Chapter 3 – Cohomology

3.0 – Cohomology Introduction

Let \( X \) be a 1-dimensional \( \Delta \)-complex, so \( X \) is an oriented graph (or pseudo-graph if edges that begin and end at the same vertex are allowed). If \( v_0 \) and \( v_1 \) are two vertices in \( X \) and \( e \) is an oriented edge from \( v_0 \) to \( v_1 \), then notationally we’ll denote \( e \) by \([v_0, v_1]\). Recall that, formally, a “vertex” is a map \( \sigma_v : \Delta^0 \to X \), and an “edge” is a map \( \tau_e : \Delta^1 \to X \) that, when restricted to each endpoint (or “face”) of \( \Delta^1 \), becomes one of the maps \( \sigma_v \) when pre-composed with the appropriate canonical linear homeomorphism \( \Delta^0 \to (\text{face of } \Delta^1) \). So our edge \( e \) is in fact a map \( \sigma_e : \Delta^1 \to X \) such that, if we denote \( \Delta^1 \) by \([u_0, u_1]\) (a line segment), then \( \sigma_e(u_0) = v_0 \) and \( \sigma_e(u_1) = v_1 \); thus, if indeed \( v_0 \neq v_1 \), it follows that \( \sigma_e(\Delta^1) \) is truly our edge \( e \) in \( X \) in the graphical sense—is homeomorphic to the standard 1-simplex \( \Delta^1 \), and so it makes sense to represent \( e \) using the simplex notation \([v_0, v_1]\) (especially since it conveys information about the orientation of \( e \)). If \( v_0 = v_1 \) it still makes sense to represent \( e \) by \([v_0, v_1] = [v_0, v_0] \) to maintain consistent notation even though the corresponding edge is not homeomorphic to any kind of simplex.\(^1\)

Let \( G \) be an abelian group (not necessarily free), \( V \) the set of vertices of \( X \), and \( E \) the set of edges of \( X \). Define

\[
\Delta^0(X; G) = \{ \varphi : V \to G \mid \varphi \text{ is a function} \}
\]

and

\[
\Delta^1(X; G) = \{ \psi : E \to G \mid \psi \text{ is a function} \}
\]

Note \( \Delta^0(X; G) \) forms an abelian group: if \( \varphi_1, \varphi_2 \in \Delta^0(X; G) \), then \( \varphi_1 + \varphi_2 \) given by

\[
(\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v)
\]

shows that \( \varphi_1 + \varphi_2 \in \Delta^0(X; G) \) since \( \varphi_1(v) \in G \) and \( \varphi_2(v) \in G \) implies that \( \varphi_1(v) + \varphi_2(v) \in G \).

In similar fashion \( \Delta^1(X; G) \) is also an abelian group.

Now define a homomorphism \( \delta_1 : \Delta^0(X; G) \to \Delta^1(X; G) \) as follows: for \( \varphi \in \Delta^0(X; G) \) let \( \delta_1 \varphi \in \Delta^1(X; G) \) be such that, for \([v_0, v_1] \in E \), \( \delta_1 \varphi([v_0, v_1]) = \varphi(v_1) - \varphi(v_0) \). Set up a chain complex

\[
\cdots \to 0 \xrightarrow{\delta_0} \Delta^0(X; G) \xrightarrow{\delta_1} \Delta^1(X; G) \xrightarrow{\delta_2} 0 \to \cdots
\]

By definition the homology groups associated with this chain complex are the simplicial cohomology groups \( H^\Delta_\bullet(X; G) \) of \( X \). In particular we have

\[
H_\Delta^0(X; G) = \frac{\operatorname{Ker}[\delta_1 : \Delta^0(X; G) \to \Delta^1(X; G)]}{\operatorname{Im}[\delta_0 : 0 \to \Delta^0(X; G)]} \cong \operatorname{Ker} \delta_1
\]

(1)

since \( \operatorname{Im} \delta_0 = 0 \), and

\[
H_\Delta^1(X; G) = \frac{\operatorname{Ker}[\delta_2 : \Delta^1(X; G) \to 0]}{\operatorname{Im}[\delta_1 : \Delta^0(X; G) \to \Delta^1(X; G)]} \cong \frac{\Delta^1(X; G)}{\operatorname{Im}[\delta_1 : \Delta^0(X; G) \to \Delta^1(X; G)]}.
\]

\(^1\)Recall that the map \( \sigma_e \) is also called a 1-simplex.
So $H^0_\Delta(X; G) \cong \{ \varphi \in \Delta^0(X; G) : \delta_1 \varphi = 0 \}$, where $0(e) := 0 \in G$ for every $e \in E$. Thus $\varphi \in H^0_\Delta(X; G)$ implies that

$$\delta_1 \varphi([v_1, v_0]) = \varphi(v_1) - \varphi(v_0) = 0,$$

or $\varphi(v_1) = \varphi(v_0)$ for every vertex $v_0$ and $v_1$ that is connected by an edge in $X$. This in turn implies that $\varphi$ must be constant on each component of $X$. Let $\{X_\alpha\}_{\alpha \in I}$ be the components of $X$, and let $V_\alpha$ be the set of vertices for the subgraph $X_\alpha$. Then

$$\text{Ker } \delta_1 = \{ \varphi \in \Delta^0(X; G) : \forall \alpha \in I \text{ } \varphi \text{ is constant on } V_\alpha \}$$

$$= \{ \varphi \in \Delta^0(X; G) : \forall \alpha \in I \exists g_{\alpha} \in G \text{ s.t. } \varphi \equiv g_{\alpha} \text{ on } V_\alpha \}.$$

It follows that each element of the group Ker $\delta_1$ corresponds to some $\{g_{\alpha}\}_{\alpha \in I} \in \prod_{\alpha \in I} G$.² Since it may well be that $g_{\alpha} \neq 0$ for an infinite number of index values $\alpha$. The one-to-one correspondence Ker $\delta_1 \mapsto \prod_{\alpha \in I} G$ can easily be shown to be an isomorphism, so therefore from (1) we obtain

$$H^0_\Delta(X; G) \cong \text{Ker } \delta_1 \cong \prod_{\alpha \in I} G.$$

Next, from (2) is can be seen that $H^1_\Delta(X; G) = 0$ iff $\delta_1$ is surjective, which is to say that for each $\psi \in \Delta^1(X; G)$ there exists some $\varphi \in \Delta^0(X; G)$ such that $\delta_1 \varphi = \psi$. This will be the case whenever the components $\{X_\alpha\}_{\alpha \in I}$ of $X$ are trees since the path between any two vertices in a tree is unique: for a given $\alpha \in I$, $[v_0, v_1] = e \in E_\alpha$ (the set of edges in $X_\alpha$) and $\psi \in \Delta^1(X; G)$, we need only define $\varphi \in \Delta^0(X; G)$ such that $\varphi(v_1) - \varphi(v_0) = \psi(e)$, with the choice of definition being unique up to a constant; then the values of $\varphi$ at all other vertices in $X_\alpha$ are set as dictated by the values of $\psi$ on the edges of the unique paths leading to those vertices. The process is repeated for the other components of $X$ to get $\delta_1 \varphi = \psi$.

If a particular component $X_\alpha$ of $X$ is not a tree, then we designate a maximal tree that is a subgraph of $X_\alpha$ which includes all of its vertices but not all edges. It is a fact that, for any choice of maximal tree for a graph, the cardinality of the set of edges omitted from the tree will be the same. For the sake of argument suppose that $X$ is a connected graph that is not a tree, let $Y \subset X$ be a maximal tree, let $E'$ be the set of edges not in $Y$, and let $E''$ be the set of edges in $Y$. The claim will be that

$$H^1_\Delta(X; G) \cong \prod_{e \in E'} G.$$

For the construction of the appropriate isomorphism, note that for any $\psi \in \Delta^1(X; G)$ there can be found some $\varphi \in \Delta^0(X; G)$ (unique up to a constant) such that $\delta_1 \varphi|_{E''} = \psi|_{E''}$. Now define $F : H^1_\Delta(X; G) \rightarrow \prod_{e \in E'} G$ by

$$F(\psi + \text{Im } \delta_1) = \{(\psi - \delta_1 \varphi)(e)\}_{e \in E'}$$

for some $\varphi$ such that $\delta_1 \varphi|_{E''} = \psi|_{E''}$ (3)

²Recall that formally $\{g_{\alpha}\}_{\alpha \in I}$ is a function $g : I \rightarrow G$ given by $g(\alpha) = g_{\alpha}$ for each $\alpha \in I$. 
The choice for \( \varphi \) is irrelevant since the difference must be a constant: if \( \hat{\varphi} = \varphi + g \) for some fixed \( g \in G \) (more precisely \( g : E \to \{g\} \)), then
\[
\delta_1 \hat{\varphi}([v_0, v_1]) = (\varphi + g)(v_1) - (\varphi + g)(v_0)
= (\varphi(v_1) + g(v_1)) - (\varphi(v_0) + g(v_0))
= \varphi(v_1) + g - \varphi(v_0) - g = \delta_1 \varphi([v_0, v_1]),
\]
and so \( \psi - \delta_1 \hat{\varphi} = \psi - \delta_1 \varphi \).

It should first be verified that \( F \) is well-defined. We start with a simple case when \( X \) is a graph with three vertices and three edges as shown in the figure, so \( E = \{e_0, e_1, e_2\} \). To construct a maximal tree \( Y \) we need only omit \( e_0 \), so that \( E' = \{e_0\} \) and \( E'' = \{e_1, e_2\} \). Suppose \( \psi_1 + \text{Im} \delta_1 = \psi_2 + \text{Im} \delta_1 \). There exist \( \varphi_1, \varphi_2 \in \Delta^0(X; G) \) such that \( \delta_1 \varphi_1|_{E''} = \psi_1|_{E''} \) and \( \delta_1 \varphi_2|_{E''} = \psi_2|_{E''} \). To show
\[
F(\psi_1 + \text{Im} \delta_1) = F(\psi_2 + \text{Im} \delta_1)
\]
means to show that
\[
(\psi_1 - \delta_1 \varphi_1)(e_0) = (\psi_2 - \delta_1 \varphi_2)(e_0),
\]
or
\[
(\psi_1 - \psi_2)(e_0) = [(\varphi_1(v_1) - \varphi_1(v_0)] - [(\varphi_2(v_1) - \varphi_2(v_0)]. \tag{4}
\]
Now, \( \psi_1 + \text{Im} \delta_1 = \psi_2 + \text{Im} \delta_1 \) implies that \( \psi_1 - \psi_2 \in \text{Im} \delta_1 \), and so there exists some \( \varphi : V \to G \) such that
\[
(\delta_1 \varphi)(e_i) = (\psi_1 - \psi_2)(e_i)
\]
for all \( e_i \in E \). Hence
\[
\psi_1(e_0) - \psi_2(e_0) = \varphi(v_1) - \varphi(v_0) \tag{5}
\]
\[
\psi_1(e_1) - \psi_2(e_1) = \varphi(v_2) - \varphi(v_0) \tag{6}
\]
\[
\psi_1(e_2) - \psi_2(e_2) = \varphi(v_2) - \varphi(v_1), \tag{7}
\]
while from \( \delta_1 \varphi_1|_{E''} = \psi_1|_{E''} \) and \( \delta_1 \varphi_2|_{E''} = \psi_2|_{E''} \) we obtain
\[
\psi_1(e_1) = \varphi_1(v_2) - \varphi_1(v_0) \quad \text{and} \quad \psi_2(e_1) = \varphi_2(v_2) - \varphi_2(v_0) \tag{8}
\]
\[
\psi_1(e_2) = \varphi_1(v_2) - \varphi_1(v_1) \quad \text{and} \quad \psi_2(e_2) = \varphi_2(v_2) - \varphi_2(v_1). \tag{9}
\]
Combining (6) and (8) gives
\[
\varphi(v_2) - \varphi(v_0) = [(\varphi_1(v_2) - \varphi_1(v_0)] - [(\varphi_2(v_2) - \varphi_2(v_0)], \tag{10}
\]
and combining (7) and (9) gives
\[
\varphi(v_2) - \varphi(v_1) = [(\varphi_1(v_2) - \varphi_1(v_1)) - [(\varphi_2(v_2) - \varphi_2(v_1)]. \tag{11}
\]
Now, if we subtract (11) from (10) we obtain
\[
\varphi(v_1) - \varphi(v_0) = [(\varphi_1(v_1) - \varphi_1(v_0)] - [(\varphi_2(v_1) - \varphi_2(v_0)]. \tag{12}
\]
We now put (12) into (5) and get precisely (4), as desired.

