Chapter 1 – The Fundamental Group

1.1 – Basic Group Theory

Given a group G we usually denote the identity element of G by e, and denote the identity map $G \to G$ on G by $\mathbb{1}$. If two groups G and H are being considered, it is convenient to let e_G and e_H denote the identity elements of G and H, respectively, and let $\mathbb{1}_G$ and $\mathbb{1}_H$ denote identity maps.

Proposition 1.1. *For groups G and H, let $\varphi : G \to H$ and $\psi : H \to G$ be homomorphisms. If $\psi \circ \varphi = \mathbb{1}_G$ and $\varphi \circ \psi = \mathbb{1}_H$, then φ is an isomorphism.*

Proof. Suppose that $\psi \circ \varphi = \mathbb{1}_G$ and $\varphi \circ \psi = \mathbb{1}_H$. Let $x \in \mathrm{Ker}(\varphi)$, so $\varphi(x) = e_H$. Now,

$$x = \mathbb{1}_G(x) = (\psi \circ \varphi)(x) = \psi(\varphi(x)) = \psi(e_H) = e_G,$$

so $\mathrm{Ker}(\varphi) = \{e_G\}$ and it follows that φ is injective.

Now let $y \in H$. Then $\psi(y) \in G$ such that $\varphi(\psi(y)) = (\varphi \circ \psi)(y) = \mathbb{1}_H(y) = y$, which shows that φ is surjective.

Therefore φ is an isomorphism. ∎

Notice that, by the symmetry of the situation in this proposition, we can conclude that ψ is an isomorphism as well.