Chapter 3 – Cohomology

3.0 – Cohomology Introduction

Let X be a 1-dimensional Δ-complex, so X is an oriented graph (or pseudo-graph if edges that begin and end at the same vertex are allowed). If v_0 and v_1 are two vertices in X and e is an oriented edge from v_0 to v_1, then notationally we'll denote e by $[v_0, v_1]$. Recall that, formally, a "vertex" is a map $\sigma_\alpha : \Delta^0 \to X$, and an "edge" is a map $\tau_\beta : \Delta^1 \to X$ that, when restricted to each endpoint (or "face") of Δ^1, becomes one of the maps σ_α when pre-composed with the appropriate canonical linear homeomorphism $\Delta^0 \mapsto$ (face of Δ^1). So our edge e is in fact a map $\sigma_e : \Delta^1 \to X$ such that, if we denote Δ^1 by $[u_0, u_1]$ (a line segment), then $\sigma_e(u_0) = v_0$ and $\sigma_e(u_1) = v_1$; thus, if indeed $v_0 \neq v_1$, it follows that $\sigma_e(\Delta^1)$—truly our edge e in X in the graphical sense—is homeomorphic to the standard 1-simplex Δ^1, and so it makes sense to represent e using the simplex notation $[v_0, v_1]$ (especially since it conveys information about the orientation of e). If $v_0 = v_1$ it still makes sense to represent e by $[v_0, v_1] = [v_0, v_0]$ to maintain consistent notation even though the corresponding edge is not homeomorphic to any kind of simplex.[1]

Let G be an abelian group (not necessarily free), V the set of vertices of X, and E the set of edges of X. Define

$$\Delta^0(X; G) = \{\varphi : V \to G \mid \varphi \text{ is a function}\}$$

and

$$\Delta^1(X; G) = \{\psi : E \to G \mid \psi \text{ is a function}\}$$

Note $\Delta^0(X; G)$ forms an abelian group: if $\varphi_1, \varphi_2 \in \Delta^0(X; G)$, then $\varphi_1 + \varphi_2$ given by

$$(\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v)$$

shows that $\varphi_1 + \varphi_2 \in \Delta^0(X; G)$ since $\varphi_1(v) \in G$ and $\varphi_2(v) \in G$ implies that $\varphi_1(v) + \varphi_2(v) \in G$. In similar fashion $\Delta^1(X; G)$ is also an abelian group.

Now define a homomorphism $\delta_1 : \Delta^0(X; G) \to \Delta^1(X; G)$ as follows: for $\varphi \in \Delta^0(X; G)$ let $\delta_1\varphi \in \Delta^1(X; G)$ be such that, for $[v_0, v_1] \in E$, $\delta_1\varphi([v_0, v_1]) = \varphi(v_1) - \varphi(v_0)$. Set up a chain complex

$$\cdots \longrightarrow 0 \xrightarrow{\delta_0} \Delta^0(X; G) \xrightarrow{\delta_1} \Delta^1(X; G) \xrightarrow{\delta_2} 0 \longrightarrow \cdots .$$

By definition the homology groups associated with this chain complex are the **simplicial cohomology groups $H^n_\Delta(X; G)$ of X**. In particular we have

$$H^0_\Delta(X; G) = \frac{\operatorname{Ker}[\delta_1 : \Delta^0(X; G) \to \Delta^1(X; G)]}{\operatorname{Im}[\delta_0 : 0 \to \Delta^0(X; G)]} \cong \operatorname{Ker} \delta_1 \tag{1}$$

since $\operatorname{Im} \delta_0 = 0$, and

$$H^1_\Delta(X; G) = \frac{\operatorname{Ker}[\delta_2 : \Delta^1(X; G) \to 0]}{\operatorname{Im}[\delta_1 : \Delta^0(X; G) \to \Delta^1(X; G)]} \cong \frac{\Delta^1(X; G)}{\operatorname{Im}[\delta_1 : \Delta^0(X; G) \to \Delta^1(X; G)]}. \tag{2}$$

[1] Recall that the map σ_e is also called a 1-simplex.

So $H^0_\Delta(X;G) \cong \{\varphi \in \Delta^0(X;G) \;:\; \delta_1\varphi = 0\}$, where $0(e) := 0 \in G$ for every $e \in E$. Thus $\varphi \in H^0_\Delta(X;G)$ implies that

$$\delta_1\varphi([v_1, v_0]) = \varphi(v_1) - \varphi(v_0) = 0,$$

or $\varphi(v_1) = \varphi(v_0)$ for every vertex v_0 and v_1 that is connected by an edge in X. This in turn implies that φ must be constant on each *component* of X. Let $\{X_\alpha\}_{\alpha\in I}$ be the components of X, and let V_α be the set of vertices for the subgraph X_α. Then

$$\begin{aligned}\operatorname{Ker}\delta_1 &= \{\varphi \in \Delta^0(X;G) \;:\; \forall\,\alpha \in I \;\; \varphi \text{ is constant on } V_\alpha\}\\ &= \{\varphi \in \Delta^0(X;G) \;:\; \forall\,\alpha \in I \;\; \exists\, g_\alpha \in G \text{ s.t. } \varphi \equiv g_\alpha \text{ on } V_\alpha\}.\end{aligned}$$

It follows that each element of the group $\operatorname{Ker}\delta_1$ corresponds to some $\{g_\alpha\}_{\alpha\in I} \in \prod_{\alpha\in I} G$,[2] where it is necessary to consider the direct *product* of copies of G as opposed to the direct *sum* since it may well be that $g_\alpha \neq 0$ for an infinite number of index values α. The one-to-one correspondence $\operatorname{Ker}\delta_1 \mapsto \prod_{\alpha\in I} G$ can easily be shown to be an isomorphism, so therefore from (1) we obtain

$$H^0_\Delta(X;G) \cong \operatorname{Ker}\delta_1 \cong \prod_{\alpha\in I} G.$$

Next, from (2) is can be seen that $H^1_\Delta(X;G) = 0$ iff δ_1 is surjective, which is to say that for each $\psi \in \Delta^1(X;G)$ there exists some $\varphi \in \Delta^0(X;G)$ such that $\delta_1\varphi = \psi$. This will be the case whenever the components $\{X_\alpha\}_{\alpha\in I}$ of X are trees since the path between any two vertices in a tree is unique: for a given $\alpha \in I$, $[v_0, v_1] = e \in E_\alpha$ (the set of edges in X_α) and $\psi \in \Delta^1(X;G)$, we need only define $\varphi \in \Delta^0(X;G)$ such that $\varphi(v_1) - \varphi(v_0) = \psi(e)$, with the choice of definition being unique *up to a constant*; then the values of φ at all other vertices in X_α are set as dictated by the values of ψ on the edges of the unique paths leading to those vertices. The process is repeated for the other components of X to get $\delta_1\varphi = \psi$.

If a particular component X_α of X is not a tree, then we designate a *maximal tree* that is a subgraph of X_α which includes all of its vertices but not all edges. It is a fact that, for any choice of maximal tree for a graph, the cardinality of the set of edges omitted from the tree *will be the same.* For the sake of argument suppose that X is a connected graph that is not a tree, let $Y \subset X$ be a maximal tree, let E' be the set of edges not in Y, and let E'' be the set of edges in Y. The claim will be that

$$H^1_\Delta(X;G) \cong \prod_{e\in E'} G.$$

For the construction of the appropriate isomorphism, note that for any $\psi \in \Delta^1(X;G)$ there can be found some $\varphi \in \Delta^0(X;G)$ (unique up to a constant) such that $\delta_1\varphi|_{E''} = \psi|_{E''}$. Now define $F : H^1_\Delta(X;G) \to \prod_{e\in E'} G$ by

$$F(\psi + \operatorname{Im}\delta_1) = \{(\psi - \delta_1\varphi)(e)\}_{e\in E'} \;\text{ for some } \varphi \text{ such that } \delta_1\varphi|_{E''} = \psi|_{E''} \tag{3}$$

[2]Recall that formally $\{g_\alpha\}_{\alpha\in I}$ is a function $g : I \to G$ given by $g(\alpha) = g_\alpha$ for each $\alpha \in I$.

The choice for φ is irrelevant since the difference must be a constant: if $\hat{\varphi} = \varphi + g$ for some fixed $g \in G$ (more precisely $g : E \to \{g\}$), then

$$\begin{aligned} \delta_1\hat{\varphi}([v_0, v_1]) &= (\varphi + g)(v_1) - (\varphi + g)(v_0) \\ &= (\varphi(v_1) + g(v_1)) - (\varphi(v_0) + g(v_0)) \\ &= \varphi(v_1) + g - \varphi(v_0) - g = \delta_1\varphi([v_0, v_1]), \end{aligned}$$

and so $\psi - \delta_1\hat{\varphi} = \psi - \delta_1\varphi$.

It should first be verified that F is well-defined. We start with a simple case when X is a graph with three vertices and three edges as shown in the figure, so $E = \{e_0, e_1, e_2\}$. To construct a maximal tree Y we need only omit e_0, so that $E' = \{e_0\}$ and $E'' = \{e_1, e_2\}$. Suppose $\psi_1 + \operatorname{Im}\delta_1 = \psi_2 + \operatorname{Im}\delta_1$. There exist $\varphi_1, \varphi_2 \in \Delta^0(X; G)$ such that $\delta_1\varphi_1|_{E''} = \psi_1|_{E''}$ and $\delta_1\varphi_2|_{E''} = \psi_2|_{E''}$. To show

$$F(\psi_1 + \operatorname{Im}\delta_1) = F(\psi_2 + \operatorname{Im}\delta_1)$$

means to show that

$$(\psi_1 - \delta_1\varphi_1)(e_0) = (\psi_2 - \delta_1\varphi_2)(e_0),$$

or

$$(\psi_1 - \psi_2)(e_0) = \big[(\varphi_1(v_1) - \varphi_1(v_0)\big] - \big[(\varphi_2(v_1) - \varphi_2(v_0)\big]. \tag{4}$$

Now, $\psi_1 + \operatorname{Im}\delta_1 = \psi_2 + \operatorname{Im}\delta_1$ implies that $\psi_1 - \psi_2 \in \operatorname{Im}\delta_1$, and so there exists some $\varphi : V \to G$ such that

$$(\delta_1\varphi)(e_i) = (\psi_1 - \psi_2)(e_i)$$

for all $e_i \in E$. Hence

$$\psi_1(e_0) - \psi_2(e_0) = \varphi(v_1) - \varphi(v_0) \tag{5}$$

$$\psi_1(e_1) - \psi_2(e_1) = \varphi(v_2) - \varphi(v_0) \tag{6}$$

$$\psi_1(e_2) - \psi_2(e_2) = \varphi(v_2) - \varphi(v_1), \tag{7}$$

while from $\delta_1\varphi_1|_{E''} = \psi_1|_{E''}$ and $\delta_1\varphi_2|_{E''} = \psi_2|_{E''}$ we obtain

$$\psi_1(e_1) = \varphi_1(v_2) - \varphi_1(v_0) \quad \text{and} \quad \psi_2(e_1) = \varphi_2(v_2) - \varphi_2(v_0) \tag{8}$$

$$\psi_1(e_2) = \varphi_1(v_2) - \varphi_1(v_1) \quad \text{and} \quad \psi_2(e_2) = \varphi_2(v_2) - \varphi_2(v_1). \tag{9}$$

Combining (6) and (8) gives

$$\varphi(v_2) - \varphi(v_0) = \big[(\varphi_1(v_2) - \varphi_1(v_0)\big] - \big[(\varphi_2(v_2) - \varphi_2(v_0)\big], \tag{10}$$

and combining (7) and (9) gives

$$\varphi(v_2) - \varphi(v_1) = \big[(\varphi_1(v_2) - \varphi_1(v_1)\big] - \big[(\varphi_2(v_2) - \varphi_2(v_1)\big]. \tag{11}$$

Now, if we subtract (11) from (10) we obtain

$$\varphi(v_1) - \varphi(v_0) = \big[(\varphi_1(v_1) - \varphi_1(v_0)\big] - \big[(\varphi_2(v_1) - \varphi_2(v_0)\big]. \tag{12}$$

We now put (12) into (5) and get precisely (4), as desired.

