Matrix Analysis

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Eigenvalues and Eigenvectors

1.1 – The Eigenvalues of a Matrix

For $A \in \mathbb{C}^{n\times n}$, if $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \setminus \{0\}$ are such that $Ax = \lambda x$, then λ is an **eigenvalue** of A, x is an **eigenvector** of A associated with λ, and (λ, x) is an **eigenpair** for A. The **spectrum** of A, denoted by $\sigma(A)$, is the set of all eigenvalues of A.

Proposition 1.1. *For any $A \in \mathbb{C}^{n\times n}$,*

$$\sigma(\bar{A}) = \overline{\sigma(A)}.$$

Proof. Suppose $\lambda \in \sigma(\bar{A})$, so $\bar{A}x = \lambda x$ for some $x \neq 0$. Then

$$A\bar{x} = \overline{\bar{A}x} = \overline{\lambda x} = \bar{\lambda}\bar{x},$$

and so $\bar{\lambda} \in \sigma(A)$. Since $\overline{\sigma(A)} = \{\bar{\mu} : \mu \in \sigma(A)\}$, it follows that $\lambda = \bar{\bar{\lambda}} \in \overline{\sigma(A)}$ and hence $\sigma(\bar{A}) \subseteq \overline{\sigma(A)}$.

Next suppose that $\lambda \in \overline{\sigma(A)}$, so $\lambda = \bar{\mu}$ for some $\mu \in \sigma(A)$. Thus there exists $x \neq 0$ such that $Ax = \mu x = \bar{\lambda}x$, so that $\bar{A}\bar{x} = \lambda\bar{x}$ and then $\lambda \in \sigma(\bar{A})$. Hence $\overline{\sigma(A)} \subseteq \sigma(\bar{A})$. ■

If $A \in \mathbb{R}^{n\times n}$, then $\bar{A} = A$ and we have $\overline{\sigma(A)} = \sigma(A)$ by Proposition 1.1. Thus $\lambda \in \sigma(A)$ implies $\lambda \in \overline{\sigma(A)}$, so that $\lambda = \bar{\mu}$ for some $\mu \in \sigma(A)$, and therefore $\bar{\lambda} \in \sigma(A)$.

Theorem 1.2. *Let $p(t)$ be a polynomial of degree k and $A \in \mathbb{C}^{n\times n}$.*

1. *If (λ, x) is an eigenpair for A, then $(p(\lambda), x)$ is an eigenpair for $p(A)$.*
2. *If $k \geq 1$ and $\mu \in \sigma(p(A))$, then there exists some $\lambda \in \sigma(A)$ such that $\mu = p(\lambda)$.*

Proposition 1.3. *For any $A \in \mathbb{C}^{n\times n}$ and $\lambda, \mu \in \mathbb{C}$, $\lambda \in \sigma(A)$ if and only if $\lambda + \mu \in \sigma(A + \mu I)$.*

Proof. Suppose $\lambda \in \sigma(A)$, so $Ax = \lambda x$ for some $x \neq 0$. Then

$$(A + \mu I)x = Ax + \mu Ix = \lambda x + \mu x = (\lambda + \mu)x$$

shows that $\lambda + \mu \in \sigma(A + \mu I)$.

Next suppose that $\lambda+\mu \in \sigma(A+\mu I)$, so there exists some $x \neq 0$ such that $(A+\mu I)x = (\lambda+\mu)x$. This gives $Ax + \mu x = \lambda x + \mu x$, and finally $Ax = \lambda x$. That is, $\lambda \in \sigma(A)$. ■

Problem 1.4. Suppose $A \in \mathbb{C}^{n\times n}$ is nonsingular. Show that if (λ, x) is an eigenpair for A, then (λ^{-1}, x) is an eigenpair for A^{-1}.

Solution. Suppose (λ, x) is an eigenpair for A. Then $Ax = \lambda x$, with $\lambda \neq 0$ since A is nonsingular, and $x \neq 0$ since x is an eigenvector. Now,

$$Ax = \lambda x \quad\Rightarrow\quad A^{-1}(Ax) = A^{-1}(\lambda x) \quad\Rightarrow\quad x = \lambda(A^{-1}x)\lambda^{-1}x = A^{-1}x,$$

and so (λ^{-1}, x) is an eigenpair for A^{-1}. ■

Problem 1.5. Let $A \in \mathbb{C}^{n\times n}$ and $e = [1, \ldots, 1]^\top \in \mathbb{C}^n$.