A simpler analysis can be employed to show that $F$ is well-defined in the case when a maximal tree is formed by deleting one edge and retaining one edge, which becomes the “base case” for an inductive argument that will establish that $F$ is well-defined when one edge is deleted and $n$ edges are retained, $n \in \mathbb{N}$ arbitrary. This result, once obtained, in turn becomes the base case for another inductive argument that establishes the well-definedness of $F$ in the general case when $m$ edges are deleted and $n$ edges are retained in the forming of a maximal tree, $m, n \in \mathbb{N}$ both arbitrary. All of this can be done under the assumption that $X$ is connected (i.e. has just one component), after which it is easy to extend to an arbitrary number of components.

To show that $F$ is a homomorphism of groups, along with addition and integer multiplication of cosets in a quotient group we assume the usual (componentwise) definitions for addition and integer multiplication of elements in a direct product of groups. In what follows $X$ is not assumed to be connected, so $E''$ is taken to be the set of edges included in the maximal tree for some component of $X$, and $E' = E - E''$. Let $m,n \in \mathbb{Z}$ and $\psi, \hat{\psi} \in \Delta^1(X;G)$. Then there exist $\varphi, \hat{\varphi} \in \Delta^0(X;G)$ such that $\delta_1 \varphi = \psi$ and $\delta_1 \hat{\varphi} = \hat{\psi}$. Now, for $e \in E''$ it’s easy to see that

$$
\delta_1(m \varphi + n \hat{\varphi})|_{E''} = (m \psi + n \hat{\psi})|_{E''}
$$

since $\delta_1$ is a homomorphism, and so by (3) we obtain

$$
F(m(\psi + \text{Im} \delta_1) + n(\hat{\psi} + \text{Im} \delta_1)) = F((m \psi + n \hat{\psi}) + \text{Im} \delta_1)
$$

$$
= \{((m \psi + n \hat{\psi}) - \delta_1(m \varphi + n \hat{\varphi}))(e)\}_{e \in E'}
$$

$$
= \{(m \psi + n \hat{\psi})(e) - (m \delta_1 \varphi + n \delta_1 \hat{\varphi})(e)\}_{e \in E'}
$$

$$
= \{m \psi(e) + m \delta_1 \varphi(e) + n \hat{\psi}(e) - n \delta_1 \hat{\varphi}(e)\}_{e \in E'}
$$

$$
= \{m(\psi - \delta_1 \varphi)(e) + n(\hat{\psi} - \delta_1 \hat{\varphi})(e)\}_{e \in E'}
$$

$$
= m\{(\psi - \delta_1 \varphi)(e)\}_{e \in E'} + n\{(\hat{\psi} - \delta_1 \hat{\varphi})(e)\}_{e \in E'}
$$

$$
= m F(\psi + \text{Im} \delta_1) + n F(\hat{\psi} + \text{Im} \delta_1).
$$

Hence $F$ is a homomorphism.

Let $\{g_e\}_{e \in E'} \in \prod_{e \in E'} G$. Define $\psi : E \to G$ by $\psi(e) = g_e$ for all $e \in E'$ and $\psi(e) = 0$ for all $e \in E''$. Let $\varphi : V \to G$ be any constant function, so there is some $g_0 \in G$ such that $\varphi(v) = g_0$ for all $v \in V$. Then $\psi|_{E''} = \delta_1 \varphi|_{E''} \equiv 0$ and in fact $\delta_1 \varphi \equiv 0$ everywhere, and by (3)

$$
F(\psi + \text{Im} \delta_1) = \{(\psi - \delta_1 \varphi)(e)\}_{e \in E'} = \{\psi(e) - \delta_1 \varphi(e)\}_{e \in E'} = \{\psi(e) - 0\}_{e \in E'} = \{g_e\}_{e \in E'},
$$

which shows that $F$ is surjective.

Finally, it remains to show that $\ker F = \{\text{Im} \delta_1\}$. We have

$$
F(\text{Im} \delta_1) = F(0 + \text{Im} \delta_1) = \{(0 - \delta_1 \varphi)(e)\}_{e \in E'}
$$

for any constant function $\varphi$ (so that $\delta_1 \varphi|_{E''} = 0|_{E''}$ as required), and so

$$
F(\text{Im} \delta_1) = \{0(e)\}_{e \in E'} = \{0\}_{e \in E'}.
$$
and we obtain \( \{ \text{Im} \delta_1 \} \subset \text{Ker} \, F \). Now, supposing that \( \psi + \text{Im} \delta_1 \in \text{Ker} \, F \), we have
\[
F(\psi + \text{Im} \delta_1) = \{(\psi - \delta_1 \varphi)(e)\}_{e \in E'} = \{0\}_{e \in E'}
\]
for some \( \varphi \) such that \( \delta_1 \varphi|_{E'} = \psi|_{E'} \); but then it is clear that \( \delta_1 \varphi|_{E'} = \psi|_{E'} \) as well, and so \( \delta_1 \varphi = \psi \) on all \( E \) and we find that \( \psi \in \text{Im} \delta_1 \). Therefore \( \psi + \text{Im} \delta_1 = \text{Im} \delta_1 \) and we have \( \text{Ker} \, F \subset \{ \text{Im} \delta_1 \} \). Since the kernel of \( F \) is trivial, \( F \) is injective.

It has been established at last that \( F \) is an isomorphism, and therefore
\[
H^1_\Delta(X;G) \cong \prod_{e \in E'} G.
\]

Now, suppose that \( X \) is a two-dimensional \( \Delta \)-complex. Let \( S_2 \) be the set of 2-simplices of \( X \), so
\[
S_2 = \{ \sigma_\alpha : \Delta^2 \to X \}_{\alpha \in A},
\]
and let \( \Delta^2(X;G) = \{ \omega : S_2 \to G \} \). Adhering to the notational conventions above, we define the homomorphism \( \delta_2 : \Delta^1(X;G) \to \Delta^2(X;G) \) by
\[
\delta_2 \psi([v_0, v_1, v_2]) = \psi([v_0, v_1]) - \psi([v_0, v_2]) + \psi([v_1, v_2])
\]
for each \( \psi \in \Delta^1(X;G) \), where \([v_0, v_1, v_2] := \sigma \in S_2 \) is a map that maps the vertices of \( \Delta^2 \) to \( v_0 \), \( v_1 \), and \( v_2 \). It’s worthwhile to be more precise here: if we let \( \Delta^2 = [u_0, u_1, u_2] \) then \( \sigma(u_i) = v_i \) for each \( i \), and moreover each \([v_i, v_j]\) denotes \( \sigma \) restricted to the face \([u_i, u_j]\) and precomposed by the canonical linear homeomorphism \( \Delta^1 \to [u_i, u_j] \). So more explicitly (13) can be written
\[
\delta_2 \psi(\sigma) = \psi|_{[u_0, u_1]} - \psi|_{[u_0, u_2]} + \psi|_{[u_1, u_2]}
\]
(14)

At last we arrive at the general case of an \( n \)-dimensional \( \Delta \)-complex \( X \). For \( 0 \leq i \leq n \) let \( S_i \) be the set of \( i \)-simplices \( \Delta^i \to X \) of \( X \), and let \( \Delta^i(X;G) \) be the set of functions \( S_i \to G \). We define the map \( \delta_i : \Delta^{i-1}(X;G) \to \Delta^i(X;G) \) by generalizing (14): for each \( \psi \in \Delta^{i-1}(X;G) \) the function \( \delta_i \psi \) is such that, for each \( \sigma : \Delta^i \to X \) in \( S_i \),
\[
\delta_i \psi(\sigma) = \sum_{j=0}^{i} (-1)^j \psi|_{[u_0, \ldots, u_j, \ldots, u_i]},
\]
(15)
where in general \( \Delta^i = [u_0, \ldots, u_i] \). In this way we obtain a chain complex
\[
\cdots \xleftarrow{\delta_{i+2}} \Delta^{i+1}(X;G) \xleftarrow{\delta_{i+1}} \Delta^i(X;G) \xleftarrow{\delta_i} \Delta^{i-1}(X;G) \xleftarrow{\delta_{i-1}} \cdots
\]
(16)

There’s a natural way to identify the abelian group \( \Delta^i(X;G) \) with the group \( \text{Hom}(\Delta_i(X),G) \) of homomorphisms \( \Delta_i(X) \to G \). In particular each \( \psi \in \Delta^i(X;G) \) can be made to correspond via a fixed isomorphism to \( \psi \in \text{Hom}(\Delta_i(X),G) \) given by
\[
\hat{\psi}(\sum_{\alpha} n_\alpha \sigma_\alpha) = \sum_{\alpha} n_\alpha \psi(\sigma_\alpha).
\]
Identifying \( \psi \) with \( \hat{\psi} \), then, we find from (15) that \( \delta_i \psi(\sigma) = \psi(\partial_i(\sigma)) \) and therefore \( \delta_i \psi = \psi \partial_i \). By definition this means that \( \delta_i \) is the dual map, called the coweboundary map, of \( \partial_i \). Going

\footnote{If we want to be fussy we can write \( \delta_i \psi = \psi \circ \partial_i \) to stress that \( \delta_i \psi \) is not a composition of functions.}
a step further we designate $\Delta^i(X; G)$ (identified with $\text{Hom}(\Delta_i(X), G)$) to be the dual cochain group of $\Delta_i(X)$ so that (16) is the dual cochain complex of the chain complex

$$\cdots \xrightarrow{\partial_{i+2}} \Delta_{i+1}(X) \xrightarrow{\partial_{i+1}} \Delta_i(X) \xrightarrow{\partial_i} \Delta_{i-1}(X) \xrightarrow{\partial_{i-1}} \cdots \tag{17}$$

The operation of passing from (17) to (16) can be characterized as the action of a contravariant functor $\Delta_i(X) \mapsto \text{Hom}(\Delta_i(X), G)$, or more generally $C \mapsto \text{Hom}(C, G)$ for any chain $C$, which sometimes is denoted by $\text{Hom}(-, G)$. The categories involved here are the category $\mathcal{C}$ of objects $\Delta_i(X)$ and morphisms $\partial_i$ (which will be the zero homomorphism for chain groups $\Delta_j(X)$ and $\Delta_k(X)$ with $|j - k| > 1$), and the category $\mathcal{D}$ of objects $\text{Hom}(\Delta_i(X), G)$ and morphisms $\delta_i$.