A simpler analysis can be employed to show that F is well-defined in the case when a maximal tree is formed by deleting one edge and retaining one edge, which becomes the "base case" for an inductive argument that will establish that F is well-defined when one edge is deleted and n edges are retained, $n \in \mathbb{N}$ arbitrary. This result, once obtained, in turn becomes the base case for another inductive argument that establishes the well-definedness of F in the general case when m edges are deleted and n edges are retained in the forming of a maximal tree, $m, n \in \mathbb{N}$ both arbitrary. All of this can be done under the assumption that X is connected (i.e. has just one component), after which it is easy to extend to an arbitrary number of components.

To show that F is a homomorphism of groups, along with addition and integer multiplication of cosets in a *quotient* group we assume the usual (componentwise) definitions for addition and integer multiplication of elements in a *direct product* of groups. In what follows X is not assumed to be connected, so E'' is taken to be the set of edges included in the maximal tree for some component of X, and $E' = E - E''$. Let $m, n \in \mathbb{Z}$ and $\psi, \hat{\psi} \in \Delta^1(X; G)$. Then there exist $\varphi, \hat{\varphi} \in \Delta^0(X; G)$ such that $\delta_1\varphi = \psi$ and $\delta_1\hat{\varphi} = \hat{\psi}$. Now, for $e \in E''$ it's easy to see that

$$\delta_1(m\varphi + n\hat{\varphi})|_{E''} = (m\psi + n\hat{\psi})|_{E''}$$

since δ_1 is a homomorphism, and so by (3) we obtain

$$\begin{aligned}
F(m(\psi + \operatorname{Im}\delta_1) + n(\hat{\psi} + \operatorname{Im}\delta_1)) &= F((m\psi + n\hat{\psi}) + \operatorname{Im}\delta_1) \\
&= \{((m\psi + n\hat{\psi}) - \delta_1(m\varphi + n\hat{\varphi}))(e)\}_{e\in E'} \\
&= \{(m\psi + n\hat{\psi})(e) - (m\delta_1\varphi + n\delta_1\hat{\varphi})(e)\}_{e\in E'} \\
&= \{m\psi(e) + m\delta_1\varphi(e) + n\hat{\psi}(e) - n\delta_1\hat{\varphi}(e)\}_{e\in E'} \\
&= \{m(\psi - \delta_1\varphi)(e) + n(\hat{\psi} - \delta_1\hat{\varphi})(e)\}_{e\in E'} \\
&= m\{(\psi - \delta_1\varphi)(e)\}_{e\in E'} + n\{(\hat{\psi} - \delta_1\hat{\varphi})(e)\}_{e\in E'} \\
&= mF(\psi + \operatorname{Im}\delta_1) + nF(\hat{\psi} + \operatorname{Im}\delta_1).
\end{aligned}$$

Hence F is a homomorphism.

Let $\{g_e\}_{e\in E'} \in \prod_{e\in E'} G$. Define $\psi : E \to G$ by $\psi(e) = g_e$ for all $e \in E'$ and $\psi(e) = 0$ for all $e \in E''$. Let $\varphi : V \to G$ be any constant function, so there is some $g_0 \in G$ such that $\varphi(v) = g_0$ for all $v \in V$. Then $\psi|_{E''} = \delta_1\varphi|_{E''} \equiv 0$ and in fact $\delta_1\varphi \equiv 0$ everywhere, and by (3)

$$F(\psi + \operatorname{Im}\delta_1) = \{(\psi - \delta_1\varphi)(e)\}_{e\in E'} = \{\psi(e) - \delta_1\varphi(e)\}_{e\in E'} = \{\psi(e) - 0\}_{e\in E'} = \{g_e\}_{e\in E'},$$

which shows that F is surjective.

Finally, it remains to show that $\operatorname{Ker} F = \{\operatorname{Im}\delta_1\}$. We have

$$F(\operatorname{Im}\delta_1) = F(0 + \operatorname{Im}\delta_1) = \{(0 - \delta_1\varphi)(e)\}_{e\in E'}$$

for any constant function φ (so that $\delta_1\varphi|_{E''} = 0|_{E''}$ as required), and so

$$F(\operatorname{Im}\delta_1) = \{0(e)\}_{e\in E'} = \{0\}_{e\in E'}$$

and we obtain $\{\operatorname{Im}\delta_1\} \subset \operatorname{Ker} F$. Now, supposing that $\psi + \operatorname{Im}\delta_1 \in \operatorname{Ker} F$, we have

$$F(\psi + \operatorname{Im}\delta_1) = \{(\psi - \delta_1\varphi)(e)\}_{e\in E'} = \{0\}_{e\in E'}$$

for some φ such that $\delta_1\varphi|_{E''} = \psi|_{E''}$; but then it is clear that $\delta_1\varphi|_{E'} = \psi|_{E'}$ as well, and so $\delta_1\varphi = \psi$ on all E and we find that $\psi \in \operatorname{Im}\delta_1$. Therefore $\psi + \operatorname{Im}\delta_1 = \operatorname{Im}\delta_1$ and we have $\operatorname{Ker} F \subset \{\operatorname{Im}\delta_1\}$. Since the kernel of F is trivial, F is injective.

It has been established at last that F is an isomorphism, and therefore

$$H^1_\Delta(X;G) \cong \prod_{e\in E'} G.$$

Now, suppose that X is a two-dimensional Δ-complex. Let S_2 be the set of 2-simplices of X, so

$$S_2 = \{\sigma_\alpha : \Delta^2 \to X\}_{\alpha\in A},$$

and let $\Delta^2(X;G) = \{\omega : S_2 \to G\}$. Adhering to the notational conventions above, we define the homomorphism $\delta_2 : \Delta^1(X;G) \to \Delta^2(X;G)$ by

$$\delta_2\psi([v_0, v_1, v_2]) = \psi([v_0, v_1]) - \psi([v_0, v_2]) + \psi([v_1, v_2]) \tag{13}$$

for each $\psi \in \Delta^1(X;G)$, where $[v_0, v_1, v_2] := \sigma \in S_2$ is a map that maps the vertices of Δ^2 to v_0, v_1, and v_2. It's worthwhile to be more precise here: if we let $\Delta^2 = [u_0, u_1, u_2]$ then $\sigma(u_i) = v_i$ for each i, and moreover each $[v_i, v_j]$ denotes σ restricted to the face $[u_i, u_j]$ and precomposed by the canonical linear homeomorphism $\Delta^1 \mapsto [u_i, u_j]$. So more explicitly (13) can be written

$$\delta_2\psi(\sigma) = \psi(\sigma|_{[u_0,u_1]}) - \psi(\sigma|_{[u_0,u_2]}) + \psi(\sigma|_{[u_1,u_2]}) \tag{14}$$

At last we arrive at the general case of an n-dimensional Δ-complex X. For $0 \le i \le n$ let S_i be the set of i-simplices $\Delta^i \mapsto X$ of X, and let $\Delta^i(X;G)$ be the set of functions $S_i \mapsto G$. We define the map $\delta_i : \Delta^{i-1}(X;G) \to \Delta^i(X;G)$ by generalizing (14): for each $\psi \in \Delta^{i-1}(X;G)$ the function $\delta_i\psi$ is such that, for each $\sigma : \Delta^i \to X$ in S_i,

$$\delta_i\psi(\sigma) = \sum_{j=0}^{i} (-1)^j \psi(\sigma|_{[u_0,\ldots,\hat{u}_j,\ldots,u_i]}), \tag{15}$$

where in general $\Delta^i = [u_0, ..., u_i]$. In this way we obtain a chain complex

$$\cdots \xleftarrow{\delta_{i+2}} \Delta^{i+1}(X;G) \xleftarrow{\delta_{i+1}} \Delta^i(X;G) \xleftarrow{\delta_i} \Delta^{i-1}(X;G) \xleftarrow{\delta_{i-1}} \cdots \tag{16}$$

There's a natural way to identify the abelian group $\Delta^i(X;G)$ with the group $\operatorname{Hom}(\Delta_i(X), G)$ of homomorphisms $\Delta_i(X) \mapsto G$. In particular each $\psi \in \Delta^i(X;G)$ can be made to correspond via a fixed isomorphism to $\hat{\psi} \in \operatorname{Hom}(\Delta_i(X), G)$ given by

$$\hat{\psi}(\sum_\alpha n_\alpha\sigma_\alpha) = \sum_\alpha n_\alpha\psi(\sigma_\alpha).$$

Identifying ψ with $\hat{\psi}$, then, we find from (15) that $\delta_i\psi(\sigma) = \psi(\partial_i(\sigma))$ and therefore $\delta_i\psi = \psi\partial_i$.[3] By definition this means that δ_i is the dual map, called the **coboundary map**, of ∂_i. Going

[3]If we want to be fussy we can write $\delta_i\psi = \psi \circ \partial_i$ to stress that $\delta_i\psi$ is not a composition of functions.

a step further we designate $\Delta^i(X;G)$ (identified with $\operatorname{Hom}(\Delta_i(X),G)$) to be the dual **cochain group** of $\Delta_i(X)$ so that (16) is the dual **cochain complex** of the chain complex

$$\cdots \xrightarrow{\partial_{i+2}} \Delta_{i+1}(X) \xrightarrow{\partial_{i+1}} \Delta_i(X) \xrightarrow{\partial_i} \Delta_{i-1}(X) \xrightarrow{\partial_{i-1}} \cdots \tag{17}$$

The operation of passing from (17) to (16) can be characterized as the action of a contravariant functor $\Delta_i(X) \mapsto \operatorname{Hom}(\Delta_i(X),G)$, or more generally $C \mapsto \operatorname{Hom}(C,G)$ for any chain C, which sometimes is denoted by $\operatorname{Hom}(-,G)$. The categories involved here are the category C of objects $\Delta_i(X)$ and morphisms ∂_i (which will be the zero homomorphism for chain groups $\Delta_j(X)$ and $\Delta_k(X)$ with $|j-k|>1$), and the category D of objects $\operatorname{Hom}(\Delta_i(X),G)$ and morphisms δ_i.

3.1 – Cohomology of Chain Complexes

Starting with a chain complex C not associated with any topological space,

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots,$$

where each C_n is a free abelian group, we dualize by applying $\operatorname{Hom}(-,G)$ for some abelian group G to obtain the cochain

$$\cdots \xleftarrow{\delta_{n+2}} \operatorname{Hom}(C_{n+1},G) \xleftarrow{\delta_{n+1}} \operatorname{Hom}(C_n,G) \xleftarrow{\delta_n} \operatorname{Hom}(C_{n-1},G) \xleftarrow{\delta_{n-1}} \cdots$$

In general $\delta_n\varphi := \delta_n(\varphi) := \varphi \circ \partial_n$. For each homology group $H_n(C) = \operatorname{Ker}\partial_n/\operatorname{Im}\partial_{n+1}$ there is a corresponding cohomology group

$$H^n(C;G) = \frac{\operatorname{Ker}\delta_{n+1}}{\operatorname{Im}\delta_n}.$$

An element of $H^n(C;G)$ is $\varphi + \operatorname{Im}\delta_n$, where $\varphi \in \operatorname{Ker}\delta_{n+1}$ implies that $\varphi\circ\partial_{n+1}$ is the trivial homomorphism: for all $x \in C_{n+1}$, $(\varphi\circ\partial_{n+1})(x) = 0$ in G.