(a) Show that the sum of the entries in each row of A is 1 if and only if $(1, e)$ is an eigenpair for A.

(b) Suppose that the sum of the entries in each row of A is 1. If A is nonsingular, show that the sum of the entries in each row of A^{-1} is also 1.

(c) Suppose that the sum of the entries in each row of A is 1. For any polynomial $p(t)$ show that the sums of the entries in each row of $p(A)$ are equal.

Solution.

(a) Suppose the sum of the entries in each row of A is 1, so $\sum_{j=1}^{n}[A]_{ij} = 1$ for each $1 \leq i \leq n$. Then

$$[Ae]_{i1} = \sum_{j=1}^{n}[A]_{ij}[e]_{j1} = \sum_{j=1}^{n}[A]_{ij} = 1$$

for each i, which is to say $Ae = e$ and thus $(1, e)$ is an eigenpair for A.

Next suppose that $(1, e)$ is an eigenpair for A, so $Ae = e$. Then for each $1 \leq i \leq n$,

$$\sum_{j=1}^{n}[A]_{ij}[e]_{j1} = [Ae]_{i1} = [e]_{i1} = 1,$$

and therefore the sum of the entries in each row of A is 1.

(b) Suppose A is nonsingular. By part (a), $(1, e)$ is an eigenpair of A, and so $(1, e)$ is an eigenpair for A^{-1} by Problem 1.4. It then follows by part (a) that the sum of the entries in each row of A^{-1} is 1.

(c) By part (a), $(1, e)$ is an eigenpair for A, and thus $(p(1), e)$ is an eigenpair for $p(A)$ by Theorem 1.2, so that $p(A)e = p(1)e$. For each $1 \leq i \leq n$,

$$p(1) = [p(1)e]_{i1} = [p(A)e]_{i1} = \sum_{j=1}^{n}[p(A)]_{ij}[e]_{j1} = \sum_{j=1}^{n}[p(A)]_{ij},$$

and so the ith-row entries of $p(A)$ add to $p(1)$ for each i. ■

Problem 1.6. Consider the block diagonal matrix

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad A_{ii} \in \mathbb{C}^{n_i}.$$

Show that $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$.

Solution. Suppose that $\lambda \in \sigma(A)$, so there exists nonzero $x \in \mathbb{C}^{n_1+n_2}$ such that $Ax = \lambda x$. In particular $x = [x_1 \;\; x_2]^\top$ for some $x_1 \in \mathbb{C}^{n_1}$ and $x_2 \in \mathbb{C}^{n_2}$. Now,

$$\begin{bmatrix} A_{11}x_1 \\ A_{22}x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax = \lambda x = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix},$$

so $A_{11}x_1 = \lambda x_1$ and $A_{22}x_2 = \lambda x_2$. Since $x \neq 0$, either $x_1 \neq 0$ or $x_2 \neq 0$, and thus either $\lambda \in \sigma(A_{11})$ or $\lambda \in \sigma(A_{22})$. Therefore $\lambda \in \sigma(A_{11}) \cup \sigma(A_{22})$, showing $\sigma(A) \subseteq \sigma(A_{11}) \cup \sigma(A_{22})$.

Now suppose that $\lambda \in \sigma(A_{11}) \cup \sigma(A_{22})$, so either $\lambda \in \sigma(A_{11})$ or $\lambda \in \sigma(A_{22})$. Assume $\lambda \in \sigma(A_{11})$. Then there exists nonzero $y \in \mathbb{C}^{n_1}$ such that $A_{11}y = \lambda y$. Let $\hat{y} = [y \;\; 0]^\top \in \mathbb{C}^{n_1+n_2}$ (so $0 \in \mathbb{C}^{n_2}$). Then $\hat{y} \neq 0$ with

$$A\hat{y} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11}y \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda y \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} y \\ 0 \end{bmatrix} = \lambda \hat{y},$$

implying that $\lambda \in \sigma(A)$.