### 3.1 – Cohomology of Chain Complexes

Starting with a chain complex $C$ not associated with any topological space,

$$\cdots \xrightarrow{\delta_{n+2}} C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots,$$

where each $C_n$ is a free abelian group, we dualize by applying $\text{Hom}(-, G)$ for some abelian group $G$ to obtain the cochain

$$\cdots \xleftarrow{\delta_{n+2}} \text{Hom}(C_{n+1}, G) \xleftarrow{\delta_{n+1}} \text{Hom}(C_n, G) \xleftarrow{\delta_n} \text{Hom}(C_{n-1}, G) \xleftarrow{\delta_{n-1}} \cdots$$

In general $\delta_n \varphi := \delta_n(\varphi) := \varphi \circ \partial_n$. For each homology group $H_n(C) = \ker \partial_n / \text{im} \partial_{n+1}$ there is a corresponding cohomology group

$$H^n(C; G) = \frac{\ker \delta_{n+1}}{\text{im} \delta_n}.$$

An element of $H^n(C; G)$ is $\varphi + \text{im} \delta_n$, where $\varphi \in \ker \delta_{n+1}$ implies that $\varphi \circ \partial_{n+1}$ is the trivial homomorphism: for all $x \in C_{n+1}$, $(\varphi \circ \partial_{n+1})(x) = 0$ in $G$.

Fix $\varphi \in \ker \delta_{n+1}$. Now, since $\varphi : C_n \rightarrow G$ and $\ker \partial_n \subset C_n$, we can define $\varphi_0 = \varphi|_{\ker \partial_n}$, which in turn induces a map $\bar{\varphi}_0 : H_n(C) \rightarrow G$ given by

$$\bar{\varphi}_0(z + \text{im} \partial_{n+1}) = \varphi_0(z).$$

Finally, define $h : H^n(C; G) \rightarrow \text{Hom}(H_n(C), G)$ by

$$h(\varphi + \text{im} \delta_n) = \bar{\varphi}_0.$$

It needs to be shown that $h$ is well-defined.

Suppose that $\varphi + \text{im} \delta_n = \varphi' + \text{im} \delta_n$. Then $(\varphi - \varphi') + \text{im} \delta_n = \text{im} \delta_n$ implies $\varphi - \varphi' \in \text{im} \delta_n$, and so there exists some $\psi \in \text{Hom}(C_{n-1}, G)$ such that $\delta_n(\psi) = \varphi - \varphi'$, whence $\psi \circ \partial_n = \varphi - \varphi'$ and we obtain $\varphi' = \varphi - \psi \circ \partial_n$. Now, for any $z \in \ker \partial_n$ we have

$$\varphi'_0(z) = \varphi'(z) = (\varphi - \psi \circ \partial_n)(z) = \varphi(z) - (\psi \circ \partial_n)(z) = \varphi(z) - \psi(0) = \varphi(z) = \varphi_0(z),$$

and so for any $z + \text{im} \partial_{n+1} \in H_n(C)$

$$\bar{\varphi}_0(z + \text{im} \partial_{n+1}) = \varphi_0(z) = \varphi'_0(z) = \varphi'_0(z + \text{im} \partial_{n+1}).$$
Therefore

\[ h(\varphi + \text{Im}\, \delta_n) = \varphi_0 = \varphi'_0 = h(\varphi' + \text{Im}\, \delta_n) \]

and \( h \) is well-defined. Moreover it is clear that \( h \) is a homomorphism.

Next it will be shown that \( h \) is surjective. Let \( f \in \text{Hom}(H_n(C); G) \). We must find some \( \varphi + \text{Im}\, \delta_n \in H^n(C; G) \) such that \( h(\varphi + \text{Im}\, \delta_n) = \varphi_0 = f \), which is to say that for each \( z \in \text{Ker}\, \partial_n \) we have

\[ f(z + \text{Im}\, \partial_{n+1}) = \varphi_0(z + \text{Im}\, \partial_{n+1}). \]

Start by defining \( \varphi_0 : \text{Ker}\, \partial_n \to G \) by

\[ \varphi_0(z) = f(z + \text{Im}\, \partial_{n+1}). \]

The task is to extend \( \varphi_0 \) to a map \( \varphi : C_n \to G \) such that \( \varphi \in \text{Ker}\, \delta_{n+1} \).

Defining \( i : \text{Ker}\, \partial_n \to C_n \) to be the inclusion map, observe that the sequence

\[
0 \rightarrow \text{Ker}\, \partial_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} \text{Im}\, \partial_n \rightarrow 0
\]

is exact. Since \( \text{Im}\, \partial_n \) is a free group the sequence splits, and so by the Splitting Lemma there exists a homomorphism \( p : C_n \to \text{Ker}\, \partial_n \) such that

\[ p \circ i_n = 1 : \text{Ker}\, \partial_n \to \text{Ker}\, \partial_n. \]

Define \( \varphi = \varphi_0 \circ p : C_n \to G \), which clearly is a homomorphism. Now, for any \( z \in \text{Ker}\, \partial_n \),

\[ \varphi(z) = \varphi_0(p(z)) = \varphi_0(p(i_n(z))) = \varphi_0((p \circ i_n)(z)) = \varphi_0(1(z)) = \varphi_0(z) = f(z + \text{Im}\, \partial_{n+1}) \]

shows that \( \varphi \) is an extension of \( \varphi_0 \) to \( C_n \).

Fix \( x \in C_{n+1} \). Then

\[
(\varphi \circ \partial_{n+1})(x) = \varphi_0(p(\partial_{n+1}x)) = \varphi_0(p(i_n(\partial_{n+1}x))) = \varphi_0(1(\partial_{n+1}x)) \\
= \varphi_0(\partial_{n+1}x) = f(\partial_{n+1}x + \text{Im}\, \partial_{n+1}) = f(\text{Im}\, \partial_{n+1}) = 0 \in G,
\]

where the second equality holds since \( \partial_{n+1}(x) \in \text{Ker}\, \partial_n \) and the last holds since \( \text{Im}\, \partial_{n+1} \) is the zero element of \( H_n(C) \). Hence \( \delta_{n+1}(\varphi) = \varphi \circ \partial_{n+1} \equiv 0 \), implying that \( \varphi \in \text{Ker}\, \delta_{n+1} \) and therefore \( \varphi + \text{Im}\, \delta_n \in H^n(C; G) \).

By definition

\[ h(\varphi + \text{Im}\, \delta_n) = \bar{\varphi}|_{\text{Ker}\, \partial_n}, \]

where for any \( z + \text{Im}\, \partial_{n+1} \in H_n(C) \) we obtain

\[
\bar{\varphi}|_{\text{Ker}\, \partial_n}(z + \text{Im}\, \partial_{n+1}) = (\varphi_0 \circ p)|_{\text{Ker}\, \partial_n}(z) = \varphi_0(p(z)) = \varphi_0(z) = f(z + \text{Im}\, \partial_{n+1}),
\]

using the fact that \( z \in \text{Ker}\, \partial_n \) implies \( p(z) = p(i_n(z)) = 1(z) = z \). Therefore \( h(\varphi + \text{Im}\, \delta_n) = f \) and \( h \) is surjective.
To determine the conditions in which \( h \) may be injective we analyze \( \text{Ker} \ h \). Start with the commutative diagram of short exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker} \partial_{n+1} & \overset{i_{n+1}}\longrightarrow & C_{n+1} & \overset{\partial_{n+1}}\longrightarrow & \text{Im} \partial_{n+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker} \partial_n & \overset{i_n}\longrightarrow & C_n & \overset{\partial_n}\longrightarrow & \text{Im} \partial_n & \longrightarrow & 0 \\
\end{array}
\]  

(19)

where the map \( \text{Ker} \partial_{n+1} \rightarrow \text{Ker} \partial_n \) is \( \partial_{n+1}|_{\text{Ker} \partial_{n+1}} \), and \( \text{Im} \partial_{n+1} \rightarrow \text{Im} \partial_n \) is \( \partial_n|_{\text{Im} \partial_{n+1}} \). We dualize (19) by applying \( \text{Hom}(-, G) \) to obtain

\[
\begin{array}{cccccc}
0 & \longleftarrow & \text{Hom}(\text{Ker} \partial_{n+1}, G) & \overset{i^*_{n+1}}\longleftarrow & \text{Hom}(C_{n+1}, G) & \overset{\partial^*_{n+1}}\longleftarrow & \text{Hom}(\text{Im} \partial_{n+1}, G) & \longleftarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longleftarrow & \text{Hom}(\text{Ker} \partial_n, G) & \overset{i^*_n}\longleftarrow & \text{Hom}(C_n, G) & \overset{\partial^*_n}\longleftarrow & \text{Hom}(\text{Im} \partial_n, G) & \longleftarrow & 0 \\
\end{array}
\]  

(20)

remembering that \( 0^* = 0 \). It will be shown that the rows of (20) are split short exact sequences.

**Proposition 3.1.** *The dual of a split short exact sequence is a split short exact sequence.*

**Proof.** Suppose that

\[
0 \longrightarrow A \overset{i}\longrightarrow B \overset{j}\longrightarrow C \longrightarrow 0
\]

(21)

is a split short exact sequence. It must be shown that the sequence

\[
0 \longrightarrow \text{Hom}(C, G) \overset{j^*}\longrightarrow \text{Hom}(B, G) \overset{i^*}\longrightarrow \text{Hom}(A, G) \longrightarrow 0
\]

(22)

is exact and splits.

For \( \varphi \in \text{Hom}(C, G) \) suppose that \( j^*(\varphi) = 0 \), so \( \varphi \circ j = 0 \). Fix \( c \in C \). Since \( j \) is surjective there exists some \( b \in B \) such that \( j(b) = c \), and thus

\[
\varphi(c) = \varphi(j(b)) = (\varphi \circ j)(b) = 0.
\]

This shows that for any \( \varphi \in \text{Ker} j^* \) we have \( \varphi = 0 \), so \( \text{Ker} j^* = \{0\} \).

Let \( \varphi \in \text{Im} j^* \), so there exists some \( \psi \in \text{Hom}(C, G) \) such that \( j^*(\psi) = \varphi \), or equivalently \( \psi \circ j = \varphi \). Now, for any \( a \in A \) we obtain

\[
(\varphi \circ i)(a) = (\psi \circ j \circ i)(a) = \psi(j(i(a))) = \psi(0) = 0,
\]

where \( \text{Im} i = \text{Ker} j \) implies that \( i(a) \in \text{Ker} j \). Thus \( i^*(\varphi) = \varphi \circ i = 0 \), implying \( \varphi \in \text{Ker} i^* \) and so \( \text{Im} j^* \subset \text{Ker} i^* \).

Let \( \varphi \in \text{Ker} i^* \), so \( \varphi : B \rightarrow G \) such that \( i^* \varphi = 0 \), or equivalently \( \varphi \circ i = 0 \) which informs us that \( \varphi \) vanishes on \( \text{Im} i \). By the Splitting Lemma there exists some \( s : C \rightarrow B \) such that \( j \circ s = 1 : C \rightarrow C \). Let \( \psi = \varphi \circ s \). Fix \( b \in B \). Then \( (s \circ j)(b) - b \in B \) with

\[
j((s \circ j)(b) - b) = (j \circ s \circ j)(b) - j(b) = (1 \circ j)(b) - j(b) = j(b) - j(b) = 0,
\]
so that \((s \circ j)(b) - b \in \text{Ker}\ j = \text{Im}\ i\) and there is some \(a \in A\) such that \(i(a) = (s \circ j)(b) - b\). Since \(\varphi\) vanishes on \(\text{Im}\ i\) it follows that \((\varphi \circ i)(a) = 0\), whence \(\varphi((s \circ j)(b) - b) = 0\) leads to \(\varphi((s \circ j)(b)) = \varphi(b)\). Now,

\[(\psi \circ j)(b) = (\varphi \circ s \circ j)(b) = \varphi((s \circ j)(b)) = \varphi(b)\]

shows that \(j^*(\psi) = \psi \circ j = \varphi\), so \(\varphi \in \text{Im}\ j^*\) and we obtain \(\text{Ker}\ i^* \subset \text{Im}\ j^*\).

