 (i.e. $\check{\tau}_k$ loops through the kth hole of M_g and $\check{\sigma}_k$ loops around it). For each k we can define

$$[\check{\tau}_k] = ((0,0), \ldots, \underbrace{(1,0)}_{k\text{th pair}}, \ldots, (0,0)) \ \text{ and } \ [\check{\sigma}_k] = ((0,0), \ldots, \underbrace{(0,1)}_{k\text{th pair}}, \ldots, (0,0)).$$

By definition $\Phi_1 = i_*^A \oplus (-i_*^B)$, or alternatively $(i_*^A, -i_*^B)$, where i_*^A and i_*^B are homomorphisms induced by inclusions $i^A : \mathcal{M}_A \cap \mathcal{M}_B \hookrightarrow \mathcal{M}_A$ and $i^B : \mathcal{M}_A \cap \mathcal{M}_B \hookrightarrow \mathcal{M}_B$ as before. Now, each $\check{\tau}_k$ becomes nulhomotopic when embedded in $\mathcal{M}_A$ or $\mathcal{M}_B$, so for each k

$$\begin{aligned} \Phi_1([\check{\tau}_k]) &= (i_*^A, -i_*^B)((0,0), \ldots, \overbrace{(1,0)}^{k\text{th pair}}, \ldots, (0,0)) \\ &= (i_*^A((0,0), \ldots, (1,0), \ldots, (0,0)),\ -i_*^B((0,0), \ldots, (1,0), \ldots, (0,0))) \\ &= (\underbrace{(0, \ldots, 0)}_{g\text{-tuple}}, \underbrace{(0, \ldots, 0)}_{g\text{-tuple}}). \end{aligned}$$

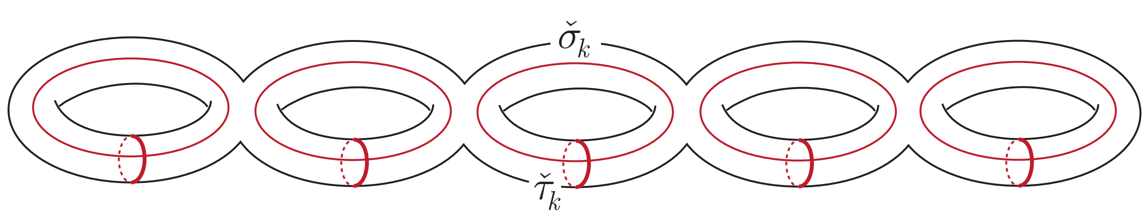


FIGURE 8. The cycles $\check{\tau}_k$ and $\check{\sigma}_k$ in M_g.

In a similar vein, a close examination of Φ_1 in (24) as well as the $g=1$ case should make it clear that for each k

$$\Phi_1([\check{\sigma}_k]) = (i_*^A((0,0),\ldots,(0,1),\ldots,(0,0))\,,\,-i_*^B((0,0),\ldots,(0,1),\ldots,(0,0)))$$

$$= (\,\underbrace{(0,\ldots,\overbrace{1}^{k\text{th entry}},\ldots,0)}_{g\text{-tuple}}\,,\,\underbrace{(0,\ldots,\overbrace{-1}^{k\text{th entry}},\ldots,0)}_{g\text{-tuple}}\,).$$

Letting $H_1(\mathcal{M}_A) = \mathrm{Ab}\langle a_1,\ldots,a_g\rangle$ and $H_1(\mathcal{M}_B) = \mathrm{Ab}\langle b_1,\ldots,b_g\rangle$, it's seen that $\operatorname{Im}\Phi_1 = \mathrm{Ab}\langle a_1-b_1,\ldots,a_g-b_g\rangle$, and then

$$H_1(X) \cong \frac{H_1(\mathcal{M}_A)\oplus H_1(\mathcal{M}_B)}{\operatorname{Im}\Phi_1} \cong \frac{\mathrm{Ab}\langle a_1,\ldots,a_g,b_1,\ldots,b_g\rangle}{\mathrm{Ab}\langle a_1-b_1,\ldots,a_g-b_g\rangle} \cong \bigoplus_{i=1}^{g}\mathbb{Z}.$$

As before,

$$\operatorname{Ker}\partial_2 = \operatorname{Im}\Psi_2 = 0 \;\Rightarrow\; \partial_2 \text{ is 1-1} \;\Rightarrow\; H_2(X)\cong \operatorname{Im}\partial_2 = \operatorname{Ker}\Phi_1;$$

thus since it ought to be clear that $\operatorname{Ker}\Phi_1 \cong \bigoplus_{i=1}^{g}\mathbb{Z}$, it follows immediately that

$$H_2(X) \cong \bigoplus_{i=1}^{g}\mathbb{Z}.$$

Next, we extend the sequence (24) to the left to get

$$\cdots \longrightarrow H_3(\mathcal{M}_A\cap\mathcal{M}_B) \xrightarrow{\Phi_3} \underbrace{H_3(\mathcal{M}_A)\oplus H_3(\mathcal{M}_B)}_{0} \xrightarrow{\Psi_3} H_3(X) \xrightarrow{\partial_3} H_2(\mathcal{M}_A\cap\mathcal{M}_B)$$
$$\xrightarrow{\Phi_2} \underbrace{H_2(\mathcal{M}_A)\oplus H_2(\mathcal{M}_B)}_{0} \longrightarrow \cdots,$$

which informs us that $H_3(X)\cong H_2(\mathcal{M}_A\cap\mathcal{M}_B)\cong H_2(M_g)\cong\mathbb{Z}$. Therefore

$$H_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0,3 \\ \bigoplus_{i=1}^{g}\mathbb{Z}, & \text{if } n = 1,2 \\ 0, & \text{if } n \geq 4. \end{cases}$$