Fix $\varphi \in \operatorname{Ker}\delta_{n+1}$. Now, since $\varphi : C_n \to G$ and $\operatorname{Ker}\partial_n \subset C_n$, we can define $\varphi_0 = \varphi|_{\operatorname{Ker}\partial_n}$, which in turn induces a map $\bar\varphi_0 : H_n(C) \to G$ given by

$$\bar\varphi_0(z + \operatorname{Im}\partial_{n+1}) = \varphi_0(z).$$

Finally, define $h : H^n(C;G) \to \operatorname{Hom}(H_n(C),G)$ by

$$h(\varphi + \operatorname{Im}\delta_n) = \bar\varphi_0.$$

It needs to be shown that h is well-defined.

Suppose that $\varphi + \operatorname{Im}\delta_n = \varphi' + \operatorname{Im}\delta_n$. Then $(\varphi-\varphi') + \operatorname{Im}\delta_n = \operatorname{Im}\delta_n$ implies $\varphi - \varphi' \in \operatorname{Im}\delta_n$, and so there exists some $\psi \in \operatorname{Hom}(C_{n-1},G)$ such that $\delta_n(\psi) = \varphi - \varphi'$, whence $\psi\circ\partial_n = \varphi - \varphi'$ and we obtain $\varphi' = \varphi - \psi\circ\partial_n$. Now, for any $z \in \operatorname{Ker}\partial_n$ we have

$$\varphi_0'(z) = \varphi'(z) = (\varphi - \psi\circ\partial_n)(z) = \varphi(z) - (\psi\circ\partial_n)(z) = \varphi(z) - \psi(0) = \varphi(z) = \varphi_0(z),$$

and so for any $z + \operatorname{Im}\partial_{n+1} \in H_n(C)$

$$\bar\varphi_0(z + \operatorname{Im}\partial_{n+1}) = \varphi_0(z) = \varphi_0'(z) = \bar\varphi_0'(z + \operatorname{Im}\partial_{n+1}).$$

Therefore

$$h(\varphi + \operatorname{Im}\delta_n) = \bar{\varphi}_0 = \bar{\varphi}_0' = h(\varphi' + \operatorname{Im}\delta_n)$$

and h is well-defined. Moreover it is clear that h is a homomorphism.

Next it will be shown that h is surjective. Let $f \in \operatorname{Hom}(H_n(C), G)$. We must find some $\varphi + \operatorname{Im}\delta_n \in H^n(C; G)$ such that $h(\varphi + \operatorname{Im}\delta_n) = \bar{\varphi}_0 = f$, which is to say that for each $z \in \operatorname{Ker}\partial_n$ we have

$$f(z + \operatorname{Im}\partial_{n+1}) = \bar{\varphi}_0(z + \operatorname{Im}\partial_{n+1}).$$

Start by defining $\varphi_0 : \operatorname{Ker}\partial_n \to G$ by

$$\varphi_0(z) = f(z + \operatorname{Im}\partial_{n+1}).$$

The task is to extend φ_0 to a map $\varphi : C_n \to G$ such that $\varphi \in \operatorname{Ker}\delta_{n+1}$.

Defining $i : \operatorname{Ker}\partial_n \to C_n$ to be the inclusion map, observe that the sequence

$$0 \longrightarrow \operatorname{Ker}\partial_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} \operatorname{Im}\partial_n \longrightarrow 0 \tag{18}$$

is exact. Since $\operatorname{Im}\partial_n$ is a free group the sequence splits, and so by the Splitting Lemma there exists a homomorphism $p : C_n \to \operatorname{Ker}\partial_n$ such that

$$p \circ i_n = \mathbb{1} : \operatorname{Ker}\partial_n \to \operatorname{Ker}\partial_n.$$

Define $\varphi = \varphi_0 \circ p : C_n \to G$, which clearly is a homomorphism. Now, for any $z \in \operatorname{Ker}\partial_n$,

$$\varphi(z) = \varphi_0(p(z)) = \varphi_0(p(i_n(z))) = \varphi_0((p \circ i_n)(z)) = \varphi_0(\mathbb{1}(z)) = \varphi_0(z) = f(z + \operatorname{Im}\partial_{n+1})$$

shows that φ is an extension of φ_0 to C_n.

Fix $x \in C_{n+1}$. Then

$$\begin{aligned}(\varphi \circ \partial_{n+1})(x) &= \varphi_0(p(\partial_{n+1}x)) = \varphi_0(p(i_n(\partial_{n+1}x))) = \varphi_0(\mathbb{1}(\partial_{n+1}x)) \\ &= \varphi_0(\partial_{n+1}x) = f(\partial_{n+1}x + \operatorname{Im}\partial_{n+1}) = f(\operatorname{Im}\partial_{n+1}) = 0 \in G,\end{aligned}$$

where the second equality holds since $\partial_{n+1}(x) \in \operatorname{Ker}\partial_n$ and the last holds since $\operatorname{Im}\partial_{n+1}$ is the zero element of $H_n(C)$. Hence $\delta_{n+1}(\varphi) = \varphi \circ \partial_{n+1} \equiv 0$, implying that $\varphi \in \operatorname{Ker}\delta_{n+1}$ and therefore $\varphi + \operatorname{Im}\delta_n \in H^n(C; G)$.

By definition

$$h(\varphi + \operatorname{Im}\delta_n) = \overline{\varphi|_{\operatorname{Ker}\partial_n}},$$

where for any $z + \operatorname{Im}\partial_{n+1} \in H_n(C)$ we obtain

$$\overline{\varphi|_{\operatorname{Ker}\partial_n}}(z + \operatorname{Im}\partial_{n+1}) = (\varphi_0 \circ p)|_{\operatorname{Ker}\partial_n}(z) = \varphi_0(p(z)) = \varphi_0(z) = f(z + \operatorname{Im}\partial_{n+1}),$$

using the fact that $z \in \operatorname{Ker}\partial_n$ implies $p(z) = p(i_n(z)) = \mathbb{1}(z) = z$. Therefore $h(\varphi + \operatorname{Im}\delta_n) = f$ and h is surjective.

To determine the conditions in which h may be injective we analyze $\operatorname{Ker} h$. Start with the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker}\partial_{n+1} & \xrightarrow{i_{n+1}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & \operatorname{Im}\partial_{n+1} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \partial_{n+1}} & & \Big\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \operatorname{Ker}\partial_n & \xrightarrow{i_n} & C_n & \xrightarrow{\partial_n} & \operatorname{Im}\partial_n & \longrightarrow & 0 \end{array} \tag{19}$$

where the map $\operatorname{Ker}\partial_{n+1} \to \operatorname{Ker}\partial_n$ is $\partial_{n+1}|_{\operatorname{Ker}\partial_{n+1}}$, and $\operatorname{Im}\partial_{n+1} \to \operatorname{Im}\partial_n$ is $\partial_n|_{\operatorname{Im}\partial_{n+1}}$. We dualize (19) by applying $\operatorname{Hom}(-,G)$ to obtain

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \operatorname{Hom}(\operatorname{Ker}\partial_{n+1},G) & \xleftarrow{i^*_{n+1}} & \operatorname{Hom}(C_{n+1},G) & \xleftarrow{\partial^*_{n+1}} & \operatorname{Hom}(\operatorname{Im}\partial_{n+1},G) & \longleftarrow & 0 \\ & & \Big\uparrow{\scriptstyle 0} & & \Big\uparrow{\scriptstyle \partial^*_{n+1}} & & \Big\uparrow{\scriptstyle 0} & & \\ 0 & \longleftarrow & \operatorname{Hom}(\operatorname{Ker}\partial_n,G) & \xleftarrow{i^*_n} & \operatorname{Hom}(C_n,G) & \xleftarrow{\partial^*_n} & \operatorname{Hom}(\operatorname{Im}\partial_n,G) & \longleftarrow & 0 \end{array} \tag{20}$$

remembering that $0^* = 0$. It will be shown that the rows of (20) are split short exact sequences.

Proposition 3.1. *The dual of a split short exact sequence is a split short exact sequence.*

Proof. Suppose that

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0 \tag{21}$$

is a split short exact sequence. It must be shown that the sequence

$$0 \longrightarrow \operatorname{Hom}(C,G) \xrightarrow{j^*} \operatorname{Hom}(B,G) \xrightarrow{i^*} \operatorname{Hom}(A,G) \longrightarrow 0 \tag{22}$$

is exact and splits.

For $\varphi \in \operatorname{Hom}(C,G)$ suppose that $j^*(\varphi) = 0$, so $\varphi \circ j = 0$. Fix $c \in C$. Since j is surjective there exists some $b \in B$ such that $j(b) = c$, and thus

$$\varphi(c) = \varphi(j(b)) = (\varphi \circ j)(b) = 0.$$

This shows that for any $\varphi \in \operatorname{Ker} j^*$ we have $\varphi = 0$, so $\operatorname{Ker} j^* = \{0\}$.

Let $\varphi \in \operatorname{Im} j^*$, so there exists some $\psi \in \operatorname{Hom}(C,G)$ such that $j^*(\psi) = \varphi$, or equivalently $\psi \circ j = \varphi$. Now, for any $a \in A$ we obtain

$$(\varphi \circ i)(a) = (\psi \circ j \circ i)(a) = \psi(j(i(a))) = \psi(0) = 0,$$

where $\operatorname{Im} i = \operatorname{Ker} j$ implies that $i(a) \in \operatorname{Ker} j$. Thus $i^*(\varphi) = \varphi \circ i = 0$, implying $\varphi \in \operatorname{Ker} i^*$ and so $\operatorname{Im} j^* \subset \operatorname{Ker} i^*$.

Let $\varphi \in \operatorname{Ker} i^*$, so $\varphi : B \to G$ such that $i^*\varphi = 0$, or equivalently $\varphi \circ i = 0$ which informs us that φ vanishes on $\operatorname{Im} i$. By the Splitting Lemma there exists some $s : C \to B$ such that $j \circ s = \mathbb{1} : C \to C$. Let $\psi = \varphi \circ s$. Fix $b \in B$. Then $(s \circ j)(b) - b \in B$ with

$$j((s \circ j)(b) - b) = (j \circ s \circ j)(b) - j(b) = (\mathbb{1} \circ j)(b) - j(b) = j(b) - j(b) = 0,$$

so that $(s \circ j)(b) - b \in \operatorname{Ker} j = \operatorname{Im} i$ and there is some $a \in A$ such that $i(a) = (s \circ j)(b) - b$. Since φ vanishes on $\operatorname{Im} i$ it follows that $(\varphi \circ i)(a) = 0$, whence $\varphi((s \circ j)(b) - b) = 0$ leads to $\varphi((s \circ j)(b)) = \varphi(b)$. Now,

$$(\psi \circ j)(b) = (\varphi \circ s \circ j)(b) = \varphi((s \circ j)(b)) = \varphi(b)$$

shows that $j^*(\psi) = \psi \circ j = \varphi$, so $\varphi \in \operatorname{Im} j^*$ and we obtain $\operatorname{Ker} i^* \subset \operatorname{Im} j^*$.

Finally, fix $\varphi \in \operatorname{Hom}(A, G)$. The Splitting Lemma implies there is a homomorphism $p : B \to A$ such that $p \circ i = \mathbb{1} : A \to A$. For any $a \in A$,

$$(\varphi \circ p \circ i)(a) = \varphi((p \circ i)(a)) = \varphi(\mathbb{1}(a)) = \varphi(a),$$

and so $\varphi = \varphi \circ p \circ i$. But $\varphi \circ p \in \operatorname{Hom}(B, G)$ such that $i^*(\varphi \circ p) = \varphi \circ p \circ i$, so $\varphi \in \operatorname{Im} i^*$ and it follows that $\operatorname{Im} i^* = \operatorname{Hom}(A, G)$.