Finally, fix \(\varphi \in \text{Hom}(A, G)\). The Splitting Lemma implies there is a homomorphism \(p : B \rightarrow A\) such that \(p \circ i 1 : A \rightarrow A\). For any \(a \in A\),

\[(\varphi \circ p \circ i)(a) = \varphi((p \circ i)(a)) = \varphi(1(a)) = \varphi(a),\]

and so \(\varphi = \varphi \circ p \circ i\). But \(\varphi \circ p \in \text{Hom}(B, G)\) such that \(i^*(\varphi \circ p) = \varphi \circ p \circ i\), so \(\varphi \in \text{Im}\ i^*\) and it follows that \(\text{Im}\ i^* = \text{Hom}(A, G)\).

Moving on, since (21) splits there is an isomorphism \(\Phi\) such that the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow f & & \downarrow \Phi \\
A \oplus C & \rightarrow & C \\
\downarrow g & & \\
& B & \rightarrow & 0
\end{array}
\]

is commutative. The dualization of this diagram is

\[
\begin{array}{ccc}
0 & \leftarrow & \text{Hom}(A, G) \\
\downarrow f^* & & \downarrow \Phi^* \\
\text{Hom}(A \oplus C, G) & \leftarrow & \text{Hom}(B, G) \\
\downarrow g^* & & \downarrow \Phi^* \\
& \text{Hom}(C, G) & \leftarrow & 0
\end{array}
\]

where \(\Phi^*\) is an isomorphism since the dual of any isomorphism is again an isomorphism. It’s easily verified that \(g \circ \Phi = j\) implies \(\Phi^* \circ g^* = j^*\) and \(\Phi \circ i = f\) implies \(i^* \circ \Phi^* = f^*\) (in general \((\varphi \circ \psi)^* = \psi^* \circ \varphi^*\)), so the dualized diagram is commutative. Finally, there’s a natural isomorphism

\[
\Omega : \text{Hom}(A \oplus C, G) \rightarrow \text{Hom}(A, G) \oplus \text{Hom}(C, G)
\]

defined by

\[
\Omega(\varphi(\cdot, \cdot)) = (\varphi(\cdot, 0), \varphi(0, \cdot)),
\]

so if we define \(\bar{\Phi}^* = f^* \circ \Omega^{-1}, \bar{g}^* = g^* \circ \Omega, \text{ and } \bar{\Phi}^* = \Phi^* \circ \Omega^{-1}, \text{ then we obtain the commutative diagram}

\[
\begin{array}{ccc}
0 & \leftarrow & \text{Hom}(A, G) \\
\downarrow f^* & & \downarrow \Phi^* \\
\text{Hom}(A, G) \oplus \text{Hom}(C, G) & \leftarrow & \text{Hom}(B, G) \\
\downarrow g^* & & \downarrow \Phi^* \\
& \text{Hom}(C, G) & \leftarrow & 0
\end{array}
\]

which shows that the sequence (22) splits.

Let

\[B_n = \text{Im} \partial_{n+1} \text{ and } Z_n = \text{Ker} \partial_n,\]
and let

\[ C_n^* = \text{Hom}(C_n, G), \quad Z_n^* = \text{Hom}(Z_n, G), \quad B_n^* = \text{Hom}(B_n, G). \]

Finally, let \( \delta_n : C_n^* \to C_{n-1}^* \) be the dual of \( \partial_n : C_n \to C_{n-1} \) as before, and let \( \varrho_n : B_n^* \to C_n^* \) be the dual of \( \varrho_n : C_n \to B_{n-1} \). The diagram (20) can be extended to a short exact sequence of chain complexes

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
\cdots & Z_n^* & \leftarrow 0 & Z_n^* & \leftarrow 0 & Z_{n-1}^* & \leftarrow \cdots \\
& i_{n+1}^* & \uparrow & i_n^* & \uparrow & i_{n-1}^* \\
\cdots & \leftarrow C_{n+1}^* & \xrightarrow{\delta_{n+1}} & C_n^* & \xrightarrow{\delta_n} C_{n-1}^* & \leftarrow \cdots \\
& \varrho_{n+1} & \uparrow & \varrho_n & \uparrow & \varrho_{n-1} \\
\cdots & \leftarrow B_n^* & 0 & B_{n-1}^* & 0 & B_{n-2}^* & \leftarrow \cdots \\
& & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & 0 & 0 \\
\end{array}
\]

Associated with this diagram is a long exact sequence of cohomology groups

\[
\begin{array}{c}
\cdots \leftarrow H^{n+1}(Z^*) \xleftarrow{i_{n+1}^*} H^{n+1}(C; G) \xleftarrow{\varrho_{n+1}^*} H^{n+1}(B^*) \xleftarrow{d_n} H^n(Z^*) \xleftarrow{i_n^*} H^n(C; G) \\
& \xleftarrow{\varrho_n^*} H^n(B^*) \xleftarrow{d_{n-1}} H^{n-1}(Z^*) \xleftarrow{\cdots} ,
\end{array}
\]

(24)

where each \( i_n^* \) and \( \varrho_n^* \) is a homomorphism induced by \( i_n^* \) and \( \varrho_n \), and each \( d_n \) is a connecting homomorphism which will be examined shortly. By definition

\[ H^n(B^*) = \frac{\text{Ker}[0 : B_{n-1}^* \to B_n^*]}{\text{Im}[0 : B_{n-2}^* \to B_{n-1}^*]} \quad \text{and} \quad H^n(Z^*) = \frac{\text{Ker}[0 : Z_n^* \to Z_{n+1}^*]}{\text{Im}[0 : Z_{n-1}^* \to Z_n^*]}, \]

so \( H^n(B^*) \cong B_{n-1}^* \) and \( H^n(Z^*) \cong Z_n^* \), and (24) can be written as

\[
\begin{array}{c}
\cdots \leftarrow Z_{n+1} \xleftarrow{i_{n+1}^*} H^{n+1}(C; G) \xleftarrow{\varrho_{n+1}^*} B_n^* \xleftarrow{d_n} Z_n^* \xleftarrow{i_n^*} H^n(C; G) \\
& \xleftarrow{\varrho_n^*} B_{n-1}^* \xleftarrow{d_{n-1}} Z_{n-1}^* \xleftarrow{\cdots}
\end{array}
\]

(25)

with appropriate adjustments to the definitions of \( i_n^* \) and \( \varrho_n^* \); in particular, let \( i_n^*(\sigma + \text{Im} \delta_n) = i_n^*(\sigma) \) and \( \varrho_n^*(\varphi) = \varrho_n(\varphi) + \text{Im} \delta_n \) (it’s easily verified that \( \varrho_n(\varphi) = \varphi \circ \partial_n \in \text{Ker} \delta_{n+1} \)).

We define \( d_n \) in reference to (24). Let \( \gamma \in Z_n^* \) be a cycle, so it represents a cohomology class \( [\gamma] \in H^n(Z^*) \) (note that in fact every element of \( Z_n^* \) is a cycle). Since \( i_n^* \) is surjective, there exists some \( \beta \in C_n^* \) such that \( i_n^*(\beta) = \gamma \). Exploiting commutativity in (23) gives

\[ i_{n+1}^*(\delta_{n+1}(\beta)) = 0(i_n^*(\beta)) = 0, \]
so \( \delta_{n+1}(\beta) \in \text{Ker} \iota_n^* \) and there must exist some \( \alpha \in B_n^* \) such that \( \varrho_{n+1}(\alpha) = \delta_{n+1}(\beta) \). Since \( \alpha \in \text{Ker}[0 : B_n^* \to B_{n+1}^*] \), \( \alpha \) represents a cohomology class \( [\alpha] \in H^{n+1}(B^*) \). Define \( d_n([\gamma]) = [\alpha] \). Since \( H^{n+1}(B^*) \cong B_n^* \) and \( H^n(Z^*) \cong Z_n^* \), in reference to (25) we can simply define \( d_n(\gamma) = \alpha \).

It turns out that \( \alpha = \gamma|_{B_n} \). From \( \varrho_{n+1}(\alpha) = \delta_{n+1}(\beta) \) comes \( \alpha \circ \partial_{n+1} = \beta \circ \partial_{n+1} \), which shows that \( \alpha = \beta|_{B_n} \) (recall that \( \alpha : B_n \to G \)). But we also have \( \beta \circ \iota_n = \gamma \) for \( \iota_n : Z_n \hookrightarrow C_n \), so \( \gamma = \beta|_{Z_n} \). Since \( B_n \subset Z_n \) it follows that \( \gamma|_{B_n} = \beta|_{B_n} = \alpha \). Hence \( d_n(\gamma) = \gamma|_{B_n} \), and so if \( \iota_n : B_n \hookrightarrow Z_n \) is the inclusion map then it’s seen that \( d_n \) is nothing more than \( \iota_n^* : Z_n^* \to B_n^* \), the dual of \( \iota_n \).

The process of verifying that (25) is exact is the same as for any long exact sequence in the previous chapter. From this sequence we can pass to a new sequence

\[
0 \leftarrow \text{Ker} \iota_n^* \leftarrow H^n(C; G) \leftarrow \zeta \text{Coker} \iota_{n-1}^* \leftarrow 0 \quad (26)
\]

where

\[
\text{Coker} \iota_{n-1}^* = \frac{B_{n-1}^*}{\text{Im} \iota_{n-1}^*}
\]

and \( \zeta \) works in the expected fashion: for any \( \varphi \in B_{n-1}^* \),

\[
\zeta(\varphi + \text{Im} \iota_{n-1}^*) = \varrho_n(\varphi) = \varrho_n(\varphi) + \text{Im} \delta_n.
\]

It’s worth verifying that \( \zeta \) is well-defined, so suppose

\[
\varphi_1 + \text{Im} \iota_{n-1}^* = \varphi_2 + \text{Im} \iota_{n-1}^*.
\]

Then \( \varphi_1 - \varphi_2 \in \text{Im} \iota_{n-1}^* = \text{Ker} \varrho_n^* \), using the exactness of (25). Now,

\[
\zeta(\varphi_1 + \text{Im} \iota_{n-1}^*) - \zeta(\varphi_2 + \text{Im} \iota_{n-1}^*) = (\varrho_n(\varphi_1) + \text{Im} \delta_n) - (\varrho_n(\varphi_2) + \text{Im} \delta_n)
\]

\[
= (\varrho_n(\varphi_1) - \varrho_n(\varphi_2)) + \text{Im} \delta_n
\]

\[
= \varrho_n(\varphi_1 - \varphi_2) + \text{Im} \delta_n
\]

\[
= \varrho_n^*(\varphi_1 - \varphi_2) = \text{Im} \delta_n,
\]

since \( \varphi_1 - \varphi_2 \in \text{Ker} \varrho_n^* \). That is,

\[
(\varrho_n(\varphi_1) - \varrho_n(\varphi_2)) + \text{Im} \delta_n = \text{Im} \delta_n,
\]

which implies

\[
\varrho_n(\varphi_1) + \text{Im} \delta_n = \varrho_n(\varphi_2) + \text{Im} \delta_n,
\]

or

\[
\zeta(\varphi_1 + \text{Im} \iota_{n-1}^*) = \zeta(\varphi_2 + \text{Im} \iota_{n-1}^*).
\]

It’s clear that \( \zeta \) is a homomorphism.