At last we compute the relative homology groups $H_n(\mathcal{M}, M_g)$. The relevant long exact sequence is

$$\cdots \longrightarrow \underbrace{\widetilde{H}_3(M_g)}_{0} \xrightarrow{i_*^3} \underbrace{\widetilde{H}_3(\mathcal{M})}_{0} \xrightarrow{j_*^3} \widetilde{H}_3(\mathcal{M},M_g) \xrightarrow{\partial_*^3} \underbrace{\widetilde{H}_2(M_g)}_{\mathbb{Z}} \xrightarrow{i_*^2} \underbrace{\widetilde{H}_2(\mathcal{M})}_{0} \xrightarrow{j_*^2} \widetilde{H}_2(\mathcal{M},M_g)$$
$$\xrightarrow{\partial_*^2} \underbrace{\widetilde{H}_1(M_g)}_{\bigoplus_{i=1}^{2g}\mathbb{Z}} \xrightarrow{i_*^1} \underbrace{\widetilde{H}_1(\mathcal{M})}_{\bigoplus_{i=1}^{g}\mathbb{Z}} \xrightarrow{j_*^1} \widetilde{H}_1(\mathcal{M},M_g) \xrightarrow{\partial_*^1} \underbrace{\widetilde{H}_0(M_g)}_{0} \xrightarrow{i_*^0} \underbrace{\widetilde{H}_0(\mathcal{M})}_{0} \longrightarrow \cdots,$$

which straightaway implies that $H_3(\mathcal{M},M_g)\cong H_2(M_g)\cong\mathbb{Z}$.

Next,

$$\operatorname{Im} j_*^2 = 0 \;\Rightarrow\; \operatorname{Ker}\partial_*^2 = 0 \;\Rightarrow\; \partial_*^2 \text{ is 1-1} \;\Rightarrow\; H_2(\mathcal{M},M_g)\cong\operatorname{Im}\partial_*^2 = \operatorname{Ker} i_*^1.$$

The map

$$i_*^1 : H_1(M_g) \cong \bigoplus_{i=1}^{2g} \mathbb{Z} \to H_1(\mathcal{M}) \cong \bigoplus_{i=1}^{g} \mathbb{Z}$$

works the same way as i_*^A above, so

$$i_*^1((0,0),\ldots,(1,0),\ldots,(0,0)) = \underbrace{(0,\ldots,0)}_{g\text{-tuple}}$$

and

$$i_*^1((0,0),\ldots,(0,1),\ldots,(0,0)) = \underbrace{(0,\ldots,\overbrace{1}^{k\text{th entry}},\ldots,0)}_{g\text{-tuple}},$$

and we find that $\operatorname{Ker} i_*^1 \cong \bigoplus_{i=1}^{g} \mathbb{Z}$, whence

$$H_2(\mathcal{M}, M_g) \cong \bigoplus_{i=1}^{g} \mathbb{Z}.$$

Moving on, $H_1(\mathcal{M}, M_g) = \operatorname{Ker} \partial_*^1 = \operatorname{Im} j_*^1$ implies that j_*^1 is surjective, whence we get

$$H_1(\mathcal{M}, M_g) \cong H_1(\mathcal{M}) / \operatorname{Ker} j_*^1 = H_1(\mathcal{M}) / \operatorname{Im} i_*^1.$$

However i_*^1 is surjective as well: for any

$$(n_1, \ldots, n_g) \in \bigoplus_{i=1}^{g} \mathbb{Z} \cong H_1(\mathcal{M})$$

we have

$$((0,n_1), \ldots, (0,n_g)) \in \bigoplus_{i=1}^{2g} \mathbb{Z} \cong H_1(M_g)$$

such that

$$\begin{aligned} i_*^1((0,n_1), \ldots, (0,n_g)) &= i_*^1(n_1((0,1), \ldots, (0,0)) + \cdots + n_g((0,0), \ldots, (0,1))) \\ &= n_1 i_*^1((0,1), \ldots, (0,0)) + \cdots + n_g i_*^1((0,0), \ldots, (0,1)) \\ &= n_1(1,0,\ldots,0) + \cdots + n_g(0,\ldots,0,1) \\ &= (n_1, \ldots, n_g), \end{aligned}$$

and therefore

$$H_1(\mathcal{M}, M_g) \cong H_1(\mathcal{M})/H_1(\mathcal{M}) \cong 0.$$

Finally, noting that $(\mathcal{M}, M_g)$ is a good pair, we have $H_0(\mathcal{M}, M_g) \cong \widetilde{H}_0(\mathcal{M}/M_g) = 0$ since the quotient space $\mathcal{M}/M_g$ is path-connected. Therefore

$$H_n(\mathcal{M}, M_g) \cong \begin{cases} 0, & \text{if } n = 0, 1 \text{ or } n \geq 4 \\ \bigoplus_{i=1}^{g} \mathbb{Z}, & \text{if } n = 2 \\ \mathbb{Z}, & \text{if } n = 3 \end{cases}$$

■