Moving on, since (21) splits there is an isomorphism Φ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[dr, "f"'] & B \arrow[r, "j"] \arrow[d, "\Phi", "\cong"'] & C \arrow[r] & 0 \\
& & A \oplus C \arrow[ur, "g"'] & &
\end{tikzcd}$$

is commutative. The dualization of this diagram is

$$\begin{tikzcd}
0 & \operatorname{Hom}(A,G) \arrow[l] & \operatorname{Hom}(B,G) \arrow[l, "i^*"'] & \operatorname{Hom}(C,G) \arrow[l, "j^*"'] \arrow[dl, "g^*"] & 0 \arrow[l] \\
& & \operatorname{Hom}(A \oplus C, G) \arrow[ul, "f^*"] \arrow[u, "\Phi^*"', "\cong"] & &
\end{tikzcd}$$

where Φ^* is an isomorphism since the dual of any isomorphism is again an isomorphism. It's easily verified that $g \circ \Phi = j$ implies $\Phi^* \circ g^* = j^*$ and $\Phi \circ i = f$ implies $i^* \circ \Phi^* = f^*$ (in general $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$), so the dualized diagram is commutative. Finally, there's a natural isomorphism

$$\Omega : \operatorname{Hom}(A \oplus C, G) \to \operatorname{Hom}(A, G) \oplus \operatorname{Hom}(C, G)$$

defined by

$$\Omega(\varphi(\cdot\,, \cdot)) = (\varphi(\cdot\,, 0), \varphi(0, \cdot)),$$

so if we define $\bar{f}^* = f^* \circ \Omega^{-1}$, $\bar{g}^* = \Omega \circ g^*$, and $\bar{\Phi}^* = \Phi^* \circ \Omega^{-1}$, then we obtain the commutative diagram

$$\begin{tikzcd}
0 & \operatorname{Hom}(A,G) \arrow[l] & \operatorname{Hom}(B,G) \arrow[l, "i^*"'] & \operatorname{Hom}(C,G) \arrow[l, "j^*"'] \arrow[dl, "\bar{g}^*"] & 0 \arrow[l] \\
& & \operatorname{Hom}(A,G) \oplus \operatorname{Hom}(C,G) \arrow[ul, "\bar{f}^*"] \arrow[u, "\bar{\Phi}^*"', "\cong"] & &
\end{tikzcd}$$

which shows that the sequence (22) splits. ■

Let

$$B_n = \operatorname{Im} \partial_{n+1} \quad \text{and} \quad Z_n = \operatorname{Ker} \partial_n,$$

and let

$$C_n^* = \operatorname{Hom}(C_n, G), \quad Z_n^* = \operatorname{Hom}(Z_n, G), \quad B_n^* = \operatorname{Hom}(B_n, G).$$

Finally, let $\delta_n : C_{n-1}^* \to C_n^*$ be the dual of $\partial_n : C_n \to C_{n-1}$ as before, and let $\varrho_n : B_{n-1}^* \to C_n^*$ be the dual of $\partial_n : C_n \to B_{n-1}$. The diagram (20) can be extended to a short exact sequence of chain complexes

$$\begin{tikzcd}
 & 0 & 0 & 0 & & \\
\cdots & Z_{n+1}^* \arrow[l] \arrow[u] & Z_n^* \arrow[l, "0"'] \arrow[u] & Z_{n-1}^* \arrow[l, "0"'] \arrow[u] & \cdots \arrow[l] & (Z^*) \\
\cdots & C_{n+1}^* \arrow[l] \arrow[u, "i_{n+1}^*"] & C_n^* \arrow[l, "\delta_{n+1}"'] \arrow[u, "i_n^*"] & C_{n-1}^* \arrow[l, "\delta_n"'] \arrow[u, "i_{n-1}^*"] & \cdots \arrow[l] & (C^*) \\
\cdots & B_n^* \arrow[l] \arrow[u, "\varrho_{n+1}"] & B_{n-1}^* \arrow[l, "0"'] \arrow[u, "\varrho_n"] & B_{n-2}^* \arrow[l, "0"'] \arrow[u, "\varrho_{n-1}"] & \cdots \arrow[l] & (B^*) \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{23}$$

Associated with this diagram is a long exact sequence of cohomology groups

$$\cdots \longleftarrow H^{n+1}(Z^*) \xleftarrow{i_{n+1}^{**}} H^{n+1}(C;G) \xleftarrow{\varrho_{n+1}^*} H^{n+1}(B^*) \xleftarrow{d_n} H^n(Z^*) \xleftarrow{i_n^{**}} H^n(C;G)$$
$$\xleftarrow{\varrho_n^*} H^n(B^*) \xleftarrow{d_{n-1}} H^{n-1}(Z^*) \longleftarrow \cdots, \tag{24}$$

where each i_n^{**} and ϱ_n^* is a homomorphism induced by i_n^* and ϱ_n, and each d_n is a connecting homomorphism which will be examined shortly. By definition

$$H^n(B^*) = \frac{\operatorname{Ker}[0 : B_{n-1}^* \to B_n^*]}{\operatorname{Im}[0 : B_{n-2}^* \to B_{n-1}^*]} \quad \text{and} \quad H^n(Z^*) = \frac{\operatorname{Ker}[0 : Z_n^* \to Z_{n+1}^*]}{\operatorname{Im}[0 : Z_{n-1}^* \to Z_n^*]},$$

so $H^n(B^*) \cong B_{n-1}^*$ and $H^n(Z^*) \cong Z_n^*$, and (24) can be written as

$$\cdots \longleftarrow Z_{n+1}^* \xleftarrow{i_{n+1}^{**}} H^{n+1}(C;G) \xleftarrow{\varrho_{n+1}^*} B_n^* \xleftarrow{d_n} Z_n^* \xleftarrow{i_n^{**}} H^n(C;G)$$
$$\xleftarrow{\varrho_n^*} B_{n-1}^* \xleftarrow{d_{n-1}} Z_{n-1}^* \longleftarrow \cdots, \tag{25}$$

with appropriate adjustments to the definitions of i_n^{**} and ϱ_n^*; in particular, let $i_n^{**}(\sigma + \operatorname{Im}\delta_n) = i_n^*(\sigma)$ and $\varrho_n^*(\varphi) = \varrho_n(\varphi) + \operatorname{Im}\delta_n$ (it's easily verified that $\varrho_n(\varphi) = \varphi \circ \partial_n \in \operatorname{Ker}\delta_{n+1}$).

We define d_n in reference to (24). Let $\gamma \in Z_n^*$ be a cycle, so it represents a cohomology class $[\gamma] \in H^n(Z^*)$ (note that in fact every element of Z_n^* is a cycle). Since i_n^* is surjective, there exists some $\beta \in C_n^*$ such that $i_n^*(\beta) = \gamma$. Exploiting commutativity in (23) gives

$$i_{n+1}^*(\delta_{n+1}(\beta)) = 0(i_n^*(\beta)) = 0,$$

so $\delta_{n+1}(\beta) \in \operatorname{Ker} i_{n+1}^* = \operatorname{Im} \varrho_{n+1}$ and there must exist some $\alpha \in B_n^*$ such that $\varrho_{n+1}(\alpha) = \delta_{n+1}(\beta)$. Since $\alpha \in \operatorname{Ker}[0 : B_n^* \to B_{n+1}^*]$, α represents a cohomology class $[\alpha] \in H^{n+1}(B^*)$. Define $d_n([\gamma]) = [\alpha]$. Since $H^{n+1}(B^*) \cong B_n^*$ and $H^n(Z^*) \cong Z_n^*$, in reference to (25) we can simply define $d_n(\gamma) = \alpha$.

It turns out that $\alpha = \gamma|_{B_n}$. From $\varrho_{n+1}(\alpha) = \delta_{n+1}(\beta)$ comes $\alpha \circ \partial_{n+1} = \beta \circ \partial_{n+1}$, which shows that $\alpha = \beta|_{B_n}$ (recall that $\alpha : B_n \to G$). But we also have $\beta \circ i_n = \gamma$ for $i_n : Z_n \hookrightarrow C_n$, so $\gamma = \beta|_{Z_n}$. Since $B_n \subset Z_n$ it follows that $\gamma|_{B_n} = \beta|_{B_n} = \alpha$. Hence $d_n(\gamma) = \gamma|_{B_n}$, and so if $\iota_n : B_n \hookrightarrow Z_n$ is the inclusion map then it's seen that d_n is nothing more than $\iota_n^* : Z_n^* \to B_n^*$, the dual of ι_n.

The process of verifying that (25) is exact is the same as for any long exact sequence in the previous chapter. From this sequence we can pass to a new sequence

$$0 \xleftarrow{\iota_n^*} \operatorname{Ker} \iota_n^* \xleftarrow{i_n^{**}} H^n(C;G) \xleftarrow{\zeta} \operatorname{Coker} \iota_{n-1}^* \longleftarrow 0 \tag{26}$$

where

$$\operatorname{Coker} \iota_{n-1}^* = \frac{B_{n-1}^*}{\operatorname{Im} \iota_{n-1}^*}$$

and ζ works in the expected fashion: for any $\varphi \in B_{n-1}^*$,

$$\zeta(\varphi + \operatorname{Im} \iota_{n-1}^*) = \varrho_n^*(\varphi) = \varrho_n(\varphi) + \operatorname{Im} \delta_n.$$

It's worth verifying that ζ is well-defined, so suppose

$$\varphi_1 + \operatorname{Im} \iota_{n-1}^* = \varphi_2 + \operatorname{Im} \iota_{n-1}^*.$$

Then $\varphi_1 - \varphi_2 \in \operatorname{Im} \iota_{n-1}^* = \operatorname{Ker} \varrho_n^*$, using the exactness of (25). Now,

$$\begin{aligned}
\zeta(\varphi_1 + \operatorname{Im} \iota_{n-1}^*) - \zeta(\varphi_2 + \operatorname{Im} \iota_{n-1}^*) &= (\varrho_n(\varphi_1) + \operatorname{Im} \delta_n) - (\varrho_n(\varphi_2) + \operatorname{Im} \delta_n) \\
&= (\varrho_n(\varphi_1) - \varrho_n(\varphi_2)) + \operatorname{Im} \delta_n \\
&= \varrho_n(\varphi_1 - \varphi_2) + \operatorname{Im} \delta_n \\
&= \varrho_n^*(\varphi_1 - \varphi_2) = \operatorname{Im} \delta_n,
\end{aligned}$$

since $\varphi_1 - \varphi_2 \in \operatorname{Ker} \varrho_n^*$). That is,

$$(\varrho_n(\varphi_1) - \varrho_n(\varphi_2)) + \operatorname{Im} \delta_n = \operatorname{Im} \delta_n,$$

which implies

$$\varrho_n(\varphi_1) + \operatorname{Im} \delta_n = \varrho_n(\varphi_2) + \operatorname{Im} \delta_n,$$

or

$$\zeta(\varphi_1 + \operatorname{Im} \iota_{n-1}^*) = \zeta(\varphi_2 + \operatorname{Im} \iota_{n-1}^*).$$

It's clear that ζ is a homomorphism.

The sequence (26) is a short exact sequence. Suppose $\zeta(\varphi + \operatorname{Im} \iota_{n-1}^*) = \operatorname{Im} \delta_n$. Then $\varrho_n(\varphi) \in \operatorname{Im} \delta_n$, and so there exists some $\psi \in C_{n-1}^*$ such that $\delta_n(\psi) = \varrho_n(\varphi)$, whence $\varphi \circ \partial_n = \psi \circ \partial_n$ and thus $\psi|_{B_{n-1}} = \varphi$. Now $\psi|_{Z_{n-1}} \in Z_{n-1}^*$, and

$$\iota_{n-1}^*(\psi|_{Z_{n-1}}) = \psi|_{Z_{n-1}} \circ \iota_{n-1} = \psi|_{B_{n-1}} = \varphi$$

shows that $\varphi \in \operatorname{Im} \iota_{n-1}^*$ and hence $\varphi + \operatorname{Im} \iota_{n-1}^* = \operatorname{Im} \iota_{n-1}^*$. Therefore $\operatorname{Ker} \zeta = 0$ and ζ is injective.