The sequence (26) is a short exact sequence. Suppose \( \zeta(\varphi + \text{Im} \iota_{n-1}^*) = \text{Im} \delta_n \). Then \( \varrho_n(\varphi) \in \text{Im} \delta_n \), and so there exists some \( \psi \in C_{n-1}^* \) such that \( \delta_n(\psi) = \varrho_n(\varphi) \), whence \( \varphi \circ \partial_n = \psi \circ \partial_n \) and thus \( \psi|_{B_{n-1}} = \varphi \). Now \( \psi|_{Z_{n-1}} \in Z_{n-1}^* \), and

\[
\iota_{n-1}(\psi|_{Z_{n-1}}) = \psi|_{Z_{n-1}} \circ \iota_{n-1} = \psi|_{B_{n-1}} = \varphi
\]
shows that \( \varphi \in \text{Im} \, \iota^*_{n-1} \) and hence \( \varphi + \text{Im} \, \iota^*_{n-1} = \text{Im} \, \iota^*_{n-1} \). Therefore Ker \( \zeta = 0 \) and \( \zeta \) is injective.

Fix \( \sigma \in \text{Ker} \, \iota^*_n \), so \( \sigma : Z_n \to G \) such that \( \sigma|_{B_n} \equiv 0 \). Since (18) is exact, by the Splitting Lemma there is some \( p : C_n \to Z_n \) such that \( p \circ i_n = \mathbb{1} : Z_n \to Z_n \). Let \( \tilde{\sigma} = \sigma \circ p \), so \( \tilde{\sigma} \in C^*_n \). For any \( x \in C^*_{n+1} \),

\[
(\sigma \circ p \circ \partial_{n+1})(x) = (\sigma \circ p)(\partial_{n+1}x) = (\sigma \circ p)(i_n(\partial_{n+1}x)) = (\sigma \circ \mathbb{1})(\partial_{n+1}x) = \sigma(\partial_{n+1}x) = 0
\]

(since \( \partial_{n+1}x \in B_n \)), which shows that

\[
\delta_{n+1}(\tilde{\sigma}) = \tilde{\sigma} \circ \partial_{n+1} = \sigma \circ p \circ \partial_{n+1} \equiv 0
\]
on \( C^*_{n+1} \). Hence \( \tilde{\sigma} \in \text{Ker} \, \delta^*_{n+1} \) so that \( \tilde{\sigma} + \text{Im} \, \delta_n \in H^n(C; G) \), and since

\[
i^*_n(\tilde{\sigma} + \text{Im} \, \delta_n) = i^*_n(\tilde{\sigma}) = \tilde{\sigma} \circ i_n = \sigma \circ p \circ i_n = \sigma \circ \mathbb{1} = \sigma
\]

we find Ker \( \iota^*_n \subset \text{Im} \, i^*_n \). As for the reverse containment, note that \( \varphi \in \text{Ker} \, \delta^*_{n+1} \) implies \( \varphi|_{B_n} \equiv 0 \), so

\[
i^*_n(\varphi + \text{Im} \, \delta_n) = i^*_n(\varphi \circ i_n) = \varphi \circ i_n \circ \iota_n = \varphi|_{B_n} \equiv 0
\]

shows that \( i^*_n \) maps into Ker \( \iota^*_n \). Hence Ker \( \iota^*_n = \text{Im} \, i^*_n \) and \( i^*_n \) in (26) is surjective.

It remains to confirm that \( \text{Im} \, \zeta = \text{Ker} \, i^*_n \). Since (25) is exact we have Ker \( i^*_n = \text{Im} \, \varrho_n \). Let \( \varphi + \text{Im} \, \delta_n \in \text{Im} \, \varrho_n \), so there exists some \( \psi \in B^*_{n-1} \) such that \( \varrho_n^*(\psi) = \varphi + \text{Im} \, \delta_n \); or \( \varphi + \text{Im} \, \delta_n = \psi \circ \partial_n + \text{Im} \, \delta_n \); but then \( \psi + \text{Im} \, \iota^*_{n-1} \in \text{Coker} \, \iota^*_n \) with

\[
\zeta(\psi + \text{Im} \, \iota^*_{n-1}) = \varrho_n(\psi) + \text{Im} \, \delta_n = \psi \circ \partial_n + \text{Im} \, \delta_n = \varphi + \text{Im} \, \delta_n,
\]

which gives \( \text{Im} \, \varrho_n \subset \text{Im} \, \zeta \). On the other hand, if \( \varphi + \text{Im} \, \delta_n \in \text{Im} \, \zeta \) then there’s some \( \psi + \text{Im} \, \iota^*_{n-1} \in \text{Coker} \, \iota^*_n \) with

\[
\zeta(\psi + \text{Im} \, \iota^*_{n-1}) = \varphi + \text{Im} \, \delta_n,
\]
or equivalently \( \psi \circ \partial_n + \text{Im} \, \delta_n = \varphi + \text{Im} \, \delta_n \); but \( \psi \in B^*_{n-1} \) such that

\[
\varrho_n^*(\psi) = \varrho_n(\psi) + \text{Im} \, \delta_n = \psi \circ \partial_n + \text{Im} \, \delta_n = \varphi + \text{Im} \, \delta_n,
\]

which makes clear that \( \text{Im} \, \zeta \subset \text{Im} \, \varrho_n \) and so \( \text{Im} \, \zeta = \text{Ker} \, i^*_n \).

Therefore (26) is exact as claimed.

Now, for each \( \sigma \in \text{Ker} \, \iota^*_n \) there is a corresponding map \( \tilde{\sigma} : H_n(C) \to G \) given by \( \tilde{\sigma}(z + B_n) = \sigma(z) \). Note that if \( z_1 + B_n = z_2 + B_n \) then \( z_1 - z_2 \in B_n \), and since \( \sigma \in \text{Ker} \, \iota^*_n \) implies that \( \sigma|_{B_n} \equiv 0 \) we obtain

\[
\tilde{\sigma}(z_1 + B_n) - \tilde{\sigma}(z_2 + B_n) = \sigma(z_1) - \sigma(z_2) = \sigma(z_1 - z_2) = 0,
\]

so \( \tilde{\sigma} \) is well-defined and clearly must be in \( \text{Hom}(H_n(C), G) \). Define \( \Theta : \text{Ker} \, \iota^*_n \to \text{Hom}(H_n(C), G) \) by \( \Theta(\sigma) = \tilde{\sigma} \). Certainly \( \Theta \) is well-defined. For \( \sigma_1, \sigma_2 \in \text{Ker} \, \iota^*_n \) we have \( \Theta(\sigma_1 + \sigma_2) = \sigma_1 + \sigma_2 \), where

\[
\sigma_1 + \sigma_2(z + B_n) = (\sigma_1 + \sigma_2)(z) = \sigma_1(z) + \sigma_2(z) = \tilde{\sigma}_1(z + B_n) + \tilde{\sigma}_2(z + B_n)
\]

\[
= (\tilde{\sigma}_1 + \tilde{\sigma}_2)(z + B_n),
\]

so

\[
\frac{\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2} = \tilde{\sigma}_1 + \tilde{\sigma}_2 = \Theta(\sigma_1) + \Theta(\sigma_2).
\]
and $\Theta$ is a homomorphism.

Fix $\bar{\sigma} \in \text{Hom}(H_n(C), G)$. Define $\varphi \in Z^*_n$ by $\varphi(z) = \bar{\sigma}(z + B_n)$. For $z \in B_n$, $\varphi(z) = 0$, and so $\varphi \in \text{Ker} \iota^*_n$. Now, $\Theta(\varphi) = \bar{\varphi}$, where

$$\bar{\varphi}(z + B_n) = \varphi(z) = \bar{\sigma}(z + B_n)$$

for all $z \in Z_n$ and hence $\Theta(\varphi) = \bar{\sigma}$. So $\Theta$ is surjective.

Suppose $\sigma \in \text{Ker} \iota^*_n$ such that $\Theta(\sigma) = 0$, where $0(z + B_n) := 0$ for all $z \in Z_n$. Then $\bar{\sigma} = 0$, so for any $z \in Z_n$ we have

$$\sigma(z) = \bar{\sigma}(z + B_n) = 0(z + B_n) = 0$$

and therefore $\sigma \equiv 0$. So $\Theta$ is injective and we conclude that $\text{Ker} \iota^*_n \cong \text{Hom}(H_n(C), G)$. As a result we may pass from (26) to a new short exact sequence

$$0 \leftarrow \iota^*_n \text{Hom}(H_n(C), G) \leftarrow h H^n(C; G) \leftarrow \text{Coker} \iota^*_{n-1} \leftarrow 0,$$

where it’s easily verified that the map $h$ from above is given by $h = \Theta \circ \iota^*_n$:

$$h(\varphi + \text{Im} \delta_n) = \overline{\varphi | Z_n} = \overline{\varphi \circ \iota_n} = \Theta(\varphi \circ \iota_n) = \Theta(\iota^*_n(\varphi)) = (\Theta \circ \iota^*_n)(\varphi + \text{Im} \delta_n).$$

For each $\bar{\varphi} \in \text{Hom}(H_n(C), G)$ there is a map $\varphi_0 : Z_n \rightarrow G$ such that $\bar{\varphi}(z + B_n) = \varphi_0(z)$, and so in particular $\varphi_0 | B_n \equiv 0$. Define $s_1 : \text{Hom}(H_n(C), G) \rightarrow \text{Ker} \delta_{n+1}$ by

$$s_1(\bar{\varphi}) = \varphi_0 \circ p,$$

where $p : C_n \rightarrow Z_n$ is as defined on page 7. Note that for any $x \in C_{n+1}$,

$$(\varphi_0 \circ p \circ \partial_{n+1})(x) = (\varphi_0 \circ p)(\partial_{n+1}x) = \varphi_0(\partial_{n+1}x) = 0,$$

where the second equality holds since $\partial_{n+1}x \in Z_n$ and $p|Z_n = 1 : Z_n \rightarrow Z_n$, and so $\delta_{n+1}(\varphi_0 \circ p) = 0$ as required.

Next, define $s_2 : \text{Ker} \delta_{n+1} \rightarrow H^n(C; G)$ by $s_2(\psi) = \psi + B_{n-1}$, and let $s = s_2 \circ s_1$. For any $\bar{\varphi} \in \text{Hom}(H_n(C), G)$ with associated $\varphi_0 : Z_n \rightarrow G$,

$$(h \circ s)(\bar{\varphi}) = h(\varphi_0 \circ p + B_{n-1}) = \overline{\varphi_0 \circ p| Z_n},$$

where for each $z + B_n \in H_n(C)$

$$\varphi_0 \circ p| Z_n(z + B_n) = \varphi_0 \circ p| Z_n(z) = \varphi_0(p(z)) = \varphi_0(z) = \bar{\varphi}(z + B_n).$$

Thus $(h \circ s)(\bar{\varphi}) = \overline{\varphi_0 \circ p| Z_n} = \bar{\varphi}$, so $h \circ s = 1 : \text{Hom}(H_n(C), G) \rightarrow \text{Hom}(H_n(C), G)$ and the sequence (27) splits.