Next assume $\lambda \in \sigma(A_{22})$. Then there exists nonzero $z \in \mathbb{C}^{n_2}$ such that $A_{22}z = \lambda z$. Let $\hat{z} = [0 \;\; z]^\top \in \mathbb{C}^{n_1+n_2}$ (so $0 \in \mathbb{C}^{n_1}$). Then $\hat{z} \neq 0$ with

$$A\hat{z} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ A_{22}z \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda z \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ z \end{bmatrix} = \lambda \hat{z},$$

again implying $\lambda \in \sigma(A)$. We conclude that $\sigma(A_{11}) \cup \sigma(A_{22}) \subseteq \sigma(A)$. ■

Problem 1.7. Let $A \in \mathbb{C}^{n\times n}$ be idempotent. Show that $\sigma(A) \subseteq \{0, 1\}$. If A is nonsingular, show that $A = I$.

A matrix $A \in \mathbb{C}^{n\times n}$ is **idempotent** (or a **projection**) if $A^2 = A$.

Solution. Suppose (λ, x) is an eigenpair for A, so $x \neq 0$ is such that $Ax = \lambda x$. Now, since $A^2 = A$,

$$Ax = \lambda x \;\;\Rightarrow\;\; A^2x = \lambda Ax \;\;\Rightarrow\;\; Ax = \lambda(\lambda x) \;\;\Rightarrow\;\; \lambda x = \lambda^2 x,$$

and so $(\lambda^2 - \lambda)x = 0$. Since $x \neq 0$, it follows that $\lambda^2 - \lambda = 0$, and so λ is either 0 or 1.

Next suppose that A is nonsingular, so A^{-1} exists. Then

$$A^2 = A \;\;\Rightarrow\;\; A^{-1}A^2 = A^{-1}A \;\;\Rightarrow\;\; A = I,$$

and so I is the only nonsingular idempotent matrix. ■

A matrix $A \in \mathbb{C}^{n\times n}$ is **nilpotent** if there exists some $k \in \mathbb{N}$ such that $A^k = 0$. If A is nilpotent the Well-Ordering Principle implies there exists some $m \in \mathbb{N}$ such that $A^m = 0$ and $A^k \neq 0$ for all $1 \leq k < m$; that is, there is some minimal positive exponent m for which $A^m = 0$.

Problem 1.8. Show that all eigenvalues of a nilpotent matrix are 0. Give an example of a nonzero nilpotent matrix. Show that 0 is the only nilpotent idempotent matrix.

Solution. Suppose $A \in \mathbb{C}^{n\times n}$ be nilpotent, and let $k \in \mathbb{N}$ be such that $A^k = 0$. Let (λ, x) be an eigenpair for A, so $x \neq 0$ and $Ax = \lambda x$. If $k = 1$ then $A = 0$, in which case $\sigma(A) = \{0\}$ is clear. Assume $k \geq 2$. Now,

$$\lambda A^{k-1}x = A^{k-1}\lambda x = A^{k-1}Ax = A^k x = 0x = 0,$$

but with induction we also find that

$$\lambda A^{k-1}x = \lambda\lambda^{k-1}x = \lambda^k x.$$

Hence $\lambda^k x = 0$, so $\lambda^k = 0$ since $x \neq 0$, and finally $\lambda = 0$.

An example of a nonzero nilpotent matrix would be

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Suppose that B is a nilpotent idempotent matrix. Let $m = \min\{k \in \mathbb{N} : A^k = 0\}$. Suppose $m \geq 2$, so A^{m-2} exists (with $A^{m-2} = I$ if $m = 2$). Since $A^2 = A$, it follows that $A^{m-2}A^2 = A^{m-2}A$, and hence $A^{m-1} = A^m = 0$ for $m - 1 \geq 1$. This contradicts the minimality of m, and so $m = 1$ must be the case. Therefore $A = 0$, and we conclude that 0 is the only nilpotent idempotent matrix. ■

A matrix $A \in \mathbb{C}^{n\times n}$ is **Hermitian** if $A^* = A$, where A^* denotes the **conjugate transpose** of A. If $x \in \mathbb{C}^n$, then we define the **Euclidean norm** of x to be

$$\|x\|_2 = x^*x.$$

If $x \neq 0$ then $\|x\|_2 > 0$.

Problem 1.9. Show that if $A \in \mathbb{C}^{n\times n}$ is Hermitian, then all the eigenvalues of A are real.