Fix $\sigma \in \operatorname{Ker} \iota_n^*$, so $\sigma : Z_n \to G$ such that $\sigma|_{B_n} \equiv 0$. Since (18) is exact, by the Splitting Lemma there is some $p : C_n \to Z_n$ such that $p \circ i_n = \mathbb{1} : Z_n \to Z_n$. Let $\hat{\sigma} = \sigma \circ p$, so $\hat{\sigma} \in C_n^*$. For any $x \in C_{n+1}$,

$$(\sigma \circ p \circ \partial_{n+1})(x) = (\sigma \circ p)(\partial_{n+1} x) = (\sigma \circ p)(i_n(\partial_{n+1} x)) = (\sigma \circ \mathbb{1})(\partial_{n+1} x) = \sigma(\partial_{n+1} x) = 0$$

(since $\partial_{n+1} x \in B_n$), which shows that

$$\delta_{n+1}(\hat{\sigma}) = \hat{\sigma} \circ \partial_{n+1} = \sigma \circ p \circ \partial_{n+1} \equiv 0$$

on C_{n+1}. Hence $\hat{\sigma} \in \operatorname{Ker} \delta_{n+1}$ so that $\hat{\sigma} + \operatorname{Im} \delta_n \in H^n(C; G)$, and since

$$i_n^{**}(\hat{\sigma} + \operatorname{Im} \delta_n) = i_n^*(\hat{\sigma}) = \hat{\sigma} \circ i_n = \sigma \circ p \circ i_n = \sigma \circ \mathbb{1} = \sigma$$

we find $\operatorname{Ker} \iota_n^* \subset \operatorname{Im} i_n^{**}$. As for the reverse containment, note that $\varphi \in \operatorname{Ker} \delta_{n+1}$ implies $\varphi|_{B_n} \equiv 0$, so

$$\iota_n^*(i_n^{**}(\varphi + \operatorname{Im} \delta_n)) = \iota_n^*(\varphi \circ i_n) = \varphi \circ i_n \circ \iota_n = \varphi|_{B_n} \equiv 0$$

shows that i_n^{**} maps into $\operatorname{Ker} \iota_n^*$. Hence $\operatorname{Ker} \iota_n^* = \operatorname{Im} i_n^{**}$ and i_n^{**} in (26) is surjective.

It remains to confirm that $\operatorname{Im} \zeta = \operatorname{Ker} i_n^{**}$. Since (25) is exact we have $\operatorname{Ker} i_n^{**} = \operatorname{Im} \varrho_n^*$. Let $\varphi + \operatorname{Im} \delta_n \in \operatorname{Im} \varrho_n^*$, so there exists some $\psi \in B_{n-1}^*$ such that $\varrho_n^*(\psi) = \varphi + \operatorname{Im} \delta_n$, or $\varphi + \operatorname{Im} \delta_n = \psi \circ \partial_n + \operatorname{Im} \delta_n$; but then $\psi + \operatorname{Im} \iota_{n-1}^* \in \operatorname{Coker} \iota_{n-1}^*$ with

$$\zeta(\psi + \operatorname{Im} \iota_{n-1}^*) = \varrho_n(\psi) + \operatorname{Im} \delta_n = \psi \circ \partial_n + \operatorname{Im} \delta_n = \varphi + \operatorname{Im} \delta_n,$$

which gives $\operatorname{Im} \varrho_n^* \subset \operatorname{Im} \zeta$. On the other hand, if $\varphi + \operatorname{Im} \delta_n \in \operatorname{Im} \zeta$ then there's some $\psi + \operatorname{Im} \iota_{n-1}^* \in \operatorname{Coker} \iota_{n-1}^*$ with

$$\zeta(\psi + \operatorname{Im} \iota_{n-1}^*) = \varphi + \operatorname{Im} \delta_n,$$

or equivalently $\psi \circ \partial_n + \operatorname{Im} \delta_n = \varphi + \operatorname{Im} \delta_n$; but $\psi \in B_{n-1}^*$ such that

$$\varrho_n^*(\psi) = \varrho_n(\psi) + \operatorname{Im} \delta_n = \psi \circ \partial_n + \operatorname{Im} \delta_n = \varphi + \operatorname{Im} \delta_n,$$

which makes clear that $\operatorname{Im} \zeta \subset \operatorname{Im} \varrho_n^*$ and so $\operatorname{Im} \zeta = \operatorname{Ker} i_n^{**}$.

Therefore (26) is exact as claimed.

Now, for each $\sigma \in \operatorname{Ker} \iota_n^*$ there is a corresponding map $\bar{\sigma} : H_n(C) \to G$ given by $\bar{\sigma}(z + B_n) = \sigma(z)$. Note that if $z_1 + B_n = z_2 + B_n$ then $z_1 - z_2 \in B_n$, and since $\sigma \in \operatorname{Ker} \iota_n^*$ implies that $\sigma|_{B_n} \equiv 0$ we obtain

$$\bar{\sigma}(z_1 + B_n) - \bar{\sigma}(z_2 + B_n) = \sigma(z_1) - \sigma(z_2) = \sigma(z_1 - z_2) = 0,$$

so $\bar{\sigma}$ is well-defined and clearly must be in $\operatorname{Hom}(H_n(C), G)$. Define $\Theta : \operatorname{Ker} \iota_n^* \to \operatorname{Hom}(H_n(C), G)$ by $\Theta(\sigma) = \bar{\sigma}$. Certainly Θ is well-defined. For $\sigma_1, \sigma_2 \in \operatorname{Ker} \iota_n^*$ we have $\Theta(\sigma_1 + \sigma_2) = \overline{\sigma_1 + \sigma_2}$, where

$$\begin{aligned}\overline{\sigma_1 + \sigma_2}(z + B_n) &= (\sigma_1 + \sigma_2)(z) = \sigma_1(z) + \sigma_2(z) = \bar{\sigma}_1(z + B_n) + \bar{\sigma}_2(z + B_n) \\ &= (\bar{\sigma}_1 + \bar{\sigma}_2)(z + B_n),\end{aligned}$$

so

$$\overline{\sigma_1 + \sigma_2} = \bar{\sigma}_1 + \bar{\sigma}_2 = \Theta(\sigma_1) + \Theta(\sigma_2)$$

and Θ is a homomorphism.

Fix $\bar{\sigma} \in \operatorname{Hom}(H_n(C), G)$. Define $\varphi \in Z_n^*$ by $\varphi(z) = \bar{\sigma}(z + B_n)$. For $z \in B_n$, $\varphi(z) = 0$, and so $\varphi \in \operatorname{Ker} \iota_n^*$. Now, $\Theta(\varphi) = \bar{\varphi}$, where

$$\bar{\varphi}(z + B_n) = \varphi(z) = \bar{\sigma}(z + B_n)$$

for all $z \in Z_n$ and hence $\Theta(\varphi) = \bar{\sigma}$. So Θ is surjective.

Suppose $\sigma \in \operatorname{Ker} \iota_n^*$ such that $\Theta(\sigma) = \bar{0}$, where $\bar{0}(z + B_n) := 0$ for all $z \in Z_n$. Then $\bar{\sigma} = \bar{0}$, so for any $z \in Z_n$ we have

$$\sigma(z) = \bar{\sigma}(z + B_n) = \bar{0}(z + B_n) = 0$$

and therefore $\sigma \equiv 0$. So Θ is injective and we conclude that $\operatorname{Ker} \iota_n^* \cong \operatorname{Hom}(H_n(C), G)$. As a result we may pass from (26) to a new short exact sequence

$$0 \xleftarrow{\iota_n^*} \operatorname{Hom}(H_n(C), G) \xleftarrow{h} H^n(C; G) \xleftarrow{\zeta} \operatorname{Coker} \iota_{n-1}^* \longleftarrow 0, \tag{27}$$

where it's easily verified that the map h from above is given by $h = \Theta \circ i_n^{**}$:

$$h(\varphi + \operatorname{Im} \delta_n) = \overline{\varphi|_{Z_n}} = \overline{\varphi \circ i_n} = \Theta(\varphi \circ i_n) = \Theta(i_n^*(\varphi)) = (\Theta \circ i_n^{**})(\varphi + \operatorname{Im} \delta_n).$$

For each $\bar{\varphi} \in \operatorname{Hom}(H_n(C), G)$ there is a map $\varphi_0 : Z_n \to G$ such that $\bar{\varphi}(z + B_n) = \varphi_0(z)$, and so in particular $\varphi_0|_{B_n} \equiv 0$. Define $s_1 : \operatorname{Hom}(H_n(C), G) \to \operatorname{Ker} \delta_{n+1}$ by

$$s_1(\bar{\varphi}) = \varphi_0 \circ p,$$

where $p : C_n \to Z_n$ is as defined on page 7. Note that for any $x \in C_{n+1}$,

$$(\varphi_0 \circ p \circ \partial_{n+1})(x) = (\varphi_0 \circ p)(\partial_{n+1} x) = \varphi_0(\partial_{n+1} x) = 0,$$

where the second equality holds since $\partial_{n+1} x \in Z_n$ and $p|_{Z_n} = \mathbb{1} : Z_n \to Z_n$, and so $\delta_{n+1}(\varphi_0 \circ p) = 0$ as required.

Next, define $s_2 : \operatorname{Ker} \delta_{n+1} \to H^n(C; G)$ by $s_2(\psi) = \psi + B_{n-1}$, and let $s = s_2 \circ s_1$. For any $\bar{\varphi} \in \operatorname{Hom}(H_n(C), G)$ with associated $\varphi_0 : Z_n \to G$,

$$(h \circ s)(\bar{\varphi}) = h(\varphi_0 \circ p + B_{n-1}) = \overline{\varphi_0 \circ p|_{Z_n}},$$

where for each $z + B_n \in H_n(C)$

$$\overline{\varphi_0 \circ p|_{Z_n}}(z + B_n) = \varphi_0 \circ p|_{Z_n}(z) = \varphi_0(p(z)) = \varphi_0(z) = \bar{\varphi}(z + B_n).$$

Thus $(h \circ s)(\bar{\varphi}) = \overline{\varphi_0 \circ p|_{Z_n}} = \bar{\varphi}$, so $h \circ s = \mathbb{1} : \operatorname{Hom}(H_n(C), G) \to \operatorname{Hom}(H_n(C), G)$ and the sequence (27) splits.

A potentially useful result that may as well be established here as anywhere else is the following.

Proposition 3.2. *If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is exact, then the dual sequence*

$$\operatorname{Hom}(A, G) \xleftarrow{\alpha^*} \operatorname{Hom}(B, G) \xleftarrow{\beta^*} \operatorname{Hom}(C, G) \longleftarrow 0$$

is also exact.

Proof. Let $\varphi \in \operatorname{Im}\beta^*$, so there exists $\psi \in \operatorname{Hom}(C, G)$ such that $\beta^*(\psi) = \psi \circ \beta = \varphi$. Now,

$$\alpha^*(\varphi) = \varphi \circ \alpha = (\psi \circ \beta) \circ \alpha = \psi \circ (\beta \circ \alpha) = \psi \circ 0 \equiv 0,$$

where the fourth equality holds since $\operatorname{Im}\alpha = \operatorname{Ker}\beta$, and therefore $\varphi \in \operatorname{Ker}\alpha^*$.