A potentially useful result that may as well be established here as anywhere else is the following.

**Proposition 3.2.** If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is exact, then the dual sequence

$$\text{Hom}(A, G) \leftarrow \alpha^* \text{Hom}(B, G) \leftarrow \beta^* \text{Hom}(C, G) \leftarrow 0$$

is also exact.
**Proof.** Let $\varphi \in \text{Im} \beta^*$, so there exists $\psi \in \text{Hom}(C, G)$ such that $\beta^*(\psi) = \psi \circ \beta = \varphi$. Now,

$$\alpha^*(\varphi) = \varphi \circ \alpha = (\psi \circ \beta) \circ \alpha = \psi \circ (\beta \circ \alpha) = \psi \circ 0 \equiv 0,$$

where the fourth equality holds since $\text{Im} \alpha = \text{Ker} \beta$, and therefore $\varphi \in \text{Ker} \alpha^*$.

Let $\varphi \in \text{Ker} \alpha^*$, so $\alpha^*(\varphi) = \varphi \circ \alpha \equiv 0$ implies that $\varphi|_{\text{Im} \alpha} \equiv 0$, or equivalently $\varphi|_{\text{Ker} \beta} \equiv 0$. Since $\text{Im} \beta = C$ (i.e. $\beta$ is surjective), the map $\hat{\beta} : B/ \text{Ker} \beta \to C$ given by $\hat{\beta}(b + \text{Ker} \beta) = \beta(b)$ is an isomorphism. Let $\hat{\phi} : B/ \text{Ker} \beta \to G$ be given by $\hat{\phi}(b + \text{Ker} \beta) = \varphi(b)$, and note that $\hat{\phi}$ is well-defined:

$$b_1 + \text{Ker} \beta = b_2 + \text{Ker} \beta \iff b_1 - b_2 \in \text{Ker} \beta \iff \varphi(b_1 - b_2) = 0 \iff \varphi(b_1) - \varphi(b_2) = 0 \iff \varphi(b_1) = \varphi(b_2) \iff \hat{\phi}(b_1 + \text{Ker} \beta) = \hat{\phi}(b_2 + \text{Ker} \beta).$$

Clearly $\hat{\phi}$ is a homomorphism, so $\psi := \hat{\phi} \circ \hat{\beta}^{-1}$ is likewise a homomorphism and therefore a member of $\text{Hom}(C, G)$. Now, for any $b \in B$,

$$(\psi \circ \beta)(b) = \hat{\phi}(\hat{\beta}^{-1}(\beta(b))) = \hat{\phi}(b + \text{Ker} \beta) = \varphi(b),$$

and thus $\beta^*(\psi) = \psi \circ \beta = \varphi$ implies that $\varphi \in \text{Im} \beta^*$.

Finally, suppose that $\beta^*(\varphi) = 0$, so that $\varphi \circ \beta \equiv 0$ implies that $\varphi \in \text{Hom}(C, G)$ with $\varphi|_{\text{Im} \beta} \equiv 0$. But then $\text{Im} \beta = C$ makes clear that $\varphi \equiv 0$ on $C$.

Therefore, since $\text{Im} \beta^* = \text{Ker} \alpha^*$ and $\text{Ker} \beta^* = 0$, the dual sequence is exact. $\blacksquare$

The balance of this section will be devoted to the proof of the Universal Coefficient Theorem for cohomology and a couple of its corollaries, followed by a few examples. As a prelude to this there is a definition and a lemma.

**Definition 3.3.** A **free resolution** $F$ of an abelian group $H$ is an exact sequence

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

in which each $F_n$ is a free abelian group.

For the dual chain complex of $F$ that results from applying the functor $\text{Hom}(\cdot, G)$,

$$\cdots \leftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \xleftarrow{0},$$

define $H^n(F; G) = \text{Ker} f_{n+1}^*/\text{Im} f_n^*$.

**Lemma 3.4. (a)** Let $F$ and $F'$ be free resolutions of abelian groups $H$ and $H'$, respectively. If $\varphi : H \to H'$ is a homomorphism, then $\varphi$ can be extended to a chain map $F \to F'$:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

$\varphi_2 \downarrow \varphi_1 \downarrow \varphi_0 \downarrow \varphi$

$$\cdots \to F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} H' \to 0$$
If $\varphi_i : F_i \to F'_i$ and $\hat{\varphi}_i : F_i \to F'_i$ are two chain maps $F \to F'$ extending $\varphi : H \to H'$, then they are chain homotopic.

For any two free resolutions $F$ and $F'$ of $H$ there are canonical isomorphisms $H^n(F; G) \cong H^n(F'; G)$ for all $n$.

**Proof.** Induction will be employed to prove (a). Let $x$ be a basis element of $F_0$. Then $\varphi(f_0(x))$ is in $H'$, and since $f'_0$ is surjective there exists some $x' \in F'_0$ such that $f'_0(x') = \varphi(f_0(x))$. Define $\varphi_0 : F_0 \to F'_0$ by $\varphi_0(x) = x'$, so we have $\varphi \circ f_0 = f'_0 \circ \varphi_0$.

Now let $n \geq 0$ be arbitrary, and suppose that $\varphi_{n-1} \circ f_n = f'_n \circ \varphi_n$. (If $n = 0$ we take $\varphi_{n-1}$ to be $\varphi$.) Let $x \in F_{n+1}$ be a basis element. Now,

$$f'_n(\varphi_n(f_{n+1}(x))) = \varphi_{n-1}(f_n(f_{n+1}(x))) = \varphi_{n-1}(0) = 0$$

since $\operatorname{Im} f_{n+1} = \operatorname{Ker} f_n$, and thus we have $\varphi_n(f_{n+1}(x)) \in \operatorname{Ker} f'_n$. Since $\operatorname{Ker} f'_{n} = \operatorname{Im} f'_{n+1}$ there’s some $x' \in F'_{n+1}$ such that $f'_{n+1}(x') = \varphi_n(f_{n+1}(x))$, and we can define $\varphi_{n+1} : F_{n+1} \to F'_{n+1}$ by $\varphi_{n+1}(x) = x'$. Hence $\varphi_n \circ f_{n+1} = f'_{n+1} \circ \varphi_{n+1}$ and the induction argument is complete.

To prove (b), recall the definition of chain homotopy: if $\varphi_i : F_i \to F'_i$ and $\hat{\varphi}_i : F_i \to F'_i$ are two chain maps, then they are chain homotopic if there can be found homomorphisms $\lambda_i : F_i \to F'_{i+1}$ such that

$$\varphi_i - \hat{\varphi}_i = f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i$$

for all $i \geq 0$. Thus, suppose that $\varphi_i : F_i \to F'_i$ and $\hat{\varphi}_i : F_i \to F'_i$ are two chain maps $F \to F'$ extending $\varphi : H \to H'$. Another induction argument will be used. For the base case let $\lambda_{-1} \equiv 0$, so we need only find some $\lambda_0 : F_0 \to F'_1$ such that $\varphi_0 - \hat{\varphi}_0 = f'_1 \circ \lambda_0$. Let $x \in F_0$ be a basis element. We’ll want to define $\lambda_0(x)$ so that $f'_1(\lambda_0(x)) = (\varphi_0 - \hat{\varphi}_0)(x)$, which requires confirming that $(\varphi_0 - \hat{\varphi}_0)(x) \in \operatorname{Im} f'_1$. From

$$f'_0 \circ \varphi_0 = \varphi \circ f_0 = f'_0 \circ \hat{\varphi}_0$$

we obtain $f'_0(\varphi_0 - \hat{\varphi}_0) \equiv 0$, whence $f'_0((\varphi_0 - \hat{\varphi}_0)(x)) = 0$ shows that $(\varphi_0 - \hat{\varphi}_0)(x) \in \operatorname{Ker} f'_0 = \operatorname{Im} f'_1$. Therefore there exists $x' \in F'_1$ such that $f'_1(x') = (\varphi_0 - \hat{\varphi}_0)(x)$, so let $\lambda_0(x) = x'$.

For the inductive step, let $n \geq 0$ be arbitrary and suppose

$$\varphi_n - \hat{\varphi}_n = f'_{n+1} \circ \lambda_n + \lambda_{n-1} \circ f_n.$$ 

We want to show that there is some map $\lambda_{n+1} : F_{n+1} \to F'_{n+2}$ such that

$$\varphi_{n+1} - \hat{\varphi}_{n+1} = f'_{n+2} \circ \lambda_{n+1} + \lambda_n \circ f_{n+1}.$$ 

So, let $x$ be a basis element of $F_{n+1}$. It’s necessary to define $\lambda_{n+1}(x)$ such that

$$f'_{n+2}(\lambda_{n+1}(x)) = (\varphi_{n+1} - \hat{\varphi}_{n+1})(x) - \lambda_n(f_{n+1}(x)),$$

which requires having

$$z := (\varphi_{n+1} - \hat{\varphi}_{n+1})(x) - \lambda_n(f_{n+1}(x)) \in \operatorname{Im} f'_{n+2}.$$ 

Since $\operatorname{Im} f'_{n+2} = \operatorname{Ker} f'_{n+1}$ this is a matter of direct manipulation,

$$f'_{n+1}(z) = f'_{n+1}((\varphi_{n+1} - \hat{\varphi}_{n+1})(x)) - (f'_{n+1} \circ \lambda_n)(f_{n+1}(x)).$$
groups, and \( \varphi \) be a homomorphism. By part (a) since \( f \) is a chain map, it's straightforward to verify that \( \varphi \) extended to a chain map \( \varphi^* : H^n \to H^n \). Therefore, there exists some \( y \in F'_{n+2} \) such that \( f'_{n+2}(y) = z \), so we let \( \lambda_{n+1}(x) = y \).

We turn now to the proof of (c). Let \( F \) and \( F' \) be free resolutions of \( H \), and let \( \varphi : H \to H \) be a homomorphism. By part (a) \( \varphi \) can be extended to a chain map \( \varphi_n : F_n \to F'_n \), and dualizing gives a chain map \( \varphi^*_n : F'_n^\ast \to F_n^\ast \),

\[
\cdots \leftarrow F_2^* \leftarrow f_2^* \leftarrow F_1^* \leftarrow f_1^* \leftarrow F_0^* \leftarrow f_0^* \leftarrow H^* \leftarrow 0
\]

which in turn induces homomorphisms \( \varphi_n^* : H^n(F';G) \to H^n(F;G) \). Now, if the maps

\[
\varphi_n : F_n \to F_n'
\]

are another extension of \( \varphi \) to a chain map \( F \to F' \), then by part (b) \( \varphi_n \) and \( \hat{\varphi}_n \) are chain homotopic, meaning once again \( \varphi_n - \hat{\varphi}_n = f'_{n+1} \circ \lambda_n + \lambda_{n-1} \circ f_n \) for maps \( \lambda_n : F_n \to F'_{n+1} \). Dualizing gives

\[
\varphi^*_n - \hat{\varphi}^*_n = \lambda^*_n \circ f'_{n+1} + f^*_n \circ \lambda^*_{n-1},
\]

which shows that \( \varphi^*_n \) and \( \hat{\varphi}^*_n \) are chain-homotopic chain maps and therefore \( \varphi_n^* = \hat{\varphi}_n^* \) for all \( n \) by Proposition 2.1.