Solution. Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian, and let (λ, x) be an eigenpair for A so that $Ax = \lambda x$ for $x \neq 0$. Now,

$$Ax = \lambda x \;\Rightarrow\; (Ax)^* = (\lambda x)^* \;\Rightarrow\; x^*A^* = \bar{\lambda}x^* \;\Rightarrow\; x^*Ax = \bar{\lambda}x^*x,$$

and since $x^*x > 0$, it follows that

$$\bar{\lambda} = \frac{x^*Ax}{x^*x} = \frac{x^*\lambda x}{x^*x} = \lambda.$$

Therefore $\lambda \in \mathbb{R}$. ■

1.2 – The Characteristic Polynomial

Given $A \in \mathbb{C}^{n\times n}$, the **characteristic polynomial** of A is

$$p_A(t) = \det(tI - A),$$

and it is known that $p_A(\lambda) = 0$ if and only if $\lambda \in \sigma(A)$. Define the **algebraic multiplicity** of $\lambda \in \sigma(A)$, denoted by $\alpha_A(\lambda)$, to be the multiplicity of λ as a zero of the polynomial $p_A(t)$. If $\lambda \notin \sigma(A)$ then we define $\alpha_A(\lambda) = 0$. In general,

$$\sum_{\lambda\in\sigma(A)} \alpha_A(\lambda) = n.$$

It is known that

$$p_A(t) = t^n - (\operatorname{tr} A)t^{n-1} + \cdots + (-1)^n \det A. \tag{1.1}$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, each distinct value in the list repeated according to its algebraic multiplicity, then it is also known that

$$\operatorname{tr} A = \sum_{k=1}^{n} \lambda_k \quad \text{and} \quad \det A = \prod_{k=1}^{n} \lambda_k. \tag{1.2}$$

Alternatively we may write

$$\operatorname{tr} A = \sum_{\lambda\in\sigma(A)} \alpha_A(\lambda)\lambda \quad \text{and} \quad \det A = \prod_{\lambda\in\sigma(A)} \lambda^{\alpha_A(\lambda)}. \tag{1.3}$$

For the proof of the first proposition we recall the inductive Laplace expansion definition of a determinant: if $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, then

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det A_{ik} = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det A_{kj}, \tag{1.4}$$

where the symbol A_{ij} generally denotes the matrix in $\mathbb{C}^{(n-1)\times(n-1)}$ resulting from deleting the ith row and jth column from matrix A. From (1.4) can be derived the **Leibniz formula**

$$\det A = \sum_{\sigma\in\mathcal{S}_n} \left(\operatorname{sgn}\sigma \prod_{i=1}^{n} a_{i\sigma(i)} \right),$$

where $\mathcal{S}_n$ is the symmetric group of degree n.

Proposition 1.10. *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is an upper or lower triangular matrix, then*

$$\det A = \prod_{i=1}^{n} a_{ii}.$$

Proof. If $n = 1$ then $\det A = a_{11}$, which confirms the base case. Now fix $n \geq 1$ and suppose $\det A = \prod_{i=1}^{n} a_{ii}$ for all $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Let $A \in \mathbb{C}^{(n+1)\times(n+1)}$ be upper triangular, so $a_{ij} = 0$ whenever $i > j$. With Laplace expansion by minors along row $n+1$,

$$\det A = \sum_{k=1}^{n} (-1)^{n+1+k} a_{n+1,k} \det A_{n+1,k} = a_{n+1,n+1} \det A_{n+1,n+1}$$

$$= a_{n+1,n+1} \prod_{i=1}^{n} a_{ii} = \prod_{i=1}^{n+1} a_{ii},$$

where the last equality follows from our inductive hypothesis and the observation that $A_{n+1,n+1} \in \mathbb{C}^{n\times n}$ is upper triangular.

The argument is much the same for a lower triangular matrix, while precisely the same analysis as above applies to a diagonal matrix since such matrices are also upper triangular. ■

Given an upper triangular matrix $A \in \mathbb{C}^{n\times n}$ with diagonal elements $a_{11}, \ldots, a_{nn}$, we find $tI - A$ to be also upper triangular with diagonal entries $t - a_{11}, \ldots, t - a_{nn}$. Thus

$$p_A(t) = \det(tI - A) = \prod_{i=1}^{n} (t - a_{ii}),$$

so that $p_A(\lambda) = 0$ if and only if $\lambda = a_{ii}$ for some $1 \le i \le n$. *Thus the set of diagonal entries in an upper (or lower) triangular matrix equals the set of eigenvalues of the matrix.* Moreover, the algebraic multiplicity of $\lambda \in \sigma(A)$ will equal the number of times that λ appears as an entry on the main diagonal of A.