Let $\varphi \in \operatorname{Ker}\alpha^*$, so $\alpha^*(\varphi) = \varphi \circ \alpha \equiv 0$ implies that $\varphi|_{\operatorname{Im}\alpha} \equiv 0$, or equivalently $\varphi|_{\operatorname{Ker}\beta} \equiv 0$. Since $\operatorname{Im}\beta = C$ (i.e. β is surjective), the map $\hat{\beta} : B/\operatorname{Ker}\beta \to C$ given by $\hat{\beta}(b + \operatorname{Ker}\beta) = \beta(b)$ is an isomorphism. Let $\hat{\varphi} : B/\operatorname{Ker}\beta \to G$ be given by $\hat{\varphi}(b + \operatorname{Ker}\beta) = \varphi(b)$, and note that $\hat{\varphi}$ is well-defined:

$$\begin{aligned} b_1 + \operatorname{Ker}\beta = b_2 + \operatorname{Ker}\beta \quad &\Leftrightarrow \quad b_1 - b_2 \in \operatorname{Ker}\beta \quad \Leftrightarrow \quad \varphi(b_1 - b_2) = 0 \\ &\Leftrightarrow \quad \varphi(b_1) - \varphi(b_2) = 0 \quad \Leftrightarrow \quad \varphi(b_1) = \varphi(b_2) \\ &\Leftrightarrow \quad \hat{\varphi}(b_1 + \operatorname{Ker}\beta) = \hat{\varphi}(b_2 + \operatorname{Ker}\beta). \end{aligned}$$

Clearly $\hat{\varphi}$ is a homomorphism, so $\psi := \hat{\varphi} \circ \hat{\beta}^{-1}$ is likewise a homomorphism and therefore a member of $\operatorname{Hom}(C, G)$. Now, for any $b \in B$,

$$(\psi \circ \beta)(b) = \hat{\varphi}(\hat{\beta}^{-1}(\beta(b))) = \hat{\varphi}(b + \operatorname{Ker}\beta) = \varphi(b),$$

and thus $\beta^*(\psi) = \psi \circ \beta = \varphi$ implies that $\varphi \in \operatorname{Im}\beta^*$.

Finally, suppose that $\beta^*(\varphi) = 0$, so that $\varphi \circ \beta \equiv 0$ implies that $\varphi \in \operatorname{Hom}(C, G)$ with $\varphi|_{\operatorname{Im}\beta} \equiv 0$. But then $\operatorname{Im}\beta = C$ makes clear that $\varphi \equiv 0$ on C.

Therefore, since $\operatorname{Im}\beta^* = \operatorname{Ker}\alpha^*$ and $\operatorname{Ker}\beta^* = 0$, the dual sequence is exact. ∎

The balance of this section will be devoted to the proof of the Universal Coefficient Theorem for cohomology and a couple of its corollaries, followed by a few examples. As a prelude to this there is a definition and a lemma.

Definition 3.3. *A* ***free resolution*** *F of an abelian group H is an exact sequence*

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

in which each F_n is a free abelian group.

For the dual chain complex of F that results from applying the functor $\operatorname{Hom}(-, G)$,

$$\cdots \longleftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \longleftarrow 0,$$

define $H^n(F; G) = \operatorname{Ker} f_{n+1}^* / \operatorname{Im} f_n^*$.

Lemma 3.4. **(a)** *Let F and F' be free resolutions of abelian groups H and H', respectively. If $\varphi : H \to H'$ is a homomorphism, then φ can be extended to a chain map $F \to F'$:*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow 0 \\ & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & \\ \cdots \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H' & \longrightarrow 0 \end{array}$$

(b) *If* $\varphi_i : F_i \to F'_i$ *and* $\hat{\varphi}_i : F_i \to F'_i$ *are two chain maps* $F \to F'$ *extending* $\varphi : H \to H'$*, then they are chain homotopic.*

(c) *For any two free resolutions* F *and* F' *of* H *there are canonical isomorphisms* $H^n(F; G) \cong H^n(F'; G)$ *for all* n*.*

Proof. Induction will be employed to prove (a). Let x be a basis element of F_0. Then $\varphi(f_0(x))$ is in H', and since f'_0 is surjective there exists some $x' \in F'_0$ such that $f'_0(x') = \varphi(f_0(x))$. Define $\varphi_0 : F_0 \to F'_0$ by $\varphi_0(x) = x'$, so we have $\varphi \circ f_0 = f'_0 \circ \varphi_0$.

Now let $n \geq 0$ be arbitrary, and suppose that $\varphi_{n-1} \circ f_n = f'_n \circ \varphi_n$. (If $n = 0$ we take φ_{n-1} to be φ.) Let $x \in F_{n+1}$ be a basis element. Now,

$$f'_n(\varphi_n(f_{n+1}(x))) = \varphi_{n-1}(f_n(f_{n+1}(x))) = \varphi_{n-1}(0) = 0$$

since $\operatorname{Im} f_{n+1} = \operatorname{Ker} f_n$, and thus we have $\varphi_n(f_{n+1}(x)) \in \operatorname{Ker} f'_n$. Since $\operatorname{Ker} f'_n = \operatorname{Im} f'_{n+1}$ there's some $x' \in F'_{n+1}$ such that $f'_{n+1}(x') = \varphi_n(f_{n+1}(x))$, and we can define $\varphi_{n+1} : F_{n+1} \to F'_{n+1}$ by $\varphi_{n+1}(x) = x'$. Hence $\varphi_n \circ f_{n+1} = f'_{n+1} \circ \varphi_{n+1}$ and the induction argument is complete.

To prove (b), recall the definition of chain homotopy: if $\varphi_i : F_i \to F'_i$ and $\hat{\varphi}_i : F_i \to F'_i$ are two chain maps, then they are chain homotopic if there can be found homomorphisms $\lambda_i : F_i \to F'_{i+1}$ such that

$$\varphi_i - \hat{\varphi}_i = f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i$$

for all $i \geq 0$. Thus, suppose that $\varphi_i : F_i \to F'_i$ and $\hat{\varphi}_i : F_i \to F'_i$ are two chain maps $F \to F'$ extending $\varphi : H \to H'$. Another induction argument will be used. For the base case let $\lambda_{-1} \equiv 0$, so we need only find some $\lambda_0 : F_0 \to F'_1$ such that $\varphi_0 - \hat{\varphi}_0 = f'_1 \circ \lambda_0$. Let $x \in F_0$ be a basis element. We'll want to define $\lambda_0(x)$ so that $f'_1(\lambda_0(x)) = (\varphi_0 - \hat{\varphi}_0)(x)$, which requires confirming that $(\varphi_0 - \hat{\varphi}_0)(x) \in \operatorname{Im} f'_1$. From

$$f'_0 \circ \varphi_0 = \varphi \circ f_0 = f'_0 \circ \hat{\varphi}_0$$

we obtain $f'_0 \circ (\varphi_0 - \hat{\varphi}_0) \equiv 0$, whence $f'_0((\varphi_0 - \hat{\varphi}_0)(x)) = 0$ shows that $(\varphi_0 - \hat{\varphi}_0)(x) \in \operatorname{Ker} f'_0 = \operatorname{Im} f'_1$. Therefore there exists $x' \in F'_1$ such that $f'_1(x') = (\varphi_0 - \hat{\varphi}_0)(x)$, so let $\lambda_0(x) = x'$.

For the inductive step, let $n \geq 0$ be arbitrary and suppose

$$\varphi_n - \hat{\varphi}_n = f'_{n+1} \circ \lambda_n + \lambda_{n-1} \circ f_n.$$

We want to show that there is some map $\lambda_{n+1} : F_{n+1} \to F'_{n+2}$ such that

$$\varphi_{n+1} - \hat{\varphi}_{n+1} = f'_{n+2} \circ \lambda_{n+1} + \lambda_n \circ f_{n+1}.$$

So, let x be a basis element of F_{n+1}. It's necessary to define $\lambda_{n+1}(x)$ such that

$$f'_{n+2}(\lambda_{n+1}(x)) = (\varphi_{n+1} - \hat{\varphi}_{n+1})(x) - \lambda_n(f_{n+1}(x)),$$

which requires having

$$z := (\varphi_{n+1} - \hat{\varphi}_{n+1})(x) - \lambda_n(f_{n+1}(x)) \in \operatorname{Im} f'_{n+2}.$$

Since $\operatorname{Im} f'_{n+2} = \operatorname{Ker} f'_{n+1}$ this is a matter of direct manipulation,

$$f'_{n+1}(z) = f'_{n+1}((\varphi_{n+1} - \hat{\varphi}_{n+1})(x)) - (f'_{n+1} \circ \lambda_n)(f_{n+1}(x))$$

$$= f'_{n+1}((\varphi_{n+1} - \hat{\varphi}_{n+1})(x)) - ((\varphi_n - \hat{\varphi}_n) - (\lambda_{n-1} \circ f_n))(f_{n+1}(x))$$

$$= f'_{n+1}((\varphi_{n+1} - \hat{\varphi}_{n+1})(x)) - (\varphi_n - \hat{\varphi}_n)(f_{n+1}(x)) + \lambda_{n-1}(f_n(f_{n+1}(x)))$$

$$= f'_{n+1}(\varphi_{n+1}(x)) - f'_{n+1}(\hat{\varphi}_{n+1}(x)) - \varphi_n(f_{n+1}(x)) + \hat{\varphi}_n(f_{n+1}(x)) = 0,$$

since $f'_{n+1} \circ \varphi_{n+1} = \varphi_n \circ f_{n+1}$ and $f'_{n+1} \circ \hat{\varphi}_{n+1} = \hat{\varphi}_n \circ f_{n+1}$. Hence there exists some $y \in F'_{n+2}$ such that $f'_{n+2}(y) = z$, so we let $\lambda_{n+1}(x) = y$.

We turn now to the proof of (c). Let F and F' be free resolutions of H, and let $\varphi : H \to H$ be a homomorphism. By part (a) φ can be extended to a chain map $\varphi_n : F_n \to F'_n$, and dualizing gives a chain map $\varphi^*_n : F'^*_n \to F^*_n$,

$$\begin{tikzcd}
\cdots & F_2^* \arrow[l] & F_1^* \arrow[l, "f_2^*"'] & F_0^* \arrow[l, "f_1^*"'] & H^* \arrow[l, "f_0^*"'] & 0 \arrow[l] \\
\cdots & F_2'^* \arrow[l] \arrow[u, "\varphi_2^*"] & F_1'^* \arrow[l, "f_2'^*"'] \arrow[u, "\varphi_1^*"] & F_0'^* \arrow[l, "f_1'^*"'] \arrow[u, "\varphi_0^*"] & H^* \arrow[l, "f_0'^*"'] \arrow[u, "\varphi^*"] & 0, \arrow[l]
\end{tikzcd}$$

which in turn induces homomorphisms $\varphi^{*\star}_n : H^n(F'; G) \to H^n(F; G)$.[4] Now, if the maps

$$\hat{\varphi}_n : F_n \to F'_n$$

are another extension of φ to a chain map $F \to F'$, then by part (b) φ_n and $\hat{\varphi}_n$ are chain homotopic, meaning once again $\varphi_n - \hat{\varphi}_n = f'_{n+1} \circ \lambda_n + \lambda_{n-1} \circ f_n$ for maps $\lambda_n : F_n \to F'_{n+1}$. Dualizing gives

$$\varphi^*_n - \hat{\varphi}^*_n = \lambda^*_n \circ {f'_{n+1}}^* + f^*_n \circ \lambda^*_{n-1},$$

which shows that φ^*_n and $\hat{\varphi}^*_n$ are chain-homotopic chain maps and therefore $\varphi^{*\star}_n = \hat{\varphi}^{*\star}_n$ for all n by Proposition 2.1.