Let \( \alpha : H \to H \) be an isomorphism, with \( \beta = \alpha^{-1} : H \to H \). By part (a), \( \alpha \) can be extended to a chain map \( \alpha_n : F_n \to F'_n \), and \( \beta \) can be extended to a chain map \( \beta_n : F'_n \to F_n \). It's straightforward to verify that \( \beta_n \circ \alpha_n : F_n \to F_n \) is an extension of \( \beta \circ \alpha = 1_H : H \to H \) to a chain map, since \( \alpha_{n-1} \circ f_n = f'_n \circ \alpha_n \) and \( \beta_{n-1} \circ f'_n = f_n \circ \beta_n \) imply that

\[
\beta_{n-1} \circ \alpha_{n-1} \circ f_n = f_n \circ \beta_n \circ \alpha_n.
\]

But the identities \( 1_{F_n} : F_n \to F_n \) likewise constitute an extension of \( 1_H \) to a chain map, and so \( (\beta_n \circ \alpha_n)^* = 1_{F_n}^* \) for all \( n \). Now,

\[
(\beta_n \circ \alpha_n)^* = (\alpha_n^* \circ \beta_n^*) = (\alpha_n^* \circ \beta_n)^*
\]

and

\[
1_{F_n}^* = 1_{F_n}^* = 1_{H^n(F;G)},
\]

so

\[
\alpha_n^* \circ \beta_n^* = 1_{H^n(F;G)}.
\]

\[\text{Recall that in the present section a superscript } \ast \text{ is used to indicate an induced homomorphism of cohomology groups, and } f^{**} \text{ is defined to be } (f^\ast)^\ast.\]
A similar argument shows that

\[ \beta_{n}^{**} \circ \alpha_{n}^{**} = 1_{H^{n}(F'; G)}, \]

and therefore \( \alpha_{n}^{**} : H^{n}(F'; G) \to H^{n}(F; G) \) is an isomorphism for all \( n \). Thus so-called canonical isomorphisms \( H^{n}(F; G) \cong H^{n}(F'; G) \) result if we specify \( \alpha \) to be the isomorphism \( 1_{H} \) and extend to a chain map \( F \to F' \).

Part (c) of the lemma shows, in particular, that the first homology group deriving from a free resolution \( F \) of a group \( H \), \( H^{1}(F; G) \), depends only on \( H \) and \( G \), and not at all on the choice for \( F \). For this reason \( H^{1}(F; G) \) is often denoted by \( \text{Ext}(H, G) \), where \( \text{Ext}(H, G) \) is taken to be a fixed group determined by \( H \) and \( G \) such that \( H^{1}(F; G) \cong \text{Ext}(H, G) \) for all \( F \).

The other homology groups \( H^{n}(F; G) \) for \( n > 1 \) turn out to be trivial since, as will be verified later, any abelian group \( H \) can be put into a free resolution of the form

\[
\cdots \to 0 \xrightarrow{f_{2}} F_{1} \xrightarrow{f_{1}} F_{0} \xrightarrow{f_{0}} H \to 0. \tag{28}
\]

Moreover, since the truncated sequence

\[
F_{1} \xrightarrow{f_{1}} F_{0} \xrightarrow{f_{0}} H \to 0
\]

is exact, by Proposition 3.2 the dual is likewise exact and thus \( H^{0}(F; G) = \text{Ker} f_{1}^{*} / \text{Im} f_{0}^{*} = 0 \) as well.

**Theorem 3.5 (Universal Coefficient Theorem for Cohomology).** If a chain complex \( C \) of free abelian groups has homology groups \( H_{n}(C) \), then the cohomology groups \( H^{n}(C; G) \) of the cochain complex obtained by applying \( \text{Hom}(\cdot, G) \) are determined by split exact sequences

\[
0 \to \text{Ext}(H_{n-1}(C), G) \xrightarrow{\zeta} H^{n}(C; G) \xrightarrow{h} \text{Hom}(H_{n}(C), G) \to 0.
\]

**Proof.** For the abelian group \( H_{n-1}(C) \) there is the free resolution \( F \) given by

\[
\cdots \to 0 \to B_{n-1} \xrightarrow{t_{n-1}} Z_{n-1} \xrightarrow{q} H_{n-1}(C) \to 0,
\]

where \( t_{n-1} \) is inclusion and \( q : Z_{n-1} \to Z_{n-1}/B_{n-1} \) is the quotient map \( q(z) = z + B_{n-1} \).

Dualizing yields

\[
\cdots \leftarrow 0 \leftarrow B_{n-1}^{*} \xleftarrow{t_{n-1}^{*}} Z_{n-1}^{*} \xleftarrow{q^{*}} \text{Hom}(H_{n-1}(C), G) \leftarrow 0,
\]

so it’s seen that

\[
\text{Coker} \ t_{n-1}^{*} = B_{n-1}^{*} / \text{Im} t_{n-1}^{*} = H^{1}(F; G)
\]

and therefore \( \text{Coker} \ t_{n-1}^{*} \) depends only on \( H \) and \( G \). Setting \( \text{Ext}(H_{n-1}(C), G) \) equal to \( H^{1}(F; G) \), then, the split exact sequence (27) becomes

\[
0 \to \text{Ext}(H_{n-1}(C), G) \xrightarrow{\zeta} H^{n}(C; G) \xrightarrow{h} \text{Hom}(H_{n}(C), G) \to 0
\]

as desired.
As was mentioned, every abelian group $H$ has a free resolution of the form (28). Start by selecting a set $S$ of generators for $H$, let $F_0$ be the free abelian group with basis $S$, and define a homomorphism $f_0 : F_0 \to H$ such that $f(s) = s$ for each $s \in S$ (note that $f_0$ is surjective). Next let $F_1 = \text{Ker} f_0$ and define $f_1 : F_1 \hookrightarrow F_0$ to be inclusion. Finally, set $F_i = 0$ for all $i \geq 2$.

**Proposition 3.6.** (a) $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$.

(b) $\text{Ext}(H, G) = 0$ if $H$ is a free abelian group.

(c) $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$.

**Proof.** For the proof of (a), let (28) be a free resolution $F$ for $H$, and let

$$
\cdots \longrightarrow 0 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f_0} H' \longrightarrow 0.
$$

be a free resolution $F'$ for $H'$. Then it’s easy to check that

$$
\cdots \longrightarrow 0 \oplus 0 \xrightarrow{f_2 \oplus f'_2} F_1 \oplus F'_1 \xrightarrow{f_1 \oplus f'_1} F_0 \oplus F'_0 \xrightarrow{f_0 \oplus f'_0} H \oplus H' \longrightarrow 0 \oplus 0,
$$

where we define

$$(f_n \oplus f'_n)(x, x') = (f_n(x), f'_n(x'))$$

is a free resolution for $H \oplus H'$, which we’ll denote by $F \oplus F'$. Applying $\text{Hom}(-, G)$ to $F \oplus F'$ yields

$$
\cdots \longleftarrow 0 \xleftarrow{(f_2 \oplus f'_2)^*} (F_1 \oplus F'_1)^* \xleftarrow{(f_1 \oplus f'_1)^*} (F_0 \oplus F'_0)^* \xleftarrow{(f_0 \oplus f'_0)^*} (H \oplus H')^* \longleftarrow 0,
$$

and thus

$$H^1(F \oplus F'; G) = \frac{\text{Ker}(f_2 \oplus f'_2)^*}{\text{Im}(f_1 \oplus f'_1)^*} = \frac{(F_1 \oplus F'_1)^*}{\text{Im}(f_1 \oplus f'_1)^*}.$$

Noting that $H^1(F; G) = F_1^* / \text{Im} f_1^*$ and $H^1(F'; G) = F'_1^* / \text{Im} f'_1^*$, define

$$\Omega : H^1(F \oplus F'; G) \to H^1(F; G) \oplus H^1(F'; G)$$

by

$$\Omega(\varphi + \text{Im}(f_1 \oplus f'_1)^*) = (\varphi(\cdot, 0) + \text{Im} f_1^*, \varphi(0, \cdot) + \text{Im} f'_1^*).$$

Suppose

$$[\varphi] := \varphi + \text{Im}(f_1 \oplus f'_1)^* = \hat{\varphi} + \text{Im}(f_1 \oplus f'_1)^*:=[\hat{\varphi}],$$

so $\varphi - \hat{\varphi} \in \text{Im}(f_1 \oplus f'_1)^*$ and there exists $\psi \in (F_0 \oplus F'_0)^*$ such that $(f_1 \oplus f'_1)^*(\psi) = \varphi - \hat{\varphi}$; that is, $\psi \circ (f_1 \oplus f'_1) = \varphi - \hat{\varphi}$, so for any $(x, x') \in F_1 \oplus F'_1$,

$$(\psi \circ (f_1 \oplus f'_1))(x, x') = \psi(f_1(x), f'_1(x')) = (\varphi - \hat{\varphi})(x, x').$$

Define $\alpha \in F_0^*$ by $\alpha = \psi(\cdot, 0)$. Now, $f_1^*(\alpha) = \alpha \circ f_1$, where for each $x \in F_1$ we have

$$(\alpha \circ f_1)(x) = \alpha(f_1(x)) = \psi(f_1(x), 0) = \psi(f_1(x), f'_1(0)) = (\varphi - \hat{\varphi})(x, 0)$$

and therefore $f_1^*(\alpha) = (\varphi - \hat{\varphi})(\cdot, 0)$. Hence $\varphi(\cdot, 0) - \hat{\varphi}(\cdot, 0) \in \text{Im} f_1^*$, implying that

$$\varphi(\cdot, 0) + \text{Im} f_1^* = \hat{\varphi}(\cdot, 0) + \text{Im} f_1^*.$$
A similar argument gives
\[ \varphi(0, \cdot) + \operatorname{Im} f_1^* = \hat{\varphi}(0, \cdot) + \operatorname{Im} f_1^*, \]
whence \( \Omega([\varphi]) = \Omega([\hat{\varphi}]) \) obtains and \( \Omega \) is well-defined. That \( \Omega \) is a homomorphism is obvious, but is it an isomorphism?

Suppose that
\[ \Omega(\varphi + \operatorname{Im}(f_1 \oplus f_1'))^* = (0, 0), \]
so \( \varphi \in (F_1 \oplus F_1')^* \). Then \( \varphi(\cdot, 0) \in \operatorname{Im} f_1^* \) and \( \varphi(0, \cdot) \in \operatorname{Im} f_1'^* \), so
\[ \exists \psi \in F_0^* \text{ s.t. } f_1^*(\psi) = \psi \circ f_1^* = \varphi(\cdot, 0), \]
and
\[ \exists \chi \in F_0'^* \text{ s.t. } f_1'^*(\chi) = \chi \circ f_1'^* = \varphi(0, \cdot). \]
Define \( \gamma \in (F_0 \oplus F_0')^* \) by \( \gamma(x, x') = \psi(x) + \chi(x') \). Now,
\[ (f_1 \oplus f_1')^*(\gamma) = \gamma \circ (f_1 \oplus f_1'), \]
where for \((x, x') \in F_1 \oplus F_1'\) we have
\[ (\gamma \circ (f_1 \oplus f_1'))(x, x') = \gamma(f_1(x), f_1'(x')) = \psi(f_1(x)) + \chi(f_1'(x')) = \varphi(x, 0) + \varphi(0, x') = \varphi(x, x'), \]
which shows that \( (f_1 \oplus f_1')^*(\gamma) = \varphi \). Since \( \varphi + \operatorname{Im}(f_1 \oplus f_1')^* = 0 \) it follows that \( \ker \Omega = \{0\} \) and \( \Omega \) is injective.