In proving the next proposition, which is a generalization of the previous one, we use the fact that deleting a row and column from a block triangular matrix results again in a block triangular matrix. Note in the block matrix A below we use the symbol ♣ to denote that the entries above the main diagonal are arbitrary, while the $\mathbf{0}$ in the block matrix indicates that all entries below the main diagonal are zero.

Proposition 1.11. *For any $k \ge 1$, given the block upper triangular matrix*

$$A = [a_{ij}] = \begin{bmatrix} B_{11} & & & \clubsuit \\ & B_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & B_{kk} \end{bmatrix}, \quad \forall 1 \le i \le k \big(B_{ii} \in \mathbb{C}^{n_i \times n_i}\big),$$

we have

$$\det A = \prod_{i=1}^{k} \det B_{ii}. \tag{1.5}$$

Proof. Fix $k \ge 1$. The statement of the proposition is certainly true if $A \in \mathbb{C}^{1\times 1}$. Suppose it is true whenever $A \in \mathbb{C}^{n\times n}$ for some fixed $n \ge 1$. Let $A \in \mathbb{C}^{(n+1)\times(n+1)}$, so $\sum_{i=1}^{k} n_i = n + 1$. If $n_1 = 1$, so that $B_{11} = a_{11}$, then since $a_{i1} = 0$ for $i \ge 2$, we have

$$\det A = \sum_{i=1}^{n+1} (-1)^{i+1} a_{i1} \det A_{i1} = a_{11} \det A_{11} = \det B_{11} \prod_{j=2}^{n+1} \det B_{jj} = \prod_{j=1}^{n+1} \det B_{jj},$$

with the last equality following from our inductive hypothesis and the observation that

$$A_{11} = \begin{bmatrix} B_{22} & & \clubsuit \\ & \ddots & \\ \mathbf{0} & & B_{kk} \end{bmatrix},$$

is an $n \times n$ block upper triangular matrix. If $n_1 = n+1$, then A is upper triangular and (1.5) obtains from Proposition 1.10.

Now assume that $1 < n_1 < n+1$, so $B_{11} \in \mathbb{C}^{n_1 \times n_1}$ for $n_1 \geq 2$. Then, since $a_{i1} = 0$ for $i > n_1$,

$$\det A = \sum_{i=1}^{n+1} (-1)^{i+1} a_{i1} \det A_{i1} = \sum_{i=1}^{n_1} (-1)^{i+1} a_{i1} \det A_{i1},$$

and since

$$A_{i1} = \begin{bmatrix} (B_{11})_{i1} & & & \clubsuit \\ & B_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & B_{kk} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

is block upper triangular, by our hypothesis we obtain

$$\begin{aligned} \det A &= \sum_{i=1}^{n_1} \left((-1)^{i+1} a_{i1} \det(B_{11})_{i1} \prod_{j=2}^{k} \det B_{jj} \right) \\ &= \left(\sum_{i=1}^{n_1} (-1)^{i+1} a_{i1} \det(B_{11})_{i1} \right) \prod_{j=2}^{k} \det B_{jj} \\ &= \prod_{j=1}^{k} \det B_{jj}, \end{aligned}$$

as desired. ■

The conclusion of the proposition above applies also to block lower triangular matrices, since $\det(A^\top) = \det A$ holds for any square matrix A.

Proposition 1.12. *For the block upper triangular matrix*

$$A = [a_{ij}] = \begin{bmatrix} A_{11} & & \clubsuit \\ & \ddots & \\ \mathbf{0} & & A_{kk} \end{bmatrix}, \quad \forall 1 \leq i \leq k \big(B_{ii} \in \mathbb{C}^{n_i \times n_i} \big),$$

we have

$$p_A(t) = \prod_{i=1}^{k} p_{A_{ii}}(t),$$

and hence $\sigma(A) = \bigcup_{i=1}^{k} \sigma(A_{ii})$ *with* $\alpha_A(\lambda) = \sum_{i=1}^{k} \alpha_{A_{ii}}(\lambda)$.