Let $\alpha : H \to H$ be an isomorphism, with $\beta = \alpha^{-1} : H \to H$. By part (a), α can be extended to a chain map $\alpha_n : F_n \to F'_n$, and β can be extended to a chain map $\beta_n : F'_n \to F_n$. It's straightforward to verify that $\beta_n \circ \alpha_n : F_n \to F_n$ is an extension of $\beta \circ \alpha = \mathbb{1}_H : H \to H$ to a chain map, since $\alpha_{n-1} \circ f_n = f'_n \circ \alpha_n$ and $\beta_{n-1} \circ f'_n = f_n \circ \beta_n$ imply that

$$\beta_{n-1} \circ \alpha_{n-1} \circ f_n = f_n \circ \beta_n \circ \alpha_n.$$

But the identities $\mathbb{1}_{F_n} : F_n \to F_n$ likewise constitute an extension of $\mathbb{1}_H$ to a chain map, and so $(\beta_n \circ \alpha_n)^{*\star} = \mathbb{1}^{*\star}_{F_i}$ for all n. Now,

$$(\beta_n \circ \alpha_n)^{*\star} = (\alpha^*_n \circ \beta^*_n)^\star = \alpha^{*\star}_n \circ \beta^{*\star}_n$$

and

$$\mathbb{1}^{*\star}_{F_n} = \mathbb{1}^\star_{F^*_n} = \mathbb{1}_{H^n(F;G)},$$

so

$$\alpha^{*\star}_n \circ \beta^{*\star}_n = \mathbb{1}_{H^n(F;G)}.$$

[4] Recall that in the present section a superscript $\star$ is used to indicate an induced homomorphism of cohomology groups, and $f^{*\star}$ is defined to be $(f^*)^\star$.

A similar argument shows that

$$\beta_n^{\star\star} \circ \alpha_n^{\star\star} = \mathbb{1}_{H^n(F';G)},$$

and therefore $\alpha_n^{\star\star} : H^n(F';G) \to H^n(F;G)$ is an isomorphism for all n. Thus so-called *canonical* isomorphisms $H^n(F;G) \cong H^n(F';G)$ result if we specify α to be the isomorphism $\mathbb{1}_H$ and extend to a chain map $F \to F'$. ■

Part (c) of the lemma shows, in particular, that the first homology group deriving from a free resolution F of a group H, $H^1(F;G)$, depends only on H and G, and not at all on the choice for F. For this reason $H^1(F;G)$ is often denoted by $\operatorname{Ext}(H,G)$, where $\operatorname{Ext}(H,G)$ is taken to be a fixed group determined by H and G such that $H^1(F;G) \cong \operatorname{Ext}(H,G)$ for all F. The other homology groups $H^n(F;G)$ for $n > 1$ turn out to be trivial since, as will be verified later, any abelian group H can be put into a free resolution of the form

$$\cdots \longrightarrow 0 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0. \tag{28}$$

Moreover, since the truncated sequence

$$F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

is exact, by Proposition 3.2 the dual is likewise exact and thus $H^0(F;G) = \operatorname{Ker} f_1^* / \operatorname{Im} f_0^* = 0$ as well.

Theorem 3.5 (Universal Coefficient Theorem for Cohomology). *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex obtained by applying* $\operatorname{Hom}(-,G)$ *are determined by split exact sequences*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C),G) \xrightarrow{\zeta} H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C),G) \longrightarrow 0$$

Proof. For the abelian group $H_{n-1}(C)$ there is the free resolution F given by

$$\cdots \longrightarrow 0 \longrightarrow B_{n-1} \xrightarrow{\iota_{n-1}} Z_{n-1} \xrightarrow{q} H_{n-1}(C) \longrightarrow 0,$$

where ι_{n-1} is inclusion and $q : Z_{n-1} \to Z_{n-1}/B_{n-1}$ is the quotient map $q(z) = z + B_{n-1}$. Dualizing yields

$$\cdots \longleftarrow 0 \longleftarrow B_{n-1}^* \xleftarrow{\iota_{n-1}^*} Z_{n-1}^* \xleftarrow{q^*} \operatorname{Hom}(H_{n-1}(C),G) \longleftarrow 0,$$

so it's seen that

$$\operatorname{Coker} \iota_{n-1}^* = B_{n-1}^* / \operatorname{Im} \iota_{n-1}^* = H^1(F;G)$$

and therefore $\operatorname{Coker} \iota_{n-1}^*$ depends only on H and G. Setting $\operatorname{Ext}(H_{n-1}(C),G)$ equal to $H^1(F;G)$, then, the split exact sequence (27) becomes

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C),G) \xrightarrow{\zeta} H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C),G) \longrightarrow 0$$

as desired. ■

As was mentioned, every abelian group H has a free resolution of the form (28). Start by selecting a set S of generators for H, let F_0 be the free abelian group with basis S, and define a homomorphism $f_0 : F_0 \to H$ such that $f(s) = s$ for each $s \in S$ (note that f_0 is surjective). Next let $F_1 = \operatorname{Ker} f_0$ and define $f_1 : F_1 \hookrightarrow F_0$ to be inclusion. Finally, set $F_i = 0$ for all $i \geq 2$.

Proposition 3.6. **(a)** $\operatorname{Ext}(H \oplus H', G) \cong \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G)$.

(b) $\operatorname{Ext}(H, G) = 0$ *if* H *is a free abelian group.*

(c) $\operatorname{Ext}(\mathbb{Z}_n, G) \cong G/nG$.

Proof. For the proof of (a), let (28) be a free resolution F for H, and let

$$\cdots \longrightarrow 0 \xrightarrow{f_2'} F_1' \xrightarrow{f_1'} F_0' \xrightarrow{f_0'} H' \longrightarrow 0.$$

be a free resolution F' for H'. Then it's easy to check that

$$\cdots \longrightarrow 0 \oplus 0 \xrightarrow{f_2 \oplus f_2'} F_1 \oplus F_1' \xrightarrow{f_1 \oplus f_1'} F_0 \oplus F_0' \xrightarrow{f_0 \oplus f_0'} H \oplus H' \longrightarrow 0 \oplus 0,$$

where we define

$$(f_n \oplus f_n')(x, x') = (f_n(x), f_n'(x')),$$

is a free resolution for $H \oplus H'$, which we'll denote by $F \oplus F'$. Applying $\operatorname{Hom}(-, G)$ to $F \oplus F'$ yields

$$\cdots \longleftarrow 0 \xleftarrow{(f_2 \oplus f_2')^*} (F_1 \oplus F_1')^* \xleftarrow{(f_1 \oplus f_1')^*} (F_0 \oplus F_0')^* \xleftarrow{(f_0 \oplus f_0')^*} (H \oplus H')^* \longleftarrow 0,$$

and thus

$$H^1(F \oplus F'; G) = \frac{\operatorname{Ker}(f_2 \oplus f_2')^*}{\operatorname{Im}(f_1 \oplus f_1')^*} = \frac{(F_1 \oplus F_1')^*}{\operatorname{Im}(f_1 \oplus f_1')^*}.$$

Noting that $H^1(F; G) = F_1^* / \operatorname{Im} f_1^*$ and $H^1(F'; G) = F_1'^* / \operatorname{Im} f_1'^*$, define

$$\Omega : H^1(F \oplus F'; G) \to H^1(F; G) \oplus H^1(F'; G)$$

by

$$\Omega(\varphi + \operatorname{Im}(f_1 \oplus f_1')^*) = (\varphi(\cdot, 0) + \operatorname{Im} f_1^*\,,\ \varphi(0, \cdot) + \operatorname{Im} f_1'^*).$$

Suppose

$$[\varphi] := \varphi + \operatorname{Im}(f_1 \oplus f_1')^* = \hat{\varphi} + \operatorname{Im}(f_1 \oplus f_1')^* := [\hat{\varphi}],$$

so $\varphi - \hat{\varphi} \in \operatorname{Im}(f_1 \oplus f_1')^*$ and there exists $\psi \in (F_0 \oplus F_0')^*$ such that $(f_1 \oplus f_1')^*(\psi) = \varphi - \hat{\varphi}$; that is, $\psi \circ (f_1 \oplus f_1') = \varphi - \hat{\varphi}$, so for any $(x, x') \in F_1 \oplus F_1'$,

$$(\psi \circ (f_1 \oplus f_1'))(x, x') = \psi(f_1(x), f_1'(x')) = (\varphi - \hat{\varphi})(x, x').$$

Define $\alpha \in F_0^*$ by $\alpha = \psi(\cdot, 0)$. Now, $f_1^*(\alpha) = \alpha \circ f_1$, where for each $x \in F_1$ we have

$$(\alpha \circ f_1)(x) = \alpha(f_1(x)) = \psi(f_1(x), 0) = \psi(f_1(x), f_1'(0)) = (\varphi - \hat{\varphi})(x, 0)$$

and therefore $f_1^*(\alpha) = (\varphi - \hat{\varphi})(\cdot, 0)$. Hence $\varphi(\cdot, 0) - \hat{\varphi}(\cdot, 0) \in \operatorname{Im} f_1^*$, implying that

$$\varphi(\cdot, 0) + \operatorname{Im} f_1^* = \hat{\varphi}(\cdot, 0) + \operatorname{Im} f_1^*.$$

A similar argument gives

$$\varphi(0,\cdot\,) + \operatorname{Im} f_1'^* = \hat{\varphi}(0,\cdot\,) + \operatorname{Im} f_1'^*,$$

whence $\Omega([\varphi]) = \Omega([\hat{\varphi}])$ obtains and Ω is well-defined. That Ω is a homomorphism is obvious, but is it an isomorphism?

Suppose that

$$\Omega(\varphi + \operatorname{Im}(f_1 \oplus f_1')^*) = (0,0),$$

so $\varphi \in (F_1 \oplus F_1')^*$. Then $\varphi(\,\cdot,0) \in \operatorname{Im} f_1^*$ and $\varphi(0,\cdot\,) \in \operatorname{Im} f_1'^*$, so

$$\exists\, \psi \in F_0^* \text{ s.t. } f_1^*(\psi) = \psi \circ f_1^* = \varphi(\,\cdot,0),$$

and

$$\exists\, \chi \in F_0'^* \text{ s.t. } f_1'^*(\chi) = \chi \circ f_1'^* = \varphi(0,\cdot\,).$$

Define $\gamma \in (F_0 \oplus F_0')^*$ by $\gamma(x,x') = \psi(x) + \chi(x')$. Now,

$$(f_1 \oplus f_1')^*(\gamma) = \gamma \circ (f_1 \oplus f_1'),$$

where for $(x,x') \in F_1 \oplus F_1'$ we have

$$\begin{aligned}(\gamma \circ (f_1 \oplus f_1'))(x,x') &= \gamma(f_1(x), f_1'(x')) = \psi(f_1(x)) + \chi(f_1'(x')) \\ &= \varphi(x,0) + \varphi(0,x') = \varphi(x,x'),\end{aligned}$$

which shows that $(f_1 \oplus f_1')^*(\gamma) = \varphi$. Since $\varphi + \operatorname{Im}(f_1 \oplus f_1')^* = 0$ it follows that $\operatorname{Ker}\Omega = \{0\}$ and Ω is injective.

Next, let

$$(\varphi + \operatorname{Im} f_1^*, \psi + \operatorname{Im} f_1'^*) \in H^1(F;G) \oplus H^1(F';G),$$

so that $\varphi : F_1 \to G$ and $\psi : F_1' \to G$ are homomorphisms. Define $\omega : F_1 \oplus F_1' \to G$ by $\omega(x,x') = \varphi(x) + \psi(x')$, which is easily verified to be a homomorphism so that $\omega \in (F_1 \oplus F_1')^*$. Now, since $\omega(x,0) = \varphi(x)$ and $\omega(0,x') = \psi(x')$ for all $x \in F_1, x' \in F_1'$, it's clear that $\omega(\,\cdot,0) = \varphi$ and $\omega(0,\cdot\,) = \psi$ and so

$$\Omega(\omega + \operatorname{Im}(f_1 \oplus f_1')^*) = (\omega(\,\cdot,0) + \operatorname{Im} f_1^*, \omega(0,\cdot\,) + \operatorname{Im} f_1'^*) = (\varphi + \operatorname{Im} f_1^*, \psi + \operatorname{Im} f_1'^*).$$

Thus Ω is surjective, and we obtain

$$H^1(F \oplus F';G) \cong H^1(F;G) \oplus H^1(F';G)$$

since Ω is an isomorphism. Therefore

$$\operatorname{Ext}(H \oplus H', G) \cong \operatorname{Ext}(H,G) \oplus \operatorname{Ext}(H',G).$$

Moving on to the proof of (b), suppose that H is a free abelian group. Then the sequence

$$\cdots \longrightarrow 0 \longrightarrow H \xrightarrow{\;\mathbb{1}\;} H \longrightarrow 0$$

is a free resolution F of H. Clearly $H^1(F;G) = 0$, which implies $\operatorname{Ext}(H,G) = 0$.