Next, let
\[ (\varphi + \operatorname{Im} f_1^*, \psi + \operatorname{Im} f_1'^*) \in H^1(F; G) \oplus H^1(F'; G), \]
so that \( \varphi : F_1 \to G \) and \( \psi : F_1' \to G \) are homomorphisms. Define \( \omega : F_1 \oplus F_1' \to G \) by
\[ \omega(x, x') = \varphi(x) + \psi(x'), \]
which is easily verified to be a homomorphism so that \( \omega \in (F_1 \oplus F_1')^* \). Now, since \( \omega(x, 0) = \varphi(x) \) and \( \omega(0, x') = \psi(x') \) for all \( x \in F_1, x' \in F_1' \), it’s clear that \( \omega(\cdot, 0) = \varphi \) and \( \omega(0, \cdot) = \psi \) and so
\[ \Omega(\omega + \operatorname{Im}(f_1 \oplus f_1')^*) = (\omega(\cdot, 0) + \operatorname{Im} f_1^*, \omega(0, \cdot) + \operatorname{Im} f_1'^*) = (\varphi + \operatorname{Im} f_1^*, \psi + \operatorname{Im} f_1'^*). \]
Thus \( \Omega \) is surjective, and we obtain
\[ H^1(F \oplus F'; G) \cong H^1(F; G) \oplus H^1(F'; G) \]
since \( \Omega \) is an isomorphism. Therefore
\[ \operatorname{Ext}(H \oplus H', G) \cong \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G). \]

Moving on to the proof of (b), suppose that \( H \) is a free abelian group. Then the sequence
\[ \cdots \to 0 \to H \xrightarrow{1} H \to 0 \]
is a free resolution \( F \) of \( H \). Clearly \( H^1(F; G) = 0 \), which implies \( \operatorname{Ext}(H, G) = 0 \).
Finally we turn to the proof of (c). Fix \( n \in \mathbb{N} \). Recalling \( \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} \), define \( \pi : \mathbb{Z} \to \mathbb{Z}_n \) by \( \pi(k) = k + n\mathbb{Z} \), and note that \( \ker \pi = n\mathbb{Z} \). Letting \( i : n\mathbb{Z} \hookrightarrow \mathbb{Z} \) to be inclusion, we construct a free \( F \) resolution for \( \mathbb{Z}_n \):

\[
\cdots \to 0 \to n\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_n \to 0.
\]

Applying \( \text{Hom}(-, G) \) we get

\[
\cdots \to 0 \to n\mathbb{Z}^* \xleftarrow{i^*} \mathbb{Z}^* \xleftarrow{\pi^*} \mathbb{Z}_n^* \to 0,
\]

where of course \( H^1(F; G) = n\mathbb{Z}^* / \text{Im } i^* \).

Define \( \Upsilon : n\mathbb{Z}^* / \text{Im } i^* \to G/nG \) by

\[
\Upsilon(\varphi + \text{Im } i^*) = \varphi(n) + nG
\]

for each homomorphism \( \varphi : n\mathbb{Z} \to G \). Suppose \( \varphi_1 + \text{Im } i^* = \varphi_2 + \text{Im } i^* \). Then \( \varphi_1 - \varphi_2 \in \text{Im } i^* \) implies that \( i^*(\psi) = \varphi_1 - \varphi_2 \) for some \( \psi \in \mathbb{Z}^* \), which is to say \( \varphi_1 - \varphi_2 = \psi \circ i : n\mathbb{Z} \to G \) and thus

\[
(\varphi_1 - \varphi_2) (n) = \psi(i(n)) = \psi(n) = n\psi(1).
\]

Therefore \( \varphi_1(n) - \varphi_2(n) \in nG \), whence

\[
\Upsilon(\varphi_1 + \text{Im } i^*) = \varphi(n) + nG = \varphi_2(n) + nG = \Upsilon(\varphi_2 + \text{Im } i^*)
\]

and \( \Upsilon \) is well-defined. Obviously \( \Upsilon \) is a homomorphism.

Suppose \( \Upsilon(\varphi + \text{Im } i^*) = 0 \), so \( \varphi(n) \in nG \) and there exists some \( g_0 \in G \) such that \( \varphi(n) = n g_0 \). Define \( \psi \in \mathbb{Z}^* \) by \( \psi(k) = k g_0 \) for each \( k \in \mathbb{Z} \). Now, for each \( kn \in n\mathbb{Z} \) we have

\[
(\psi \circ i)(kn) = \psi(i(kn)) = \psi(kn) = (kn) g_0 = \varphi(kn),
\]

so \( i^*(\psi) = \psi \circ i = \varphi \). Hence \( \varphi + \text{Im } i^* = 0 \), so \( \ker \Upsilon = \{0\} \) and \( \Upsilon \) is injective.

Next, let \( g + nG \in G/nG \). Define \( \varphi : n\mathbb{Z} \to G \) to be a homomorphism such that \( \varphi(n) = g \) (so \( \varphi(kn) = k \varphi(n) = kg \) for all \( k \in \mathbb{Z} \)). Then

\[
\Upsilon(\varphi + \text{Im } i^*) = \varphi(n) + nG = g + nG.
\]

Therefore \( \Upsilon \) is surjective.

Since \( \Upsilon \) is an isomorphism it follows that

\[
G/nG \cong n\mathbb{Z}^* / \text{Im } i^* = H^1(F; G) \cong \text{Ext}(\mathbb{Z}_n, G),
\]

as desired.

If \( H \) is finitely generated it is a fact from algebra that \( H \) has a (unique) direct sum decomposition \( H = H_{\text{tor}} \oplus B \), where \( H_{\text{tor}} \) is the torsion subgroup of \( H \) and \( B \) is a free abelian group. Thus by the preceding proposition

\[
\text{Ext}(H, \mathbb{Z}) = \text{Ext}(H_{\text{tor}} \oplus B, \mathbb{Z}) = \text{Ext}(H_{\text{tor}}, \mathbb{Z}) \oplus \text{Ext}(B, \mathbb{Z}) \cong \text{Ext}(H_{\text{tor}}, \mathbb{Z}).
\]
Since $H_{tor} \subset H$ and $H$ is finitely generated, $H_{tor}$ must be a finitely generated torsion group and therefore of finite order. Thus $H_{tor} \cong \mathbb{Z}_k$ for some positive integer $k$, and it follows from part (c) of Proposition 3.6 that

$$\text{Ext}(H_{tor}, \mathbb{Z}) \cong \text{Ext}(\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}_k.$$ 

Therefore, in general, $\text{Ext}(H, \mathbb{Z}) \cong H_{tor}$.

Two additional facts from algebra are: (i) $\text{Hom}(H, \mathbb{Z})$ is isomorphic to the free part of $H$ if $H$ is a finitely generated abelian group; and (ii) if $A_1, \ldots, A_n$ are abelian groups with subgroups $B_i \subset A_i$, then

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong A_1/B_1 \times \cdots \times A_n/B_n.$$ 

We use these facts to prove the following.

**Proposition 3.7.** If the homology groups $H_n(C)$ and $H_{n-1}(C)$ of a chain complex $C$ of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n(C)$ and $T_{n-1} \subset H_{n-1}(C)$, then $H^n(C; \mathbb{Z}) \cong (H_n(C)/T_n) \oplus T_{n-1}$.

**Proof.** First, $H_n(C)$ has a direct sum decomposition $H_n(C) \cong T_n \oplus B$, where $B$ is the free part of $H_n(C)$. Also we have $\text{Ext}(H_{n-1}(C), \mathbb{Z}) \cong T_{n-1}$. By (i) above, $\text{Hom}(H_n(C), \mathbb{Z}) \cong B$; and by (ii),

$$H_n(C)/T_n \cong (T_n \oplus B)/(T_n \oplus \{0\}) \cong T_n/T_n \oplus B/\{0\} \cong \{0\} \oplus B \cong B.$$ 

(technically the first isomorphism would need to be verified.) Hence $\text{Hom}(H_n(C), \mathbb{Z}) \cong H_n(C)/T_n$, and by Theorem 3.5 we have the split short exact sequence

$$0 \longrightarrow \text{Ext}(H_{n-1}(C), \mathbb{Z}) \longrightarrow H^n(C; \mathbb{Z}) \longrightarrow \text{Hom}(H_n(C), \mathbb{Z}) \longrightarrow 0.$$ 

Therefore, by the Splitting Lemma, $H^n(C; \mathbb{Z}) \cong T_{n-1} \oplus (H_n(C)/T_n)$. \hfill \blacksquare

It’s high time to consider some examples.

**Example 3.8.** Show that the map $H \xrightarrow{n} H$ given by $x \mapsto nx$ for each $x \in H$ induces multiplication by $n$ in $\text{Ext}(H, G)$, and so too does $G \xrightarrow{n} G$.

**Solution.** Given an abelian group $H$, let (28) be a free resolution $F$ of $H$. Define $n : H \rightarrow H$ by $n(x) = nx$. Then $n$ can be extended to a chain map $n_i : F_i \rightarrow F_i$ where $n_i(x) = nx$ for each $i \geq 0$ and $x \in F_i$:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
\downarrow n_2 & & \downarrow n_1 & & \downarrow n_0 & & \downarrow n & & \downarrow n \\
\cdots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0
\end{array}$$

(It’s straightforward to verify that the diagram is commutative.) Dualizing yields
For each $i$, $n_i^*(\alpha) = \alpha \circ n_i$, where

$$(\alpha \circ n_i)(x) = \alpha(nx) = n\alpha(x) = (n\alpha)(x)$$

so that $n_i^*(\alpha) = n\alpha$. In particular the map $n_1^*$ induces

$$(n_1^*)_*: H^1(F; G) \to H^1(F; G)$$

given by

$$(n_1^*)_*(\alpha + \text{Im} f_1^*) = n_1^*(\alpha) + \text{Im} f_1^* = n\alpha + \text{Im} f_1^* = n(\alpha + \text{Im} f_1^*)$$

for each $\alpha \in \text{Ker} f_2^*$. Thus $(n_1^*)_*$ is multiplication by $n$ in $H^1(F; G)$, and since $\text{Ext}(H, G) \cong H^1(F; G)$ it’s immediate that $n_1^*$, which ultimately was “induced” by $n$, in turn induces multiplication by $n$ in $\text{Ext}(H, G)$.

Now let $n: G \to G$ be multiplication by $n$ in $G$. This map induces homomorphisms $\overline{n}_i: F_i^* \to F_i^*$ given by

$$\overline{n}_i(\alpha) = n \circ \alpha.$$ 

For each $x \in F_i,$

$$(n \circ \alpha)(x) = n(\alpha(x)) = n\alpha(x) = (n\alpha)(x),$$

so $\overline{n}_i(\alpha) = n\alpha$. The map $\overline{n}_1$ in particular induces

$$\overline{n}_1: H^1(F; G) \to H^1(F; G)$$

given by

$$\overline{n}_1(\alpha + \text{Im} f_1^*) = \overline{n}_1(\alpha) + \text{Im} f_1^* = n\alpha + \text{Im} f_1^*,$$

so $\overline{n}_1$ is multiplication by $n$ on $H^1(F; G)$, and by extension $\text{Ext}(H, G)$ as well. 

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5The mystical shape-shifting abilities of the term “induce” is common coin amongst the high priesthood of algebra, and unfortunately we just have to accept it as a symptom of human laziness.