Proof. By definition,

$$p_A(t) = \det(tI - A) = \begin{vmatrix} tI - A_{11} & & -\clubsuit \\ & \ddots & \\ \mathbf{0} & & tI - A_{kk} \end{vmatrix},$$

where $-\clubsuit$ denotes entries of opposite value as those represented by $\clubsuit$ in A, and each $tI - A_{ii}$ is a square matrix of size $n_i \times n_i$. By Proposition 1.11 it follows that

$$p_A(t) = \prod_{i=1}^{k}(tI - A_{ii}) = \prod_{i=1}^{k} p_{A_{ii}}(t).$$

The observation that $p_A(\lambda) = 0$ iff $p_{A_{ii}}(\lambda) = 0$ for some $1 \le i \le k$ iff $\lambda \in \sigma(A_{ii})$ for some $1 \le i \le k$ readily implies the final statement in the proposition. ■

Problem 1.13. For $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. If $A, S \in \mathbb{C}^{n\times n}$ with S nonsingular, show that $\operatorname{tr}(S^{-1}AS) = \operatorname{tr} A$ and $\det(S^{-1}AS) = \det A$.

Solution. For $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$ we have

$$\operatorname{tr}(BA) = \sum_{k=1}^{n}[BA]_{kk} = \sum_{k=1}^{n}\left(\sum_{j=1}^{m}[B]_{kj}[A]_{jk}\right) = \sum_{j=1}^{m}\left(\sum_{k=1}^{n}[A]_{jk}[B]_{kj}\right) = \sum_{j=1}^{m}[AB]_{jj} = \operatorname{tr}(AB),$$

and so for $A, S \in \mathbb{C}^{n\times n}$ with S nonsingular we find that

$$\operatorname{tr}(S^{-1}AS) = \operatorname{tr}\big(S^{-1}(AS)\big) = \operatorname{tr}\big((AS)S^{-1}\big) = \operatorname{tr}\big(A(SS^{-1})\big) = \operatorname{tr} A,$$

and also

$$\det(S^{-1}AS) = \det(S^{-1})\det A \det S = (\det S)^{-1}\det A \det S = \det A.$$

■

For the next problem we make use of a convenient property of products of diagonal matrices, namely

$$\operatorname{diag}(a_1, \dots, a_n)\operatorname{diag}(b_1, \dots, b_n) = \operatorname{diag}(a_1 b_1, \dots, a_n b_n),$$

which naturally extends to products of three or more diagonal matrices.

Problem 1.14. Let $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix. Compute the characteristic polynomial $p_D(t)$ and show that $p_D(D)$.

Solution. For $D = \operatorname{diag}(d_1, \dots, d_n)$, by Proposition 1.10,

$$p_D(t) = \det(tI - D) = \det\big(\operatorname{diag}(t - d_1, \dots, t - d_n)\big) = \prod_{i=1}^{n}(t - d_i).$$

Now,

$$\begin{aligned} p_D(D) &= \prod_{i=1}^{n}(D - d_i I) = \prod_{i=1}^{n}\operatorname{diag}(d_1 - d_i, \dots, d_n - d_i) \\ &= \operatorname{diag}\left(\prod_{i=1}^{n}(d_1 - d_i), \dots, \prod_{i=1}^{n}(d_n - d_i)\right) \\ &= \operatorname{diag}(0, \dots, 0) = 0. \end{aligned}$$

■

Problem 1.15. Show that the trace of a nilpotent matrix is zero. What is the characteristic polynomial of a nilpotent matrix?

Solution. Let $A \in \mathbb{C}^{n\times n}$ be nilpotent, so there exists some $k \in \mathbb{N}$ such that $A^k = 0$. Problem 1.8 gives $\sigma(A) = \{0\}$, and since $\alpha_A(0) = n$, by (1.3) we obtain

$$\operatorname{tr} A = \sum_{\lambda\in\sigma(A)} \alpha_A(\lambda)\lambda = (n)(0) = 0.$$

Now, $p_A(t) = 0$ iff $t \in \sigma(A)$, so 0 is the only root of $p_A(t)$, and by (1.1) we conclude that $p_A(t) = t^n$. ■

Problem 1.16. Let $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. Show that $p_{A+\lambda I}(t) = p_A(t-\lambda)$. What are the eigenvalues of $A + \lambda I$?