Finally we turn to the proof of (c). Fix $n \in \mathbb{N}$. Recalling $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, define $\pi : \mathbb{Z} \to \mathbb{Z}_n$ by $\pi(k) = k + n\mathbb{Z}$, and note that $\operatorname{Ker}\pi = n\mathbb{Z}$. Letting $i : n\mathbb{Z} \hookrightarrow \mathbb{Z}$ to be inclusion, we construct a free F resolution for $\mathbb{Z}_n$:

$$\cdots \longrightarrow 0 \longrightarrow n\mathbb{Z} \xrightarrow{\;i\;} \mathbb{Z} \xrightarrow{\;\pi\;} \mathbb{Z}_n \longrightarrow 0.$$

Applying $\operatorname{Hom}(-, G)$ we get

$$\cdots \longleftarrow 0 \longleftarrow n\mathbb{Z}^* \xleftarrow{\;i^*\;} \mathbb{Z}^* \xleftarrow{\;\pi^*\;} \mathbb{Z}_n^* \longleftarrow 0,$$

where of course $H^1(F; G) = n\mathbb{Z}^*/\operatorname{Im} i^*$.

Define $\Upsilon : n\mathbb{Z}^*/\operatorname{Im} i^* \to G/nG$ by

$$\Upsilon(\varphi + \operatorname{Im} i^*) = \varphi(n) + nG$$

for each homomorphism $\varphi : n\mathbb{Z} \to G$. Suppose $\varphi_1 + \operatorname{Im} i^* = \varphi_2 + \operatorname{Im} i^*$. Then $\varphi_1 - \varphi_2 \in \operatorname{Im} i^*$ implies that $i^*(\psi) = \varphi_1 - \varphi_2$ for some $\psi \in \mathbb{Z}^*$, which is to say $\varphi_1 - \varphi_2 = \psi \circ i : n\mathbb{Z} \to G$ and thus

$$(\varphi_1 - \varphi_2)(n) = \psi(i(n)) = \psi(n) = n\psi(1).$$

Therefore $\varphi_1(n) - \varphi_2(n) \in nG$, whence

$$\Upsilon(\varphi_1 + \operatorname{Im} i^*) = \varphi_1(n) + nG = \varphi_2(n) + nG = \Upsilon(\varphi_2 + \operatorname{Im} i^*)$$

and Υ is well-defined. Obviously Υ is a homomorphism.

Suppose $\Upsilon(\varphi + \operatorname{Im} i^*) = 0$, so $\varphi(n) \in nG$ and there exists some $g_0 \in G$ such that $\varphi(n) = ng_0$. Define $\psi \in \mathbb{Z}^*$ by $\psi(k) = kg_0$ for each $k \in \mathbb{Z}$. Now, for each $kn \in n\mathbb{Z}$ we have

$$(\psi \circ i)(kn) = \psi(i(kn)) = \psi(kn) = (kn)g_0 = \varphi(kn),$$

so $i^*(\psi) = \psi \circ i = \varphi$. Hence $\varphi + \operatorname{Im} i^* = 0$, so $\operatorname{Ker}\Upsilon = \{0\}$ and Υ is injective.

Next, let $g + nG \in G/nG$. Define $\varphi : n\mathbb{Z} \to G$ to be a homomorphism such that $\varphi(n) = g$ (so $\varphi(kn) = k\varphi(n) = kg$ for all $k \in \mathbb{Z}$). Then

$$\Upsilon(\varphi + \operatorname{Im} i^*) = \varphi(n) + nG = g + nG.$$

Therefore Υ is surjective.

Since Υ is an isomorphism it follows that

$$G/nG \cong n\mathbb{Z}^*/\operatorname{Im} i^* = H^1(F; G) \cong \operatorname{Ext}(\mathbb{Z}_n, G),$$

as desired. ∎

If H is finitely generated it is a fact from algebra that H has a (unique) direct sum decomposition $H = H_{tor} \oplus B$, where H_{tor} is the torsion subgroup of H and B is a free abelian group. Thus by the preceding proposition

$$\operatorname{Ext}(H, \mathbb{Z}) = \operatorname{Ext}(H_{tor} \oplus B, \mathbb{Z}) = \operatorname{Ext}(H_{tor}, \mathbb{Z}) \oplus \underbrace{\operatorname{Ext}(B, \mathbb{Z})}_{0} \cong \operatorname{Ext}(H_{tor}, \mathbb{Z}).$$

Since $H_{tor} \subset H$ and H is finitely generated, H_{tor} must be a finitely generated torsion group and therefore of finite order. Thus $H_{tor} \cong \mathbb{Z}_k$ for some positive integer k, and it follows from part (c) of Proposition 3.6 that

$$\mathrm{Ext}(H_{tor}, \mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}_k.$$

Therefore, in general, $\mathrm{Ext}(H, \mathbb{Z}) \cong H_{tor}$.

Two additional facts from algebra are: (i) $\mathrm{Hom}(H, \mathbb{Z})$ is isomorphic to the free part of H if H is a finitely generated abelian group; and (ii) if $A_1, ..., A_n$ are abelian groups with subgroups $B_i \subset A_i$, then

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong A_1/B_1 \times \cdots \times A_n/B_n.$$

We use these facts to prove the following.

Proposition 3.7. *If the homology groups $H_n(C)$ and $H_{n-1}(C)$ of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n(C)$ and $T_{n-1} \subset H_{n-1}(C)$, then $H^n(C; \mathbb{Z}) \cong (H_n(C)/T_n) \oplus T_{n-1}$.*

Proof. First, $H_n(C)$ has a direct sum decomposition $H_n(C) \cong T_n \oplus B$, where B is the free part of $H_n(C)$. Also we have $\mathrm{Ext}(H_{n-1}(C), \mathbb{Z}) \cong T_{n-1}$. By (i) above, $\mathrm{Hom}(H_n(C), \mathbb{Z}) \cong B$; and by (ii),

$$H_n(C)/T_n \cong (T_n \oplus B)/(T_n \oplus \{0\}) \cong T_n/T_n \oplus B/\{0\} \cong \{0\} \oplus B \cong B.$$

(Technically the first isomorphism would need to be verified.) Hence $\mathrm{Hom}(H_n(C), \mathbb{Z}) \cong H_n(C)/T_n$, and by Theorem 3.5 we have the split short exact sequence

$$0 \longrightarrow \underbrace{\mathrm{Ext}(H_{n-1}(C), \mathbb{Z})}_{T_{n-1}} \longrightarrow H^n(C; \mathbb{Z}) \longrightarrow \underbrace{\mathrm{Hom}(H_n(C), \mathbb{Z})}_{H_n(C)/T_n} \longrightarrow 0.$$

Therefore, by the Splitting Lemma, $H^n(C; \mathbb{Z}) \cong T_{n-1} \oplus (H_n(C)/T_n)$. ■

It's high time to consider some examples.

Example 3.8. Show that the map $H \xrightarrow{n} H$ given by $x \mapsto nx$ for each $x \in H$ induces multiplication by n in $\mathrm{Ext}(H, G)$, and so too does $G \xrightarrow{n} G$.

Solution. Given an abelian group H, let (28) be a free resolution F of H. Define $\mathtt{n} : H \to H$ by $\mathtt{n}(x) = nx$. Then $\mathtt{n}$ can be extended to a chain map $\mathtt{n}_i : F_i \to F_i$ where $\mathtt{n}_i(x) = nx$ for each $i \geq 0$ and $x \in F_i$:

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\mathtt{n}_2"'] & F_1 \arrow[r, "f_1"] \arrow[d, "\mathtt{n}_1"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\mathtt{n}_0"'] & H \arrow[r] \arrow[d, "\mathtt{n}"'] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0
\end{tikzcd}$$

(It's straightforward to verify that the diagram is commutative.) Dualizing yields

$$\begin{array}{ccccccccccc}
\cdots & \longleftarrow & F_2^* & \xleftarrow{f_2^*} & F_1^* & \xleftarrow{f_1^*} & F_0^* & \xleftarrow{f_0^*} & H^* & \longleftarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \mathtt{n}_2^*} & & \Big\uparrow{\scriptstyle \mathtt{n}_1^*} & & \Big\uparrow{\scriptstyle \mathtt{n}_0^*} & & \Big\uparrow{\scriptstyle \mathtt{n}^*} & & \\
\cdots & \longleftarrow & F_2^* & \xleftarrow{f_2^*} & F_1^* & \xleftarrow{f_1^*} & F_0^* & \xleftarrow{f_0^*} & H^* & \longleftarrow & 0
\end{array}$$

For each i, $\mathtt{n}_i^*(\alpha) = \alpha \circ \mathtt{n}_i$, where

$$(\alpha \circ \mathtt{n}_i)(x) = \alpha(nx) = n\alpha(x) = (n\alpha)(x)$$

so that $\mathtt{n}_i^*(\alpha) = n\alpha$. In particular the map $\mathtt{n}_1^*$ induces

$$(\mathtt{n}_1^*)_* : H^1(F;G) \to H^1(F;G)$$

given by

$$(\mathtt{n}_1^*)_*(\alpha + \operatorname{Im} f_1^*) = \mathtt{n}_1^*(\alpha) + \operatorname{Im} f_1^* = n\alpha + \operatorname{Im} f_1^* = n(\alpha + \operatorname{Im} f_1^*)$$

for each $\alpha \in \operatorname{Ker} f_2^*$. Thus $(\mathtt{n}_1^*)_*$ is multiplication by n in $H^1(F;G)$, and since $\operatorname{Ext}(H,G) \cong H^1(F;G)$ it's immediate that $\mathtt{n}_1^*$, which ultimately was "induced" by $\mathtt{n}$, in turn induces multiplication by n in $\operatorname{Ext}(H,G)$.[5]

Now let $\mathtt{n} : G \to G$ be multiplication by n in G. This map induces homomorphisms $\overline{\mathtt{n}}_i : F_i^* \to F_i^*$ given by

$$\overline{\mathtt{n}}_i(\alpha) = \mathtt{n} \circ \alpha.$$

For each $x \in F_i$,

$$(\mathtt{n} \circ \alpha)(x) = \mathtt{n}(\alpha(x)) = n\alpha(x) = (n\alpha)(x),$$

so $\overline{\mathtt{n}}_i(\alpha) = n\alpha$. The map $\overline{\mathtt{n}}_1$ in particular induces

$$\overline{\overline{\mathtt{n}}}_1 : H^1(F;G) \to H^1(F;G)$$

given by

$$\overline{\overline{\mathtt{n}}}_1(\alpha + \operatorname{Im} f_1^*) = \overline{\mathtt{n}}_1(\alpha) + \operatorname{Im} f_1^* = n\alpha + \operatorname{Im} f_1^*,$$

so $\overline{\overline{\mathtt{n}}}_1$ is multiplication by n on $H^1(F;G)$, and by extension $\operatorname{Ext}(H,G)$ as well. ■

[5]The mystical shape-shifting abilities of the term "induce" is common coin amongst the high priesthood of algebra, and unfortunately we just have to accept it as a symptom of human laziness.