Solution. Directly we have

$$p_{A+\lambda I}(t) = \det\big(tI - (A+\lambda I)\big) = \det\big((t-\lambda)I - A\big) = p_A(t-\lambda).$$

Now, for any $\lambda_k \in \sigma(A)$,

$$p_{A+\lambda I}(\lambda_k + \lambda) = p_A\big((\lambda_k+\lambda) - \lambda\big) = p_A(\lambda_k) = 0,$$

and so $\lambda_k + \lambda \in \sigma(A + \lambda I)$ for all $1 \le k \le n$. ■

Problem 1.17. Let $n \ge 3$, $B \in \mathbb{C}^{(n-2)\times(n-2)}$, and $\lambda, \mu \in \mathbb{C}$. Given

$$A = \begin{bmatrix} \lambda & \clubsuit & \clubsuit \\ 0 & \mu & 0 \\ 0 & \clubsuit & B \end{bmatrix},$$

show that

$$p_A(t) = (t-\lambda)(t-\mu)p_B(t)$$

Solution. By Proposition 1.11,

$$\begin{aligned} p_A(t) = \det(tI - A) &= \begin{vmatrix} t-\lambda & -\clubsuit & -\clubsuit \\ 0 & t-\mu & 0 \\ 0 & -\clubsuit & tI - B \end{vmatrix} \\ &= (t-\lambda)\begin{vmatrix} t-\mu & 0 \\ -\clubsuit & tI-B \end{vmatrix} = (t-\lambda)\det(t-\mu)\det(tI-B) \\ &= (t-\lambda)(t-\mu)p_B(t). \end{aligned}$$

Thus $\sigma(A) = \sigma(B) \cup \{\lambda, \mu\}$, in particular. ■

The next problem requires use of the **arithmetic-geometric mean inequality:** for $x_1, \dots, x_n \ge 0$,

$$\frac{1}{n}\sum_{k=1}^{n} x_k \ge \left(\prod_{k=1}^{n} x_k\right)^{1/n}, \tag{1.6}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Problem 1.18. Suppose $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is such that $a_{ij} \in \{0,1\}$ for all $1 \le i,j \le n$, with all eigenvalues $\lambda_1,\ldots,\lambda_n$ being positive real numbers. Show that $\operatorname{tr} A = n$, $\det A = 1$, $a_{kk} = 1$ for all $1 \le k \le n$, and $\sigma(A) = \{1\}$.

Solution. For each k we have $a_{kk} \in \{0,1\}$, so that $\operatorname{tr} A = \sum_{i=1}^n a_{kk} \le n$. Also, with (1.6) we obtain

$$(\det A)^{1/n} = \left(\prod_{k=1}^n \lambda_k\right)^{1/n} \le \frac{1}{n}\sum_{k=1}^n \lambda_k = \frac{\operatorname{tr} A}{n},$$

so $n(\det A)^{1/n} \le \operatorname{tr} A$. Now, from (1.4) it's easily seen that $\det A$ much equal an integer, but since $\det A = \prod_{k=1}^n \lambda_k$ with $\lambda_k > 0$ for each k, it is clear that $\det A \in \mathbb{N}$ and in particular $\det A \ge 1$. Hence $n(\det A)^{1/n} \ge n$, and our findings taken together give

$$n \le n(\det A)^{1/n} \le \operatorname{tr} A \le n.$$

This implies that $\operatorname{tr} A = n$, and also $n(\det A)^{1/n} = n$ so that $\det A = 1$. Of course, $n = \operatorname{tr} A = \sum_{k=1}^n a_{kk}$, and since $a_{kk} \in \{0,1\}$ for each k, we conclude that $a_{kk} = 1$ for all k. Finally, $\sum_{k=1}^n \lambda_k = \operatorname{tr} A = n$ and $\prod_{k=1}^n \lambda_k = \det A = 1$, giving

$$\frac{1}{n}\sum_{k=1}^n \lambda_k = 1 = \left(\prod_{k=1}^n \lambda_k\right)^{1/n}.$$

However, equality holds for (1.6) if and only if $x_1 = \cdots = x_n$, and so we may set $\lambda_k = \lambda$ for all k. Then

$$n = \sum_{k=1}^n \lambda_k = \sum_{k=1}^n \lambda = n\lambda,$$

giving $\lambda = 1$ and therefore $\sigma(A) = \{1\}$. ■